(HW#4 & HW#5)

Selected HW Solutions – Math 313

p54 | 2.4.1 | Show that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ div $\Rightarrow \sum_{n=1}^{\infty} b_n$ div.

It is assumed that $\{b_n\}$ decreasing & $b_n \geq 0 \ \forall n \in \mathbb{N}$.

Let $t_m = b_1 + b_2 + \cdots + b_m$.

Then $b_1 + b_2 + (b_3+b_4) + (b_5+b_6+b_7+b_8) = t_{2^3}$

$$\vee$$

$$b_1 + b_2 + 2b_3 + 2^2 b_5$$

No better: $b_1 + (b_2+b_3) + (b_4+b_5+b_6+b_7) = t_{2^3-1}$

$$\vee$$

$$b_1 + 2b_2 + 2^2 b_4$$

and so we see that (by induction)

$$t_{2^{n+1}-1} > \underbrace{b_1 + 2b_2 + 2^2 b_4 + \cdots + 2^n b_{2^n}}$$

This is the n-th partial sum for $\sum_{n=0}^{\infty} 2^n b_{2^n}$ & so we see that

$\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges $\Longrightarrow \sum_{n=1}^{\infty} b_n$ diverges. //

2.4.2 | $x_1 = 3$, $x_{n+1} = \dfrac{1}{4 - x_n}$

$x_1 = 3$, $x_2 = 1$, $x_3 = \frac{1}{3}$, $x_4 = \frac{3}{11}$, $x_5 = \frac{11}{41}$, ...

Prove by induction that
$$0 < x_{n+1} < x_n.$$

(next page)

$$x_{n+1} = \frac{1}{4-x_n}$$

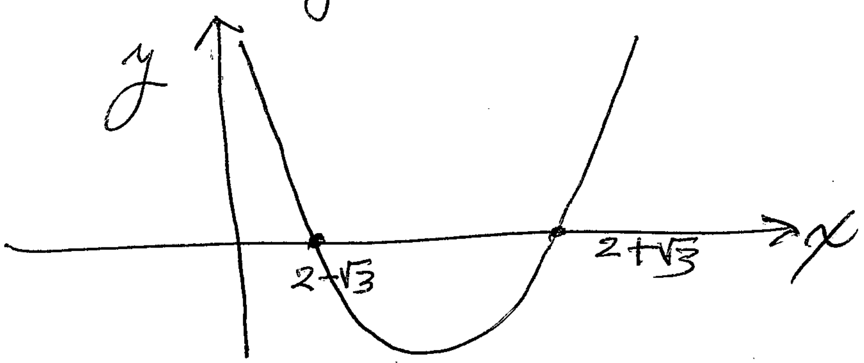
If the limit exists then
$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1}$ (easy to check this).
$\therefore$ If the limit exists & $x = \lim_{n\to\infty} x_n$, then

$$x = \frac{1}{4-x}$$

$$\Rightarrow 4x - x^2 = 1 \Rightarrow x^2 - 4x + 1 = 0$$

$$\Rightarrow x = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}.$$

Consider $y = x^2 - 4x + 1$



We expect from our computations that for $x_1 = 3$ we would have $x_n \in (2-\sqrt{3}, 2+\sqrt{3})$ for all n and that $x_{n+1} < x_n$ $\forall n$. Then we would have $\lim_{n\to\infty} x_n = 2 - \sqrt{3}$.

We must prove by induction that 1) $x_n \in (2-\sqrt{3}, 2+\sqrt{3})$
and 2) $x_{n+1} < x_n$ for $n = 1, 2, 3, \ldots$

First we prove 1).

I. $n = 1$, $x_1 = 3 \in (2-\sqrt{3}, 2+\sqrt{3})$ ✓

II. Suppose $2-\sqrt{3} < x_n < 2+\sqrt{3}$.

We must show that this implies that $\frac{1}{4-x_n} \in (2-\sqrt{3}, 2+\sqrt{3})$.

But (a) $\frac{1}{4-x_n} < 2+\sqrt{3}$ (N.B. $4-x_n > 0$)

$\iff 4-x_n > \frac{1}{2+\sqrt{3}} = \frac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}$

$\iff 4-(2-\sqrt{3}) > x_n$

$\iff 2+\sqrt{3} > x_n$ ✓

(b) $2-\sqrt{3} < \frac{1}{4-x_n}$

$\iff 4-x_n < \frac{1}{2-\sqrt{3}} = \frac{2+\sqrt{3}}{1}$

$\iff x_n > 4-2-\sqrt{3} = 2-\sqrt{3}$ ✓

Thus, it follows by induction that $2-\sqrt{3} < x_n < 2+\sqrt{3}$ for all $n = 1, 2, 3, \ldots$

Now we have $x_{n+1} < x_n$

$\iff \frac{1}{4-x_n} < x_n$

$\iff 1 < 4x_n - x_n^2$

$\iff \underline{x_n^2 - 4x_n + 1 < 0}$

But we know (using $y = x^2-4x+1$) that $2-\sqrt{3} < x < 2+\sqrt{3}$

$\iff x^2 - 4x + 1 < 0$.

$\therefore$ $x_{n+1} < x_n$ for all $n = 1, 2, 3, \ldots$

$\{x_n\}$ is a decreasing bounded sequence

$\therefore \lim_{n\to\infty} x_n$ exists.

2.4.3 omitted

2.4.4 Show that $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$ converges, and find the limit.

1° If the limit exists we could write $x = \underbrace{\sqrt{2\sqrt{2\sqrt{2\sqrt{\cdots}}}}} = \sqrt{2x}$.

So $x^2 = 2x$, whence $x = 2$ or $x = 0$.

We have $x_1 = \sqrt{2}$, $x_2 = \sqrt{2\sqrt{2}} = \sqrt{2x_1}$, and generally $x_{n+1} = \sqrt{2x_n}$.

Calculation of 1st few terms suggests $x = 2 = \lim_{n\to\infty}(x_n)$.

2° $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2x_n}$

Claim. $x_{n+1} > x_n$ for all $n = 1, 2, \ldots$

Pf. (I) $x_2 = \sqrt{2\sqrt{2}}$ and $2\sqrt{2} > 2 \Rightarrow \sqrt{2\sqrt{2}} > \sqrt{2}$

So $x_2 > x_1$.

(II) Suppose $x_n > x_{n-1}$ for some n.

Then $x_{n+1} = \sqrt{2x_n} > \sqrt{2x_{n-1}} = x_n$

$\therefore\ x_{n+1} > x_n$. //

Claim. $x_n < 2$ for all $n = 1, 2, \ldots$

Pf. $x_{n+1} > x_n\ \forall n$

$\Rightarrow \sqrt{2x_n} > x_n$

$\Rightarrow 2x_n > x_n^2 \Rightarrow 2 > x_n$. //

3° $(2 - x_{n+1}) = 2 - \sqrt{2x_n}$

$= 2 - \sqrt{4 + \frac{4(x_n - 2)}{2}}$

$(2 - x_{n+1}) = 2 - 2\sqrt{1 - \frac{(2 - x_n)}{2}}$

We want to show that if $(2 - x_n)$ is small then $(2 - x_{n+1})$ is smaller.

Consider finding k such that $k<1$ & $1-\sqrt{1-\varepsilon} < k\varepsilon$ $\quad (0<\varepsilon<1)$.

$$\iff -\sqrt{1-\varepsilon} < -1+k\varepsilon$$

$$\iff \sqrt{1-\varepsilon} > 1-k\varepsilon$$

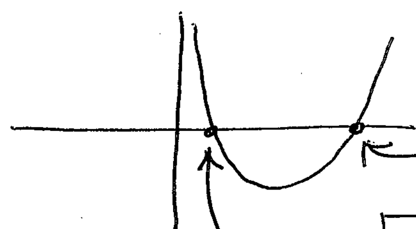
$$1-\varepsilon > 1-2k\varepsilon+k^2\varepsilon^2$$

$$\varepsilon^2 k^2 - 2\varepsilon k + \varepsilon < 0$$

$$\varepsilon k^2 - 2k + 1 < 0$$

$$k = \frac{2 \pm \sqrt{4-4\varepsilon}}{2} = 1 \pm \sqrt{1-\varepsilon}$$

are roots of $\varepsilon k^2 - 2k + 1 = 0$.



$1+\sqrt{1-\varepsilon}$

$1-\sqrt{1-\varepsilon} \approx 1-(1-\varepsilon/2) \approx \varepsilon/2$.

So we can choose small amount larger than $\varepsilon/2$. e.g.

<u>Claim</u>: $1-\sqrt{1-\varepsilon} < 2\varepsilon/3$.

<u>Pf</u>: $\iff \sqrt{1-\varepsilon} > 1-2\varepsilon/3$

$$\iff 1-\varepsilon > 1-\frac{4\varepsilon}{3}+4\varepsilon^2/9$$

$$\iff 0 > \left(1-\frac{4}{3}\right)\varepsilon + 4\varepsilon^2/9$$

$$\iff 0 > -\varepsilon/3 + 4\varepsilon^2/9$$

True for ε suff small.

2.4.5

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In[25]:= F[x_] := 1 / x + x / 2

In[26]:=
F[1]
F[F[1]]
F[F[F[1]]]
F[F[F[F[1]]]]
F[F[F[F[F[1]]]]]
F[F[F[F[F[F[1]]]]]]
N[F[1], 20]
N[F[F[1]], 20]
N[F[F[F[1]]], 20]
N[F[F[F[F[1]]]], 20]
N[F[F[F[F[F[1]]]]], 20]
N[F[F[F[F[F[F[1]]]]]], 20]
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Out[26]= $\frac{3}{2}$

Out[27]= $\frac{17}{12}$

Out[28]= $\frac{577}{408}$

Out[29]= $\frac{665\,857}{470\,832}$

Out[30]= $\frac{886\,731\,088\,897}{627\,013\,566\,048}$

Out[31]= $\frac{1\,572\,584\,048\,032\,918\,633\,353\,217}{1\,111\,984\,844\,349\,868\,137\,938\,112}$

Out[32]= 1.5000000000000000000

Out[33]= 1.4166666666666666667

Out[34]= 1.4142156862745098039

Out[35]= 1.4142135623746899106

Out[36]= 1.4142135623730950488

Out[37]= 1.4142135623730950488

$$\boxed{x_{n+1} = \frac{1}{x_n} + \frac{x_n}{2}}$$

$$\Rightarrow x_{n+1} - x_n = \frac{1}{x_n} - \frac{x_n}{2} = \frac{2 - x_n^2}{2x_n}.$$

So if $x_n > \sqrt{2}$ then $x_n^2 > 2$ & $2 - x_n^2 < 0$.

So $x_{n+1} = x_n + \left[\frac{2 - x_n^2}{2x_n}\right] < x_n$.

Now show that

$x_n > \sqrt{2} \Rightarrow x_{n+1} > \sqrt{2}$.

Pf. $x_n > \sqrt{2} \Rightarrow x_n^2 > 2$

$\Rightarrow x_n^2 - 2 > 0$

$\Rightarrow x_n^4 - 4x_n^2 + 4 > 0$

$\Rightarrow 4 + x_n^4 > 4x_n^2$

$\Rightarrow \frac{1}{x_n^2} + \frac{x_n^2}{4} > 1$

$\Rightarrow \frac{1}{x_n^2} + 1 + \frac{x_n^2}{4} > 2$

$\Rightarrow \left(\frac{1}{x_n} + \frac{x_n}{2}\right)^2 > 2$

$\Rightarrow \frac{1}{x_n} + \frac{x_n}{2} > \sqrt{2}$

$\Leftrightarrow x_{n+1} > \sqrt{2}$ ✓ //

This implies that

$x_n \longrightarrow \sqrt{2}$ as $n \to \infty$.

2.4.6 | omitted

P. 57 | 2.5.1 | omitted

2.5.2 | (a) Assume $a_1+a_2+\cdots = L$

i.e. $\lim_{n\to\infty} \underbrace{(a_1+a_2+\cdots+a_n)}_{\substack{\| \\ S_n}} = L$

Show that any re-grouping of the terms

$(a_1+a_2+\cdots+a_{n_1}) + (a_{n_1+1}+\cdots+a_{n_2}) + (a_{n_2+1}+\cdots+a_{n_3})+\cdots$

leads to a series that also converges to L.

Note that we are not rearranging.

Regrouping does not affect the limit because regrouping the finite sums does not change their values S_n & we know $\lim_{n\to\infty} S_n = L$ exists.

If L does not exist as in

$\underbrace{1-1+1-1+1-1+\cdots+(-1)^{n+1}1}_{n \text{ terms}} = S_n$

then we can regroup to get only some of the partial sums as in $(1-1)+(1-1)+(1-1)+\cdots+(1-1) = S_{2n} = 0$.

Comment: Rearrangement can change partial sums even when the limit exists.

e.g. $L = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \pm \cdots$

converges.

$$\ln(1+x) = \int_0^x \frac{dt}{1+t} = \int_0^x \left[1 - t + t^2 - t^3 \pm \cdots\right] dt$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \pm \cdots$$

& one can show this converges $\forall x > 0$. In particular,

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \pm \cdots$$

The partial sums are

1

$1 - \frac{1}{2} = \frac{1}{2}$

$1 - \frac{1}{2} + \frac{1}{3} = \frac{1}{2} + \frac{1}{3} = 5/6$

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = 5/6 - 1/4 = \frac{14}{24} = \frac{7}{12}$

$\cdots$

But the rearrangement (see P. 36 of text)

$$\frac{1}{1} + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} \pm \cdots$$

has quite different partial sums. The rules for this re-arrangement use the fact that there are ∞-ly many terms. We will discuss this again.

For the sake of future discussion here is how the book motivated the re-arrangement:

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \cdots$$

$$\frac{1}{2}S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots$$

$$\frac{3}{2}S = \underbrace{1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots}$$

2.5.3 (a) $\underset{a_1}{\frac{1}{2}}, \underset{a_2}{\frac{1}{2}+\frac{1}{4}}, \underset{a_3}{\frac{1}{4}}, \underset{a_4}{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}, \underset{a_5}{\frac{1}{8}}, \underset{a_6}{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}}, \underset{a_7}{\frac{1}{16}}, \ldots$

$\{a_1, a_3, a_5, a_7, \ldots\} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}$

$\{a_2, a_4, a_6, a_8, \ldots\} = \{\frac{1}{2}+\frac{1}{4}, \frac{1}{2}+\frac{1}{4}+\frac{1}{8}, \ldots\}$

So $\{a_{2n-1}\} \longrightarrow 0$

$\{a_{2n}\} \longrightarrow 1$.

(b) not possible

(c) Can be done. Choose sequence as follows. $\{1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\}$

This contains constant sequence

$\frac{1}{n}, \ldots, \frac{1}{n}, \ldots, \frac{1}{n}, \ldots$

converging to $\frac{1}{n}$ for each n.

(d) $\{1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \frac{1}{5}, 5, \frac{1}{6}, 6, \frac{1}{7}, \ldots\}$

(e) Cannot be done.

2.5.4 omitted.

2.5.5 omitted

2.5.6 $\{a_n\}$ bounded sequence.

$S = \left\{ x \in \mathbb{R} \mid x < a_n \text{ for infinitely many terms } a_n \right\}$.

Show $\exists$ subsequence $\{a_{n_k}\}$ converging to $s = \sup S$.

Let $\varepsilon > 0$ & consider $(s-\varepsilon, s+\varepsilon)$
Suppose only finitely many terms of $\{a_n\}$ are in this interval. We know that for any $x > s$ there are only finitely many $a_n > x$. And for any $x < s$ there are infinitely many $a_n > x$. This means that it is contradictory for $(s-\varepsilon, s+\varepsilon)$ to have only finitely many terms of $\{a_n\}$. For any $x \in (s-\varepsilon, s)$ must have ∞-ly many $a_n > x$. These are $\therefore > s+\varepsilon$ but then $\exists\, p > s$ s.t. $\exists\, \infty$-ly many $a_n > p$.
$\therefore$ $(s-\varepsilon, s+\varepsilon)$ has ∞-ly many a_n.
$\therefore$ Consider intervals $(s-\frac{1}{k}, s+\frac{1}{k})$ & choose an a_{n_k} from each. The sequence $\{a_{n_k}\}$ converges to s. //

2.6.1 (a) $\left\{\frac{(-1)^n}{2^n}\right\}$

(b) $\{1, 2, 3, 4, \ldots\}$

(c) not possible

(d) $\{\frac{1}{2}, 1, \frac{1}{4}, 2, \frac{1}{8}, 3, \frac{1}{16}, 4, \ldots\}$

2.6.2 omit

2.6.3 omit

2.6.4 omit

2.6.5 omit

2.6.6 omit

P 67 2.7.1 $\cancel{(a)}$ (a_n)

(i) $a_1 \geq a_2 \geq a_3 \geq \cdots$

& (ii) $(a_n) \to \phi$

Show: $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof. $S_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots \pm a_n$

$$\left.\begin{array}{l} S_2 = (a_1 - a_2) \\ S_4 = (a_1 - a_2) + (a_3 - a_4) \\ \ldots \end{array}\right\} \Rightarrow S_2 \leq S_4 \leq S_6 \leq \cdots$$

$S_1 = a_1$

$S_3 = a_1 - a_2 + a_3 = a_1 - (a_3 - a_2)$

$S_5 = a_1 - a_2 + a_3 - a_4 + a_5 = a_1 - (a_3 - a_2) - (a_5 - a_4)$

$\ldots$

$\Longrightarrow S_1 \geq S_3 \geq S_5 \geq S_7 \geq \cdots$

$[S_2, S_1] \supset [S_4, S_3] \supset [S_6, S_5] \supset \cdots$

nested intervals.

Lengths are $a_2 \geq a_4 \geq a_6 \geq \cdots \to \phi \Rightarrow$ converges //

2.7.2 | omit

2.7.3 | $\sum a_n$ given.

$n \in \mathbb{N}$. $\left.\begin{array}{l} p_n = a_n \text{ if } a_n \geq 0 \\ p_n = 0 \text{ if } a_n \leq 0 \end{array}\right\}$ $q_n = \begin{cases} a_n \text{ if } a_n \leq 0 \\ 0 \text{ if } a_n > 0 \end{cases}$

(a) $\sum a_n$ div $\Rightarrow$ one of $\sum p_n$ or $\sum q_n$ div.

Pf. If both $\sum p_n$ & $\sum q_n$ conv then $\sum a_n$ conv absol $\Longrightarrow$ $\sum a_n$ conv. //

(b) $\sum a_n$ conv cond $\Rightarrow$ both $\sum p_n + \sum q_n$ diverge.

Pf: $\sum a_n$ conv cond means $\sum a_n$ conv but $\sum |a_n|$ div.

~~But~~ But $\sum |a_n| = \sum p_n + \sum q_n$ & so at least one diverges. But if only one diverges, then easy to see that $\sum a_n$ must diverge. //

2.7.4 | $\sum \frac{1}{n}$ div, $\sum \frac{1}{n}$ div

$\sum \frac{1}{n^2}$ conv.

2.7.5 | (a) Given $\sum a_n$ conv absol.

$\Leftrightarrow$ $\sum |a_n|$ converges.

Then $\sum a_n^2 = \sum |a_n|^2$ and since we know $a_n \to 0$ we have eventually $|a_n|^2 < |a_n|$ & $\therefore$ $\sum |a_n|^2$ conv. //

Note $\sum_n \frac{(-1)^n}{\sqrt{n}}$ converges but not absolutely.

& $\sum_n \left(\frac{(-1)^n}{\sqrt{n}}\right)^2 = \sum_n \frac{1}{n}$ diverges.

2.7.5 (b) $\sum \frac{1}{n^2}$ converges

but $\sum_n \sqrt{\frac{1}{n^2}} = \sum \frac{1}{n}$ diverges.

2.7.6 (a) $\sum x_n$ conv absol

(y_n) bounded

$\Longrightarrow \sum x_n y_n$ converges.

Pf. $\sum |x_n|$ converges is given.

$|y_n| < B$ some bound B.

$\Rightarrow \sum |x_n y_n| < (\sum |x_n|) B$

$\Rightarrow \sum |x_n y_n|$ conv.

$\Rightarrow \sum x_n y_n$ conv. by (2.7.6) //

(b) $\sum \frac{(-1)^n}{n}$ conv. conditionally,

$((-1)^n)$ bounded seq

$x_n = \frac{(-1)^n}{n}$, $y_n = (-1)^n$

$\Rightarrow x_n y_n = 1/n$ & $\sum x_n y_n$ diverges.

2.7.7 omit.

2.7.8 omit.

2.7.9 (Ratio Test)

$\sum_{n=1}^{\infty} a_n$, $a_n \neq 0$.

If (a_n) satis $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r < 1$,

then $\sum_{n=1}^{\infty} a_n$ converges absol.

Pf. (a) Let r' satis $r < r' < 1$.

Then $\exists N$ s.t. $n \geq N \Rightarrow |a_{n+1}| \leq |a_n| r'$

Because given $\varepsilon > 0$ $\exists N$ s.t. $n \geq N$

$$\Longrightarrow \left| \left|\frac{a_{n+1}}{a_n}\right| - r \right| < \varepsilon$$

$$-\varepsilon < \left|\frac{a_{n+1}}{a_n}\right| - r < \varepsilon$$

$$-\varepsilon + r < \left|\frac{a_{n+1}}{a_n}\right| < \varepsilon + r$$

So we choose $\varepsilon = r' - r$

& conclude $\left|\frac{a_{n+1}}{a_n}\right| < \varepsilon + r = r'$

& $\therefore |a_{n+1}| < r' |a_n|$ //

(b)

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n<N} |a_n| + \sum_{n=N}^{\infty} |a_n|$$

$$< \underbrace{\sum_{n<N} |a_n|}_{\text{finite}} + \underbrace{\left(\sum_{n=N}^{\infty} (r')^{N-n}\right)}_{\text{conv}} |a_N|$$

(c) $\Longrightarrow \sum_n |a_n|$ conv.

$\Longrightarrow \sum a_n$ conv. //

2.7.10 (a) $a_n > 0$, $\lim_{n\to\infty}(n a_n) = l$, $l \neq 0$

$\Longrightarrow$ $\sum a_n$ diverges.

Pf. Make comparison with $\sum_n \frac{1}{n}$. detail omitted //

(b) $a_n > 0$, $\lim(n^2 a_n)$ $\exists$

Show $\sum a_n$ converges.

Pf: Make comparison with $\sum \frac{1}{n^2}$. detail omitted //

2.7.11 Give examples of series $\sum a_n$, $\sum b_n$ both diverge but $\sum \min\{a_n, b_n\}$ conv. + want $\{a_n\}, \{b_n\}$ both pos decr.

$$\sum_n a_n = 1 + \frac{1}{2} + \frac{1}{3^2} + \frac{1}{4} + \frac{1}{5^2} + \frac{1}{6} + \frac{1}{7^2} + \cdots$$

$$\sum_n b_n = 1 + \frac{1}{2^2} + \frac{1}{3} + \frac{1}{4^2} + \frac{1}{5} + \frac{1}{6^2} + \frac{1}{7} + \cdots$$

$$\sum_n \min\{a_n, b_n\} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \cdots$$

2.7.12 $(x_n), (y_n)$

$$S_n = x_1 + \cdots + x_n$$

$$x_j = S_j - S_{j-1}$$

Show $\displaystyle\sum_{j=m+1}^{n} x_j y_j = S_n y_{n+1} - S_m y_{m+1} + \sum_{j=m+1}^{n} S_j (y_j - y_{j+1})$

$\boxed{\text{omit}}$

2.7.13 Partial sums of $\displaystyle\sum_{n=1}^{\infty} x_n$ bounded,

(y_n) satis $y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$

$\lim_{n\to\infty} y_n = 0$.

$\Longrightarrow \displaystyle\sum_{n=1}^{\infty} x_n y_n$ converges,

omit

2.7.14 $\displaystyle\sum_{n=1}^{\infty} x_n$ conv. (y_n) satis

$y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$

$\Longrightarrow \displaystyle\sum_{n=1}^{\infty} x_n y_n$ conv.

(a) note we did not assum $y_n \to 0$ as $n \to \infty$.

b)
c) omit