Homework - Math 435 - Spring 2014

Problems due March 11, 2014 -Math 435.

1. Start with the cubic $x^3+px+q=0$. (Note I am writing this so that $x^3=-px-q$ with signs opposite from our class notes.), Express the roots in the form

$$\alpha = a+b, \beta = wa+w^2b, \gamma = w^2a+wb$$

where w is a third root of unity and you know that $1+w+w^2=0$ and that $w^3=1$. So you can use these forms of the roots in

$$D = (\alpha-\beta)(\alpha-\gamma)(\beta-\gamma).$$

(In working out this algebra it will probably be helpful to define

$$\lambda = 1-w$$

and note that

$$\bar{\lambda} = 1-w^2$$

and that

$$\lambda\bar{\lambda} = 3.$$

You should find that

$$D = w\lambda(\lambda a+\bar{\lambda}b)(\bar{\lambda}a+\lambda b)(a-b).$$

Prove by substution and calculation that

$$D = 3(w-w^2)(a^3-b^3).$$

Now you can use the quadratic equation associated with

$$a^3b^3 = -p^3/27$$

$$a^3+b^3 = -q$$

to get a formula for

$$a^3-b^3.$$

Use this to prove that

$$D^2 = -27q^2-4p^3.$$

2. Let Z_n denote the integers modulo n (clock arithmetic mod n). Recall that

$$Z_n = \{0, 1, 2, ..., n-1\}$$

where $a + b$ = remainder of the usual $a + b$ after division by n. We have seen that Z_n is a group with the operation $+$, and that the identity is 0 ($0 + x = x$ for all x). We now want to consider the multiplicative structure of Z_n. To this end define $a \times b =$ the remainder of usual $a \times b$ on division by n. Thus in Z_3 we have $2 \times 2 = 1$ and so 2 is its own multiplictive inverse. Of course 0 does not have a multiplicative inverse, but the set $\{1, 2\}$ in Z_3 forms a group of order 2. On the other hand in $Z_4 = \{0, 1, 2, 3\}$ we see that $2 \times 2 = 0$.

It can happen that a product of non-zero elements is equal to 0. You can see from this that 2 has no multiplicative inverse in Z_4, for if

$$2 \times a = 1,$$

then $0 = 2 \times 2$ implies that

$$0 = 0 \times a = (2 \times 2) \times a = 2 \times (2 \times a) = 2 \times 1 = 2$$

and this is a contradiction. An element a of Z_n is said to be a *zero divisor* if a is not zero, and there exists a b not zero such that $a \times b = 0$. Show that no zero divisor can have a multiplicative inverse in Z_n.

Let U_n denote the set of elements of Z_n that are not zero divisors. Show that every element of U_n has a multiplicative inverse in U_n and show that U_n forms a group using multiplication in Z_n, with 1 as the identity element. If you have trouble proving this, look at some examples such as Z_6 – find U_6 and look at its structure.

It is a fact that if $n = p$, a prime number,then

$$U_p = \{1, 2, 3, ..., p-1\}$$

(i.e. all the non-zero elements of Z_n). Examine directly the multiplication tables for U_n and Z_n for $n = 2, 3, 4, 5, 6, 7$ and verify that this is so in these examples.

3. Goodman. Page 23. Problems $1.5.7$ and $1.5.10$.

Goodman. Page 74.Problems $1.10.10$ and $1.10.9$.

4. Recall our examples where we wrote down the multiplication table for a group, then rearranged the columns so that the identity element appeared on the diagonal of the table. Then we made matrices for each group element by putting 1's where that element appears in the revised table, and 0's elsewhere. We illustrated in class how these matrices multiply in the pattern of the group. Rewrite your class notes for this problem, and extend your notes to include the example of the multiplcation table for the symmetries of the triangle (i.e. for the group S_3). You should obtain a nice set of six matrices that act like S_3. Try out some of their products and see how they work. We now have notes on the web that do this part. Continue and do the problem at the end of these notes to get the permutations and matrices associated with the quaternion group.