IMAGINARY VALUES IN MATHEMATICAL LOGIC

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Reprinted from PROCEEDINGS OF THE SEVENTEENTH
INTERNATIONAL SYMPOSIUM ON MULTIPLE-VALUED LOGIC
MAY 26-28, 1987 Boston, MA

Imaginary Values in Mathematical Logic

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ABSTRACT: We discuss the relationship of G. Spencer-Brown's Laws of Form with multiple-valued logic. The calculus of indications is presented as a diagrammatic formal system. This leads to new domains and values by allowing infinite and self-referential expressions that extend the system. We reformulate the Varela/Kauffman calculi for self-reference, and give a new completeness proof for the corresponding three-valued algebra (CSR).

I. INTRODUCTION.

This work is primarily devoted to looking at the mathematics of G. Spencer-Brown's Laws of Form [16] and at extensions of this work for multiple-valued logic due to myself and Francisco Varela ([2], [3], [4], [12], [17]).

I present a proof of the completeness of Varela's calculus for self-reference (denoted CSR). This clears up a gap in Varela's original paper [17]. One of the outgrowths of this discussion is a new formulation of the axioms for the Varela calculus. We determine clearly that the Varela calculus (and corresponding structures in the Kleene three-valued logic) is predicated on the appearance of exactly one value that is non-Boolean in the sense that it remains invariant under crossing (negation). This means that the Varela calculus is a minimal and complete description of the skeletal situation of the emergence of a third value.

It is a characteristic of the subject of formal logic that simple and useful applications have waited for a clearing of mathematics and epistemology before making their appearance. This is undoubtedly the case with the subject of multiple-valued logic, and particularly with the uses of Laws of Form. (Here and elsewhere I use Laws of Form without underlining to refer both to Spencer-Brown's book by that title and to the context that it connotes. This context holds discrimination as a primary act through which the patterns of mathematics and logic are seen.)

A more general completeness result (see Rasiowa [15]) and its relation to Laws of Form deserve further study. There is a deep mathematical relationship between the formal approach using imaginary values (pioneered by Varela and myself) and an intricate translation involving set-theory. Further clarification of these issues will allow us to make a more penetrating analysis of the use of these calculi in circuits and waveforms.

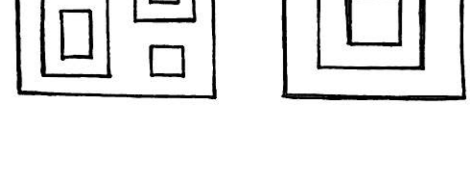
I begin the paper with a discussion of Spencer-Brown's primary arithmetic in relation to Boolean algebra. I emphasize that the primary arithmetic can be viewed as a simple example of a non-trivial diagrammatic formal system. We discuss the concept of diagrammatic formal system and its relation with the usual notion of formal system in mathematical logic. The diagrammatic formal system is more general in concept, depending on conventions and agreements for the interaction and manipulation of elements (as in the pieces and board of a well-defined game). This type of formal system deserves to be explored more deeply. The approaches to completeness theorems that we discuss use these aspects of diagrammatics crucially, and it is through this point of view that the multifold relationships to circuit design, fractals, topology and complex numbers ([2], [4], [5], [6], [8], [9], [10], [12]) come to the fore in my own work. (See also [13], [14], [18], [19], [20] for related issues in the work of Lefebvre, Varela and Goguen.)

I particularly recommend that we continue to ask questions and create clarification of the Laws of Form mathemtical structures. These structures are very deep reflections of intelligence, whose unfoldment will inevitably lead to new and useful inventions.

A general comment on approach: While Laws of Form does lead to a clarification of issues in multiple-valued logic, this is only one of a myriad of possible applications. There is enormous potential for understanding systems through the distinctions that we make in perceiving and modeling them, By searching for the underlying distinctions being used to create a system we penetrate to levels of understanding that are simply not available in any external modelling approach.

II. PRIMARY ARITHMETIC AS A DIAGRAMMATIC FORMAL SYSTEM.

I first given an equivalent version of Spencer-Brown's primary arithmetic [16]. The elements of this system are finite collections of disjoint rectangles in the plane. Any such collection is called an expression. This includes the empty collection. A typical non-empty expression is given below.



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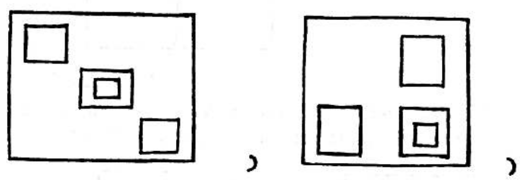
Note that typographically it is difficult to represent the empty expression. This is due to the already-present complex display of symbols on a page. Nevertheless, there is only one empty expression under consideration, and this is understood to be a plane without any rectangles drawn upon it.

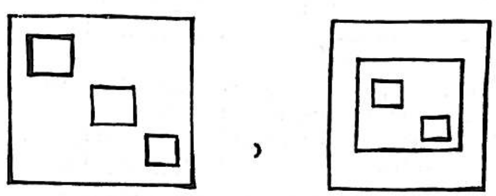
It is useful in illustrating a system of this type to simply draw representative collections of rectangles on a blackboard or sheet of paper. The analog with studying geometry and using diagrams should be obvious. However, in primary arithmetic we are concerned only with the fact that each rectangle distinguishes an inner and outer region in the plane (rather than any topological or rigid properties).

Since we can distinguish for any rectangle in a plane a bounded inside and an unbounded exterior, we speak of the inside or outside of a given rectangle, and also say whether one rectangle is inside (in the bounded part) or outside (in the unbounded part) of another.

Two expressions are said to be <u>identical</u> if all relations of inside and outside among the component rectangles are the same in each. In other words, there should be a one-to-one correspondence between the rectangles of one expression and the rectangles of the other such that these relations are preserved. For example, the two expressions indicated below are identical



while the following two expressions are not identical.

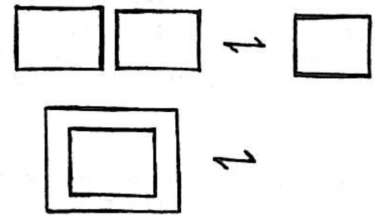


Identity of expressions is an identity of <u>form</u>. It does not depend upon any particular properties of the rectangles other than their mutual relationships in the plane.

To what extent should this description of rectangle-expressions be formalized? It can be expressed in relation to a mathematical plane in terms of sets of points. But any such formalization ultimately rests on systems of notations and communication that <u>presuppose</u> exactly the set of conventions for planar writing that we are attempting to formalize. We read script and write symbols exactly on the basis of these ideas.

With this understanding it is possible to begin to view this arithmetic of rectangles directly as built on a <u>formalism</u> of rectangle drawings that are no different in kind from any other written mathematical text. I request that the reader make this shift!

The primary arithmetic arises in the study of an equivalence on expressions generated by the following two types of moves (calling and crossing):



(calling)

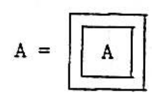
(crossing).

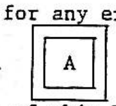
It is understood that the replacements above can be performed within any larger expression just so long as the explicit configurations being changed are identical to those above. More precisely, it must be possible to encircle a portion of the larger expression and to find <u>within</u> this encirclement the form of crossing or the form of calling. If there are no rectangles in the encirclement, then the crossing change allows us to draw two rectangles, one within the other.

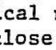
Two expressions are said to be <u>equivalent</u> if one can be obtained from the other by a sequence of moves of these two types. Note that the moves can be performed in either direction. An expression equivalent to the empty expression is said to be <u>unmarked</u>.

One then proves that any expression is either marked or unmarked. No finite expression of rectangles can be both marked and unmarked,

This is a complete description of the primary arithmetic at the level of its construction. Patterns of equivalence give rise to the next domain of description. Thus we can write



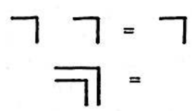
since it is indeed the case that for any expression A, A is marked exactly when  is marked.

But upon making a description of this kind, we have moved to the level of algebra. In algebra it may be convenient to create a special symbol for the unmarked state. If we do so, then we have arrived at standard Boolean algebra. If we choose to simply use  within this formalism as the natural symbol for unmarked, and otherwise to continue to respect the conventions of the primary arithmetic, then we are doing Spencer-Brown's primary algebra. This is a subtle difference between primary algebra and Boolean algebra.

The practical nature of this difference resides in the close fit between primary algebra and primary arithmetic. Primary algebra is a minimal description of primary arithmetic that returns directly to the arithmetic when variables are replaced by expressions. This has an immediate advantage for symbolic computation, and also for pattern recognition.

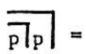
Everything we have said in relation to primary arithmetic and primary algebra as diagrammatic systems applies to many other mathematical situations. By understanding the fitting relationship of good description with that which it describes we open the possibility for creative modelling.

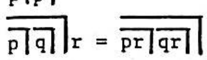
REMARK: The translation of this discussion to Spencer-Brown's notation is simply that of replacing the rectangle by an abbreviated rectangle (the mark). Thus ⌉ stands for □, and the initials for the primary arithmetic become:



(= denotes ⟷, not identity.)

The initials of the primary algebra take the form:

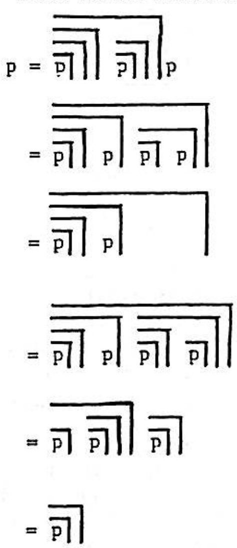
I. 

II. 

(plus implicit associativity and commutativity).

REMARK: The term initials is due to Spencer-Brown. The intials denote a starting point for the system (arithmetic or algebra in this case). Initials can be taken to be axioms or postulates, or simply as rules of the game.

One of the mathematical advantages of this notation is seen in the derivation of reflexion (p⌉⌉ = p) from these initials:



(I)

(II)

(I)

(I)

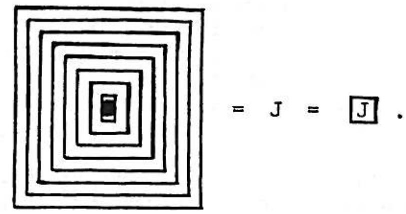
(II)

(I).

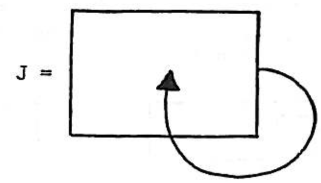
Here the combinatorial interaction of I and II just conspires to make reflexion a consequence. The minimal notation of the primary algebra gives access to this phenomena. Certain circumstances require the use of every bit of the available structure -- taking the system to its limits.

III. IMAGINARY VALUES.

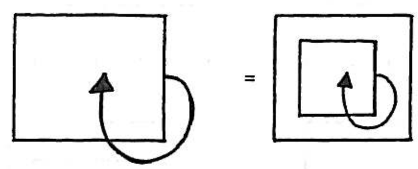
Consider the equation x = [x]. It has no solution in primary arithmetic. The notation suggests a formal solution -- an infinite collection of nested boxes:



Being infinite, this expression is outside the category of finite expressions that constitute the primary arithmetic. This suggests a possible interpretation for an "imaginary" logical value that is invariant under negation. Since J contains an identical copy of itself within itself, we can also describe J recursively or self-referentially via the suggestive notation



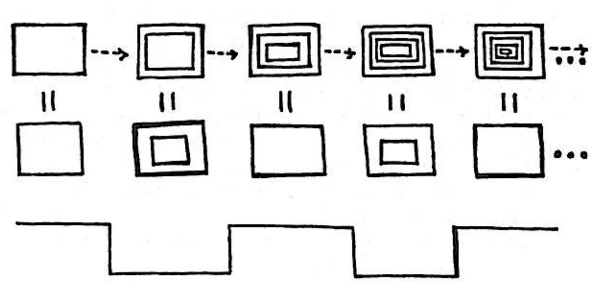
Here the arrow-head indicates where the form reenters its own indicational space. Thus



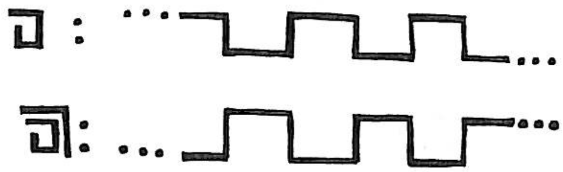
Note that on performing the recursion, the indicative line and arrow-head are erased.

In Spencer-Brown notation (see [12], [16], [17]) we write ⌉ (re-entering) for this re-entering mark. In the next section we discuss a particular algebraic context for this value.

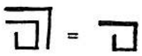
There is much to say on the interpretive side. One remark is that J = [J] can be viewed as a shorthand for the recursion J ----> [J]. Then we obtain the process

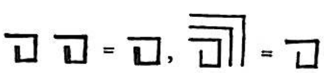


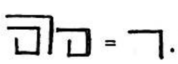
and, as indicated above, if each step in the process is evaluated as an element of primary arithmetic, then the process appears as a discrete oscillation, a wave-form. Depending upon the starting value for J, there are really two phase-shifted waveforms:



We would like to be able to make three statements here:

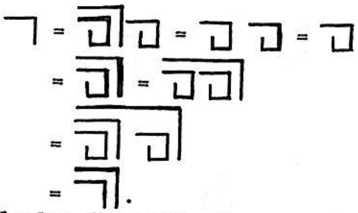
1. 

2. 

3. 

The first statement says that <u>globally</u>, the wave-form J is identical with $\overline{\overline{J}|}|$. The third statement says that J and $\overline{J}|$ <u>taken together</u> combine to form a constantly marked state. In this interpretation, J and $\overline{J}|$ represent phase-shifted forms in juxtaposition.

With the usual conventions of substitution, these statements lead to a paradox.



The calculus for self-reference of the next section is one way to resolve the paradox (by letting go of statement 3.). Another very beautiful way to avoid the paradox is the

<u>Flagg Resolution</u>. There is only one re-entering mark. If, in an expression, a given instance of J is replaced by $\overline{J}|$ then <u>all</u> instances of J must be replaced in the same manner.

This treatment of the paradox allows one to remain in the primary, Boolean context and still maintain all three statements about the wave-form. James Flagg and the author will explore this point of view elsewhere.

Another way to retain the wave-form context is to allow <u>two</u> extra values i and j, satisfying $\overline{i}| = i$, $\overline{j}| = j$, ij = ⏋. This is studied in [3] and [12]. See also [4] for a relationship with complex numbers. There is, at this level, a very remarkable correspondence with the formalism of the complex numbers. Briefly this involves introducing the <u>square root of negation</u> $\sqrt{\sim}$. With

$\sqrt{\sim}\sqrt{\sim} = \sim$ and $\sim$⏋ = $\overline{⏋}|$, $\sim\overline{⏋}|$ = ⏋

$a + \sqrt{\sim}\, b$: ... abababab ...

$b + \sqrt{\sim}\, a$: ... babababa ...

we take the identifications

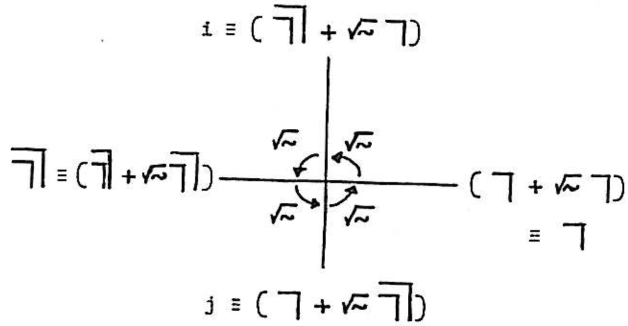
⏋ + $\sqrt{\sim}$⏋ = marked

$\overline{⏋}|$ + $\sqrt{\sim}$⏋ = possibly unmarked

$\overline{⏋}|$ + $\sqrt{\sim}\overline{⏋}|$ = unmarked

⏋ + $\sqrt{\sim}\overline{⏋}|$ = possibly marked.

and find that these values are "rotated" cyclically by the square root of negation, just as the square root of minus one operates on the complex numbers. The square root of negation is an imaginary operator that rotates from the domain of necessity (marked/unmarked, true/false) to the domain of possibility (possibly marked/possibly unmarked).



$\sqrt{\sim}\,(a + \sqrt{\sim}\, b)$

$= \sqrt{\sim}\, a + \sim b$

$= \overline{b}| + \sqrt{\sim}\, a$

$\overline{a + \sqrt{\sim}\, b}| = \overline{b}| + \sqrt{\sim}\,\overline{a}|$

$\overline{i}| = i$, $\overline{j}| = j$

REMARK: For those interested in an interpretation in the framework of quantum mechanics, $\rceil + \sqrt{\sim}\,\overline{\rceil}|$ can be thought of as a mixed state, neither marked nor unmarked and yet capab le of projecting these values. Just so, $\sqcup$ is neither marked nor is it unmarked.

NEW CSR INITIALS

I. $\overline{\overline{p}|\,q}|\,p = p$ (occultation)

II. $\overline{\overline{p}|\,\overline{q}|}|\,r = \overline{\overline{pr}|\ \overline{qr}|}|$ (transposition)

III. $\overline{\overline{p}|\,p}|\ \sqcup = \sqcup$ (location)

$\overline{\sqcup}| = \sqcup$

It is understood that the mark operates on all symbols underneath it, that the juxtaposition operation (p,q -----> pq) is commutative and associative, and that the re-entering mark is an element of this system.

We will show that if u is any element in a system satisfying the initials above, then $u = \overline{u}|$ implies that $u = \sqcup$. Thus the calculus describes the situation of unique self-reference.

We then give a proof of the equational completeness for CSR.

The first part of the task is to derive a series of consequences of initials I and II. These are common to both the above system and to Varela's calculus. His calculus differs in the third initial where he takes

III'. $\overline{p\,\sqcup}|\,p = \sqcup$, $\overline{\sqcup}| = \sqcup$.

In this terminology an algebra satisfying I and II will be called a brownian algebra [12]. This is the analog, in this context, of a De Morgan algebra [3]. It is possible to have infinitely many different x in a brownian algebra each satisfying the equation $x = \overline{x}|$. See [3] and [12] for examples.

Consequences of I and II

1. $\overline{\overline{a}|}| = a$

Dem. $\overline{\overline{a}|}| = \overline{\overline{\overline{\overline{a}|}|\,a}|\,\overline{a}|}|$ (I)

$= \overline{\overline{\overline{\overline{a}|}|\,a}|\,\overline{\overline{\overline{a}|}|\,a}|}|$ (I)

$= \overline{\rceil\ \rceil}|\ \overline{\overline{a}|}|\,a$ (II)

$= \overline{\overline{a}|}|\,a$ (I) [here take p unmarked, q marked]

$= a$ (I)

2. aa = a

Dem. $aa = \overline{\overline{a}|}|\,a = a$.

3. $\rceil a = \rceil$

Dem. $\rceil a \underset{2.}{=} \overline{\overline{\rceil a}|}| \underset{I}{=} \rceil$.

4. $\overline{\overline{\overline{a}|b}|\,c}| = \overline{ac}|\,\overline{\overline{b}|c}|$

Dem. $\overline{\overline{\overline{a}|b}|\,c}| \underset{2.}{=} \overline{\overline{\overline{a}|\,\overline{\overline{b}|}|}|\,c}| \underset{II}{=} \overline{\overline{\overline{ac}|\ \overline{\overline{b}|c}|}|}|$

$\underset{2.}{=} \overline{ac}|\,\overline{\overline{b}|c}|$.

5. $\overline{\overline{a}|r}|\,\overline{\overline{b}|r}| = \overline{\overline{ab}|r}|$.

Dem. $\overline{\overline{a}|r}|\,\overline{\overline{b}|r}| \underset{2.}{=} \overline{\overline{\overline{\overline{a}|r}|\,\overline{\overline{b}|r}|}|}|$

$\underset{II}{=} \overline{\overline{\overline{a}|\,\overline{\overline{b}|}|}|\,r}| \underset{2.}{=} \overline{\overline{ab}|r}|$.

6. $\overline{\overline{ab}|b}| = \overline{\overline{a}|b}|\,\overline{b\,\overline{b}|}|$

Dem. $\overline{\overline{ab}|b}| \underset{2.}{=} \overline{\overline{\overline{a}|\,\overline{\overline{b}|}|}|\,b}|$

$\underset{II}{=} \overline{\overline{\overline{\overline{a}|b}|\,\overline{\overline{b}|b}|}|}| = \overline{\overline{a}|b}|\,\overline{\overline{b}|b}|$.

7. $\overline{\overline{a}|b}|\,\overline{\overline{a}|\overline{b}|}| = \overline{\overline{a}|\ \overline{b\,\overline{b}|}|}|$

Dem. $\overline{\overline{a}|\,\overline{b\,\overline{b}|}|}| \underset{2.}{=} \overline{\overline{a}|\ \overline{\overline{\overline{b}|}|\,\overline{b}|}|}| \underset{II}{=} \overline{\overline{\overline{\overline{a}|\overline{b}|}|\,\overline{\overline{a}|b}|}|}|$

$\underset{2.}{=} \overline{\overline{a}|\overline{b}|}|\,\overline{\overline{a}|b}|$.

8. $\overline{\overline{a}\rceil\overline{br}\rceil\overline{cr}\rceil}\rceil = \overline{\overline{\overline{a}\rceil\overline{b}\rceil\overline{c}\rceil}\rceil\,\overline{\overline{a}\rceil\overline{r}\rceil}\rceil}\rceil$.

Dem. $\overline{\overline{a}\rceil\overline{br}\rceil\overline{cr}\rceil}\rceil = \overline{\overline{a}\rceil\,\overline{\overline{\overline{b}\rceil\overline{c}\rceil}\rceil r}\rceil}\rceil$ (2.,II)

$= \overline{\overline{\overline{a}\rceil\overline{b}\rceil\overline{c}\rceil}\rceil\,\overline{\overline{a}\rceil\overline{r}\rceil}\rceil}\rceil$ (2.,II,2.).

9. $\overline{\overline{\overline{p}\rceil\overline{q}\rceil}\rceil\,\overline{\overline{r}\rceil\overline{s}\rceil}\rceil}\rceil = \overline{\overline{pr}\rceil\overline{ps}\rceil\overline{qr}\rceil\overline{qs}\rceil}\rceil$

Dem. $\overline{\overline{\overline{p}\rceil\overline{q}\rceil}\rceil\,\overline{\overline{r}\rceil\overline{s}\rceil}\rceil}\rceil = \overline{\overline{p\,\overline{\overline{r}\rceil\overline{s}\rceil}\rceil}\rceil\,\overline{q\,\overline{\overline{r}\rceil\overline{s}\rceil}\rceil}\rceil}\rceil$ (II)

$= \overline{\overline{pr}\rceil\overline{ps}\rceil\overline{qr}\rceil\overline{qs}\rceil}\rceil$ (II,2.)

10. (crosstransposition)

$$\overline{\overline{ax}\rceil\,\overline{\overline{a}\rceil\overline{y}\rceil}\rceil\,\overline{\overline{aa}\rceil\overline{z}\rceil}\rceil}\rceil = \overline{ax}\rceil\,\overline{\overline{a}\rceil y}\rceil\,\overline{xyz}\rceil\,\overline{\overline{a}\rceil a}\rceil$$

Dem. $\overline{\overline{ax}\rceil\,\overline{\overline{a}\rceil\overline{y}\rceil}\rceil\,\overline{\overline{aa}\rceil\overline{z}\rceil}\rceil}\rceil$

$= \overline{\overline{\overline{x}\rceil\,\overline{\overline{a}\rceil\overline{y}\rceil}\rceil\,\overline{\overline{a}\rceil\overline{z}\rceil}\rceil}\rceil\,\overline{\overline{\overline{a}\rceil\overline{y}\rceil}\rceil\,\overline{a}\rceil}\rceil}\rceil$ (8.)

$= \overline{\overline{\overline{x}\rceil\overline{y}\rceil\overline{z}\rceil\,\overline{\overline{x}\rceil a}\rceil}\rceil\,\overline{\overline{\overline{a}\rceil\overline{y}\rceil}\rceil\,\overline{a}\rceil}\rceil}\rceil$ (8.)

$= \overline{xyz}\rceil\,\overline{ax}\rceil\,\overline{\overline{aa}\rceil}\rceil\,\overline{\overline{ya}\rceil}\rceil$ (2.,II,2.).

We include one structural theorem at this level:

<u>THEOREM</u>: Let x be an element in a brownian algebra satisfying $\overline{x}\rceil = x$. Then ax = bx and $\overline{a}\rceil x = \overline{b}\rceil x$ implies that a = b for a and b in this algebra.

Proof. a = $\overline{\overline{a}\rceil x}\rceil a$ (I)

$= \overline{\overline{b}\rceil x}\rceil a$ ($\overline{a}\rceil x = \overline{b}\rceil x$)

$= \overline{\overline{b}\rceil\overline{x}\rceil}\rceil a$ ($x = \overline{x}\rceil$)

$= \overline{\overline{ba}\rceil\overline{xa}\rceil}\rceil$ (II)

$= \overline{\overline{ba}\rceil\overline{bx}\rceil}\rceil$ (xa = xb)

$= \overline{\overline{a}\rceil\overline{x}\rceil}\rceil b$ (II)

$= \overline{\overline{a}\rceil x}\rceil b$ ($x = \overline{x}\rceil$)

$= \overline{\overline{b}\rceil x}\rceil b$ ($\overline{a}\rceil x = \overline{b}\rceil x$)

= b (I).

<u>Consequences with III included</u>

PROPOSITION: $\overline{p\,\overline{\sqcup}}\rceil p = \overline{\sqcup}\,p$.

Proof: $\overline{p\,\overline{\sqcup}}\rceil p = \overline{\overline{p}\rceil\,\overline{\overline{\sqcup}}\rceil}\rceil p$ (2., $\overline{\overline{\sqcup}}\rceil = \overline{\sqcup}$)

$= \overline{\overline{p}\rceil p\,\overline{\overline{\sqcup}\,p}\rceil}\rceil$ (II)

$= \overline{\overline{p}\rceil p\,\overline{\overline{\sqcup}\,\overline{p}\rceil}\rceil}\rceil$ (2., $\overline{\overline{\sqcup}}\rceil = \overline{\sqcup}$)

$= \overline{\overline{p}\rceil p\,\overline{\sqcup}}\rceil\,\overline{\overline{p}\rceil p\,\overline{p}\rceil}\rceil$ (II, 2.)

$= \overline{\overline{\sqcup}}\rceil p$ (I,2.,III)

$= \overline{\sqcup}\,p$. ($\overline{\overline{\sqcup}}\rceil = \overline{\sqcup}$)

<u>Basic Metatheorem for CSR</u>. $x = \overline{x}\rceil$ iff $x = \overline{\sqcup}$.

<u>Proof</u>. $\overline{\sqcup} = \overline{\overline{\sqcup}}\rceil$ by definition. Conversely, suppose that $x = \overline{x}\rceil$. Then

x = [x⌉ □]⌉ x   (I)

= [x □]⌉ x   (x = [x]⌉)

= x □   ([P □]⌉ P = P □)

= x⌉ □   (x = [x]⌉)

= [xx]⌉ □   (x = xx)

= [x⌉ x]⌉ □   (x = [x]⌉)

= □   (III).

<u>Completeness for CSR</u>

We sketch the proof of equational completeness for CSR. That is, the equality of two expressions in CSR is a consequence of the initials I, II, III if and only if the two expressions agree for all substitutions from the arithmetic

⌉⌉ , ⌉ , □

for corresponding variables.

The proof is based on the fact that any expression can be algebraically brought into the form

E = [xa]⌉ [x⌉ b]⌉ [[xx]⌉ c]⌉ d

where the variable x appears only as indicated. (The term [[xx]⌉ c]⌉ was neglected in [17], hence the need for a new proof of the completeness.) Think of E = E(x) as a function of x. Then

E(⌉) = b⌉ d

E(⌉⌉) = a⌉ d

E(□) = [□ a]⌉ [□ b]⌉ [□ c]⌉ d.

Similar remarks apply to a function of many variables. Our proof, being based on induction, only requires the isolation of one variable at a time.

Similarly, if

F = [xa']⌉ [x⌉ b']⌉ [[xx]⌉ c']⌉ d'

then

F(⌉) = b'⌉ d

F(⌉⌉) = a'⌉ d'

F(□) = [□ a']⌉ [□ b']⌉ [□ c']⌉ d'.

Given that E(x) = F(x) for x = ⌉, ⌉⌉, □ we wish to demonstrate that E = F. It suffices to show that E □ = F □ and that E⌉ □ = F⌉ □ in CSR. By symmetry of the argument, it actually suffices to show that E □ = F □.

Note that

[□ a]⌉ [□ b]⌉ [□ c]⌉ d □ = [□]⌉ a]⌉ [[□]⌉ b]⌉ [[□]⌉ c]⌉ d □

= d □
II

Thus we can assume:

b⌉ d = b'⌉ d'

a⌉ d = a'⌉ d'

d □ = d' □.   (*)

Then E □ = [xa]⌉ [x⌉ b]⌉ [[xx]⌉ c]⌉ d □

= [xa]⌉ [x⌉ b]⌉ [[xx]⌉ c]⌉ d [[xx]⌉]⌉ □   (III)

= [xa]⌉ [x⌉ b]⌉ [[xx]⌉]⌉ d □   (I)

= [[xa]⌉ [x⌉ b⌉]⌉ [xx]⌉]⌉ d □   (crosstransposition - 10.)

= [[xa⌉ d □]⌉ [x⌉ b⌉ d □]⌉ [xx⌉ d □]⌉]⌉   (II).

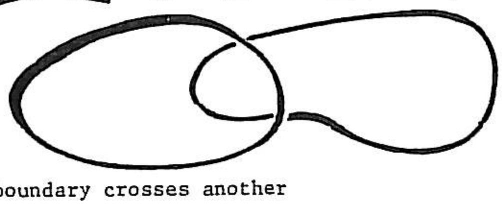
Now use the equalities (*). Substitute, and reverse steps to conclude that E □ = F □. This completes the essential inductive step in proving equational completeness for CSR.

In [12] we use a similar technique to obtain equational completeness for brownian algebras. A corresponding argument for DeMorgan algebras is given in [3]. The novelty of this approach is its use of imaginary Boolean values in the course of proving the result.

IV. EPILOGUE.

Except for the interpretation of section III I have made little mention of standard three-valued or multiple-valued logic in this paper. The relationship of the Varela system to Kleene's three valued logic is detailed in [19]. In fact, it has been my intent to direct attention to other related domains - formal diagrammatic systems, complex numbers, imaginary values. A few more words about diagrams: One way to see the emergence of a third value is in the recognition of the boundary. Thus fuzzy set theory arises by directly articulating the structure of a boundary layer that intermediates between inside and outside. In Venn diagrams we may decide to take the boundary seriously and develop diagram-notation for how one boundary can cross another:

 or . Thus, in



one boundary crosses another twice, while in

each boundary crosses and is crossed by the other. An appropriate formal diagrammatic system already exists for this situation via the theory of knots and links (See [9]). Extraordinary relationships with other domains arise in the articulation of the boundary. In the topological context, each boundary becomes an indicational space in which further distinctions and (lower-dimensional) boundaries can be explored. The plane is the lowest dimensional space in which the boundaries (one-dimensional) can have internal structure. Imaginary values intermediate between the structural, timeless geometric/topological domain and the process/creative domain of time/recursion and possibility.

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Acknowledgement: Research for this paper was partially supported by ONR grant N0014-84-K-0099 and the Stereo-Chemical Topology Project at the University of Iowa, Iowa City, Iowa 52242. Research resulting in the alternate third initial of the Varela calculus was sponsored by the Deputy for Development Planning, Aeronautical Systems Division (AFSC), United States Air Force, Wright-Patterson AFB, Ohio 45433 under contract F33615-85-C-0122 with SYSTRAN Corporation, Dayton, Ohio 45432.