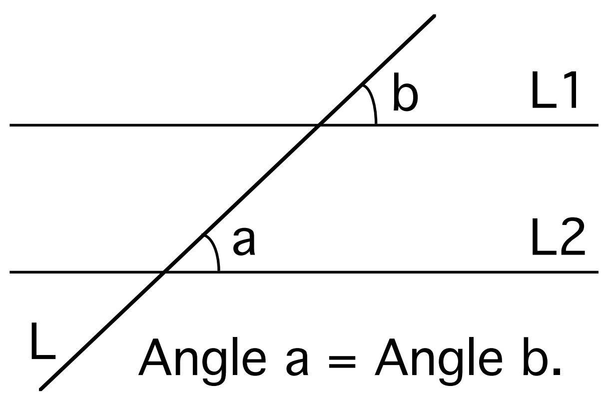
Elements of Plane Geometry
by LK

These are notes indicating just some bare essentials of plane geometry and some problems to think about. Below, we give a modified version of the axioms for Euclidean Geometry. In these axioms we do NOT take as a definition that two parallel lines never meet. Rather, we say that two lines are parallel if a line transversal to the two of them makes equal corresponding angles with each of them (see details below). This will allow us to *prove* that parallel lines do not meet.

Axioms and Definitions

1. Two points determine a line. That is, if two lines each contain distinct points p and q, then the two lines are identical. If two distinct points are given, then these points determine a line segment drawn between them. Any such line segment can be extended indefinitely to form a unique line that contains the two points.
2. Given two rays (half-lines) intersecting at a single point, there is given a measure of their separation called the *angle between the rays*. A *straight angle* occurs when the two rays lie on one straight line. All straight angles are equal (congruent). If the angle between two lines is zero, then the lines are identical with one another.
3. Given any straight line segment, a circle can be drawn having that segment as radius and one endpoint as the center. (This means that every point p on the circle lies on a segment with endpoints the center and the point p, and that all these segments are congruent to one another.)

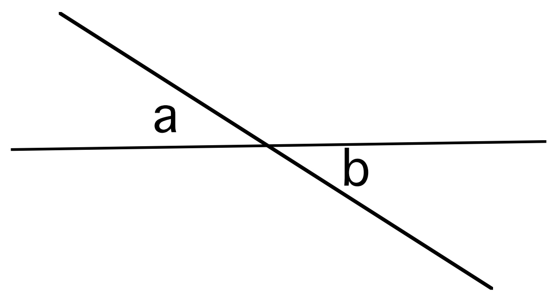
Definition: Two lines L1 and L2 are said to be *parallel* if, given a line L transverse to both L1 and L2 (i.e. L intersects L1 in one point and L intersects L2 in one point as well), then the corresponding angles between L and L1 and between L and L2 are equal.



4. Given a line L and a point p not on L, there is a unique line through p that is parallel to L.

In the exercises that follow, *use the above axioms* to prove what is asked.
Note that we have defined parallel lines in terms of the transversal angle property that corresponding angles are equal. This means that we do not assume that parallel lines never meet! We *prove* that parallel lines do not meet via the exercises below!

Exercises.
(a) Prove that vertical angles are equal.



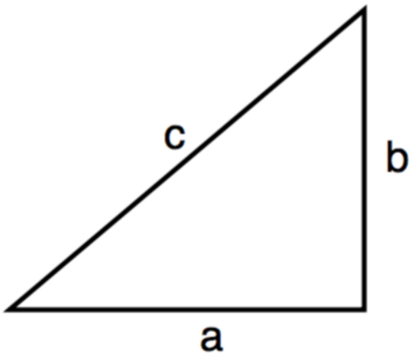
Angle a = Angle b

(b) Prove that the sum of the angles of a triangle is equal to a straight angle.
(c) Prove that distinct parallel lines never meet.

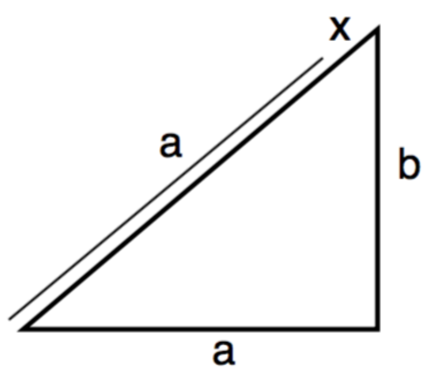
(d) Prove the Pythagorean Theorem: If a and b are the lengths of the sides of a right triangle and c is the length of the hypotenuse (the hypotenuse is the side opposite the right angle. a right angle is one half of a straight angle), then $a^2 + b^2 = c^2$.

(d') Shouldn't there be a theorem in three dimensions that says $a^2 + b^2 + c^2 = d^2$? What would the geometry be behind such a statement? What would the theorem really say?

(e) Observe that $(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$.
This formula can be used to produce triples of numbers like (3,4,5) where $3^2 + 4^2 = 5^2$. These are called Pythagorean Triples.
Investigate this phenomenon. In the figure below, I indicate one way to see how the algebraic formula above is related to the geometry of the triangle.



$$a^2 + b^2 = c^2$$



$$a^2 + b^2 = (a + x)^2$$

$$a^2 + b^2 = a^2 + 2ax + x^2$$

$$b^2 = 2ax + x^2$$

$$2ax = b^2 - x^2$$

$$a = (b^2 - x^2)/(2x)$$

$$b = b$$

$$c = a + x = (b^2 + x^2)/2x$$

Scale everything up by a factor of 2x and get.

$$A = b^2 - x^2$$

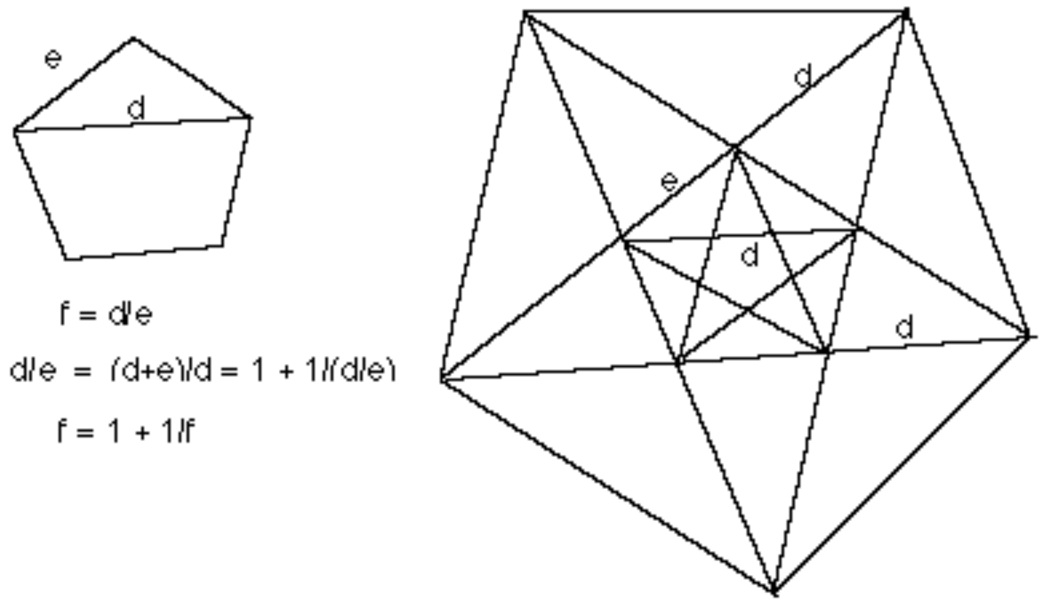
$$B = 2bx$$

$$C = b^2 + x^2$$

$$(b^2 - x^2)^2 + (2bx)^2 = (b^2 + x^2)^2$$

(f) **The Pentagon and the Golden Ratio**

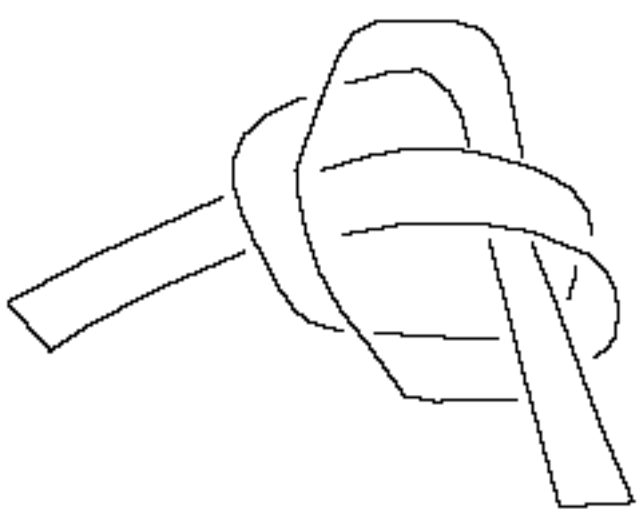
Contemplate the following figure.



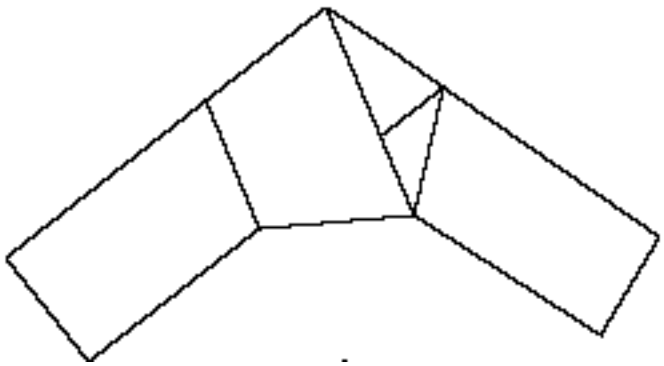
The small pentagon has its edge e and chord d labeled. We embed the small pentagon in the larger one and observe via a parallelogram and by similar triangles that
d/e = (d+e)/d (similar triangles).
Thus with F = d/e we have
d/e = 1 + e/d whence
F = 1 + 1/F.
This is sufficient to show that F is the golden ratio. Note how the golden ratio appears here through the way that a pentagon embeds in a pentagon, a pentagonal self-reference.

(g) **The Trefoil Knot, the Pentagon and the Golden Ratio**

Here is an experiment that you can do with the trefoil knot and a strip of paper. Tie the strip into a trefoil and pull it gently tight and fold it so that you obtain a flat knot.



As you pull it tight with care a pentagon will appear!



Now it is intuitively clear that this pentagonal form of the trefoil knot uses the least length of paper for a given width of paper (to make a flattened trefoil). At this writing, I do not have a proof of this statement.

Euclid's Axioms

Here are the axioms that Euclid used for geometry in 300BC. Note that for Euclid parallel lines are assumed to never meet. So his system, logically equivalent to ours, starts with different assumptions.

1. A straight **line segment** can be drawn joining any two points.
2. Any straight **line segment** can be extended indefinitely in a straight line.
3. Given any straight **line segment**, a **circle** can be drawn having the segment as **radius** and one endpoint as center.
4. All **right angles** are **congruent**.
5. If two lines are drawn which **intersect** a third in such a way that the sum of the inner angles on one side is less than two **right angles**, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is equivalent to what is known as the **parallel postulate**.

Euclid's fifth postulate cannot be proven as a theorem, although this was attempted by many people. Euclid himself used only the first four postulates ("absolute geometry") for the first 28 propositions of the Elements, but was forced to invoke the parallel postulate on the 29th. In 1823, Janos Bolyai and Nicolai Lobachevsky independently realized that entirely self-consistent "non-Euclidean geometries" could be created in which the parallel postulate *did not hold.* (Gauss had also discovered but suppressed the existence of non-Euclidean geometries.)

You are encouraged to look up the subject of Non-Euclidean Geometry in the library or on the web.

Analytic Geometry

Undoubtedly, you are already familiar with analytic geometry, sometimes called coordinate geometry. In coordinate geometry a pair of real numbers (x,y) is a *point*. Two points (a,b) and (c,d) are equal if and only if a = c and b = d. Two distinct points (a,b) and (c,d) determine the *line* consisting in all (x,y) such that (x-a)(d-b) = (y-b)(c-a). One can define angles by using vector dot products and a few facts about cosines and sines. The basic formula is that <(a,b)|(c,d)> = sqrt(a^2 + b^2)sqrt(c^2 + d^2) cos(Theta) where <(a,b)|(c,d)> = dot product of (a,b) and (c,d) = ac + bd, and Theta is the angle between the ray from (0,0) to (a,b) and the ray from (0,0) to (c,d). With these definitions of point, line and angle we can prove all our original axioms for geometry or we can prove Euclid's axioms. Thus we have a model for geometry in terms of coordinate points and one can often use the coordinates to explore and verify geometrical results. The French mathematician Rene Descartes is the originator of coordinate geometry.