Knots and Graphs I - Arc Graphs and Colorings

Frank Harary
Department of Computer Science
New Mexico State University
Las Cruces, New Mexico 88003

and

Louis H. Kauffman
Department of Mathematics
University of Illinois at Chicago
851 South Morgan St.
Chicago, Illinois 60607-7045

I. Introduction

It is well-known that combinatorial knot theory is a graph theoretic way to approach the theory of knots, links and their topology in three-dimensional space. The usual formulation of this theory via knot diagrams works with a formal system that is a mixture of diagrams and graphs. Our purpose is to introduce a specific, clear and simple graph theoretic (and hence set-theoretic) foundation for the theory of knot and link diagrams.

In the course of our work, we introduce a very natural *arc-graph* associated with any knot or link diagram, and we show how these arc-graphs can be put to use in studying the algebraic invariants of knots such as the fundamental group, the quandle or the involutory quandle. By specializing the quandle to the case of colorings with values in Z/nZ (the integers modulo n), we define a new invariant, the coloring number C(K), for knots and links. There are two main variations of this coloring number, that we denote by C(K) and CM(K). CM(K) is the least number of colors needed on a minimal diagram for K. We close with a discussion about this new invariant and conjectures related to it.

Section 2 sets the background for knot and link diagrams, and defines the arc-graph A(K) of a diagram K. We then define admissible labellings (G, L) for plane 4-regular graphs G and show that such labelled graphs are in precise correspondence with link diagrams. We illustrate this theory with specific examples and with a new proof that every such G can be given the structure of an alternating weave.

In Section 3 we discuss the Reidemeister moves on link diagrams and their translation into the language of labelled 4-regular plane graphs. Section 4 discusses the basic theory of the involutory quandle, IQ(K), of a link K and shows how the arc-graph is central to the structure of this invariant. Section 5 specializes the IQ to the case of colorings with values in a cyclic group. The new invariant C(K) is introduced and we discuss a number of conjectures in relation to this invariant. In particular, in Section 6 we discuss the conjecture that CM(K) is equal to v(K), the number of crossings of K, for K a reduced, connected, alternating knot diagram whose determinant is a prime number.

Caveat. Throughout this paper the term *graph* will refer to a graph in which loops and multiple edges are allowed.

Acknowledgment. Louis Kauffman thanks the National Science Foundation for support of this research under grant number DMS-9205277 and the NSA for partial support under grant number MSPF-96G-179. Frank Harary thanks the Departments of Computer Science and Mathematical Sciences and the Computer Research Laboratory at New Mexico State University for supporting infrastructure.

II. Knots, Links and 4-Regular Graphs

We first give a quick description of the diagrammatic approach to the theory of knots and links. We then show how to use graph theory to give this diagrammatic formal system a simple and effective set-theoretic underpinning. This translation of knot diagrams to graph theoretic concepts is significant in that it provides a direct connection between combinatorial knot theory, graph theory and the foundations of mathematics. We shall point out some of the implications of the diagrams themselves, both for graph theory and for foundations.

The formal diagrammatic system for knots and links is based on diagrams such as the ones shown in Figure 1.

Definition 2.1. Each *diagram* is composed of a finite collection of disjoint arcs in the plane. Each arc has each of its endpoints paired with an endpoint of another arc (possibly the other endpoint of the same arc) across an interior point of an arc in the collection of arcs.

The arc containing the interior point is called the *overcrossing arc*. The configuration of overcrossing arc and paired endpoints is called a *crossing*. In the pairing of endpoints at a crossing it is possible to draw a straight line segment between the two paired endpoints in such a way that this segment intersects the overcrossing arc in exactly one point. This point is the designated interior point of the overcrossing arc.

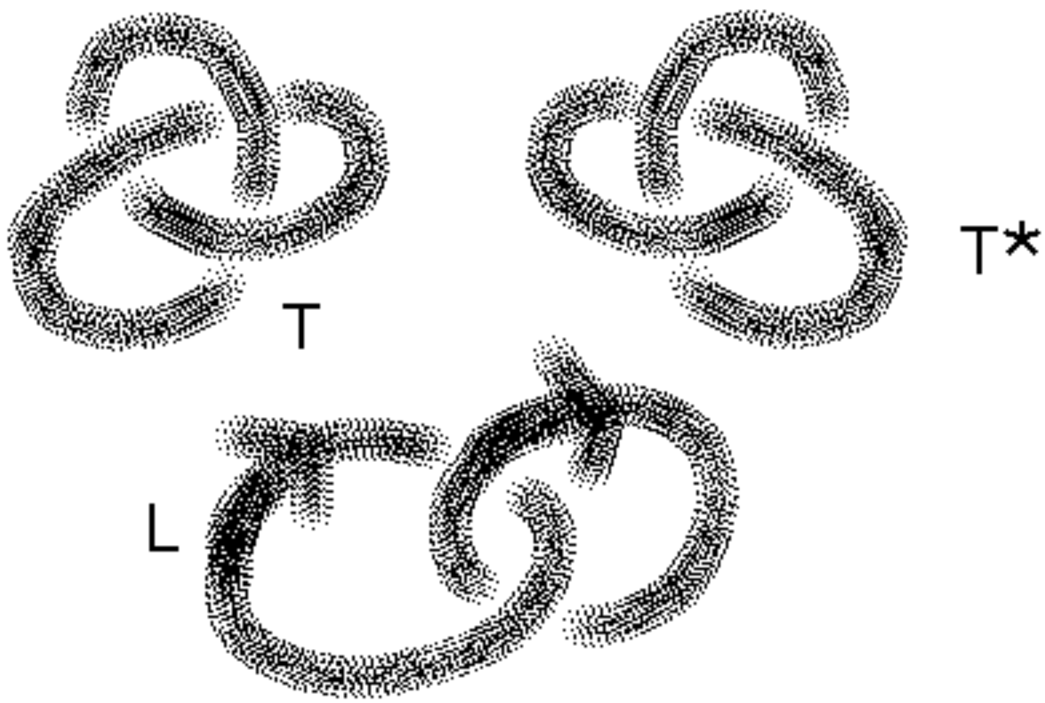

Figure 1 - Exemplary knot and link diagrams

The *mirror image* of a knot or link is obtained by reversing all of its crossings. A link is said to be *oriented* if there is a specific direction, indicated by an arrow, chosen for each of its components.
In Figure 1 we have shown the trefoil diagram T, its *reflection* or mirror image T*, and an oriented link L. Each of these diagrams may be regarded as a description of a projection of a curve (T or T*) or curves (L) from 3-dimensional space to the surface of the plane (or to the surface of a 2-dimensional sphere). The diagrams are drawn in such a way that one can interpret a continuous line segment that is flanked by two segments ending on either side of the line as the crossing of two segments of the projection. The continuous line is above (in 3-space) the line that it crosses over. See Figure 2.

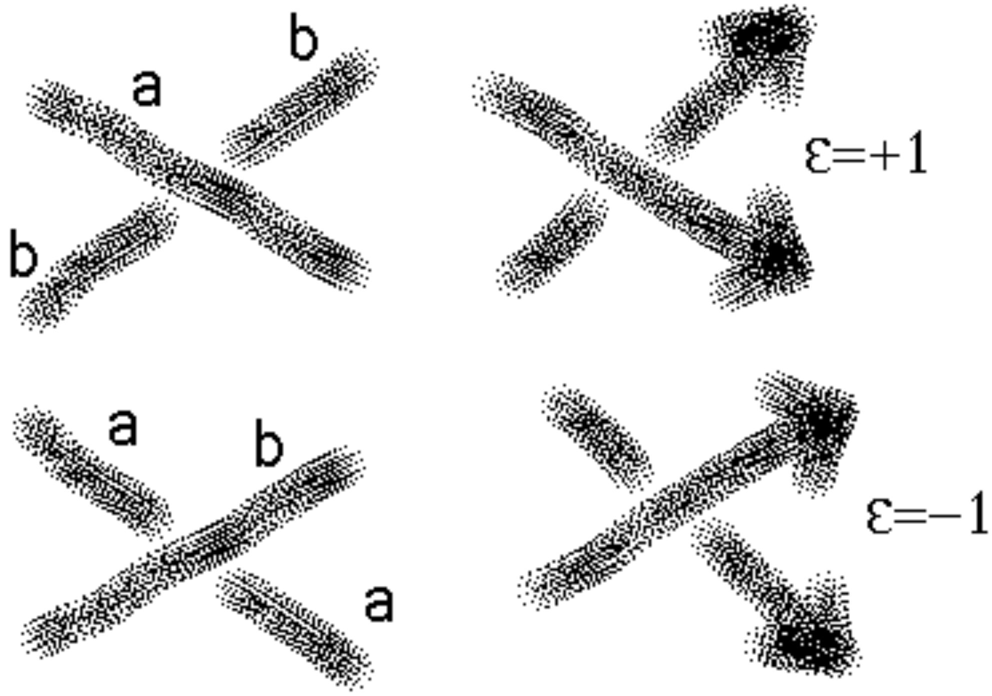

Figure 2- Crossing structure

In Figure 2 we illustrate two possibilities for such a crossing. In the first case line a crosses over line b (line b is broken into two segments in the diagram). In the second case line b crosses over line a. We have indicated possible *orientations* at these crossings. A crossing is said to be of *type +1* if the overcrossing line becomes parallel to the undercrossing line when the overcrossing line is turned counterclockwise; of *type -1* when it is turned clockwise.

Returning to Figure 1, note that each diagram is composed of a collection of arcs with their endpoints paired across interior points of (overcrossing) arcs.

Definition 2.2. For graph theoretic purposes it is fruitful to define the *arc-graph*, A(K), of an unoriented link diagram K. The arc-graph (see Figure 3) has one vertex for each arc in K, and one edge for each pairing of arc endpoints. This edge is labelled by the arc that passes between the endpoints. Each vertex is labelled by its corresponding arc.

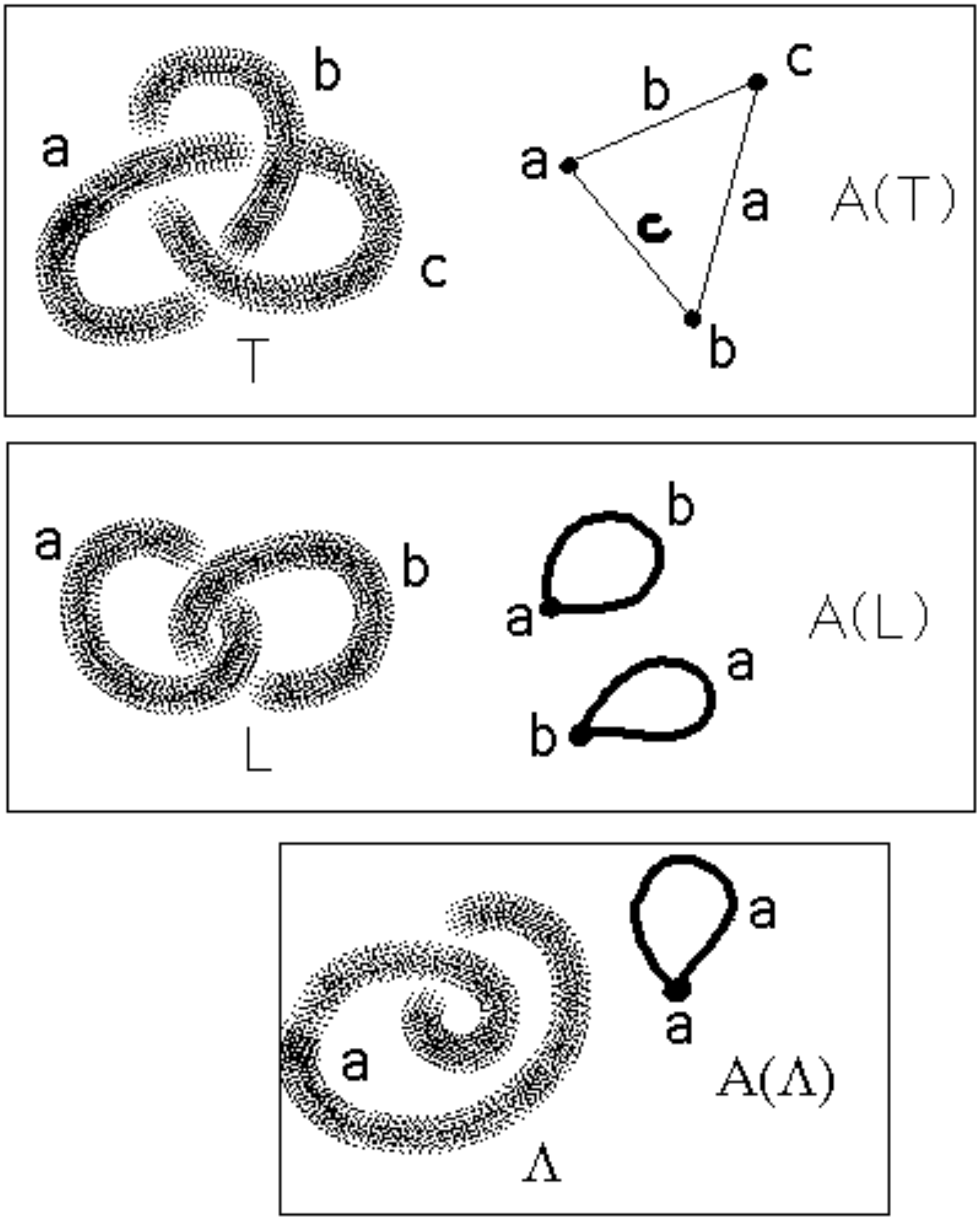

Figure 3 - Diagrams and their arc graphs

The arc-graph A(K) of link diagram K consists in a disjoint union of labelled cycle graphs, i.e. it is a regular graph of degree 2 (See [6].). The number of cycle graphs in A(K) is equal to the number of topological components in the corresponding link K. It is common topology parlance to speak of a *link diagram with n components.* By this it is meant that *A(K) has n cycles.*

It is natural to wonder to what extent one can reconstruct the diagram K from the labelled graph A(K). This is equivalent (See [1].) to the well-known (and solved) Gauss code reconstruction problem. We shall discuss this question elsewhere. For our purposes, the arc-graph of a knot or link diagram is useful because it gives a combinatorial formulation of many properties of the diagram, and because, from it, we can read certain significant topological invariants of knots and links. These invariants will be taken up in Section 4.

We now look directly at the link diagrams themselves.

Definition 2.3. The *shadow graph* G of a diagram K is obtained by taking each crossing in the diagram and adding to the diagram the segment connecting the paired endpoints at that crossing. Let S(K) denote this new diagram. This new segment intersects the overcrossing line in a single interior point. The set of these intersection points is defined to be the set of vertices of the shadow graph G. An edge in G is a path from a vertex to a vertex that consists in first going along one of the new segments to an endpoint of an arc in G, then along the arc, then along a new segment to a vertex. This gives the subset S(K) the structure of a plane graph G, the shadow graph of K. It is a 4-regular plane graph, with loop and multiple edges allowed.

Certainly a link diagram can be regarded as a 4-regular plane graph G (the shadow graph of the diagram) with extra structure. The art of translating this structure (depicted diagrammatically by broken line segments associated across other lines) is to make a perspicuous choice of labelling for the graphical edges. Here we present a new method for accomplishing that labelling, illustrated in Figure 4.

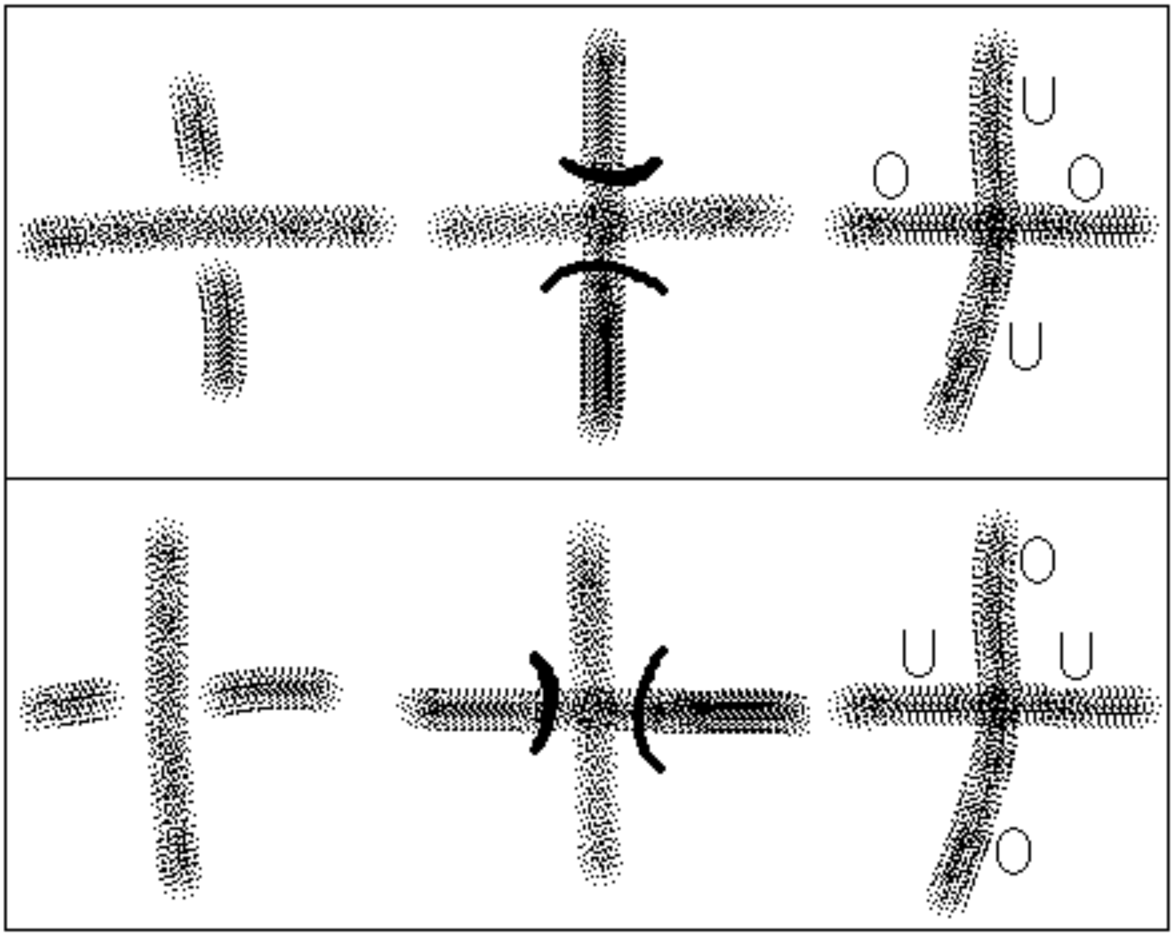

Figure 4 - Parenthetical crossing code

In Figure 4 the broken segment (or missing segment) at the crossing is filled in. In its place are two parentheses on either side of the overcrossing line. The concave sides of the parentheses face away from the overcrossing line and they are inscribed upon the edges at the vertex that corresponds to the undercrossing line. As we can see from Figure 4, there ensue four types of labels for each edge: no parentheses, one parenthesis (two possibilities here since the edge has two ends), two parentheses. Let us denote the absence of a parenthesis by O (for overcrossing) and the presence of a parenthesis by U (for undercrossing). Then the four possible edge labels are OO, OU, UO and UU. an edge labelled OO crosses over at both of its ends. An edge labelled OU or UO goes from over to under or vice versa. An edge labelled UU underpasses at both ends. Note that the double label on the edge is correlated with its endpoints. Thus if e is an edge of G and $\partial e=\{a,b\}$ is the set of endpoints in the boundary of e, then a label on e is a function $L(e): \partial e \dashrightarrow \{O,U\}$.

To be quite precise, if $\partial e=\{a\}$ then we replace ∂e by the multiset $\{a',a''\}$ so that two function values are possible. A *labelling of G* is a choice of label for each edge in G.

Now recall that G is given as a 4-regular plane graph. Hence each vertex has a cyclic order of edges given from the planar embedding. Let <1234> denote the typical order of edges e_1,e_2,e_3,e_4 incident to a vertex v of G. Then in an admissible labelling, e_1 and e_2 receive the same label at v. Furthermore, the label (U or O) received by e_1 and e_3 at v are opposite (U if O and O if U) to that received by e_2 and e_4 at v. With this notion of admissible labelling for G, we have captured precisely the structure of knot or link diagram in purely graph theoretic terms.

Note that if e_1 and e_3 receive the O label at v, then the segment e_1e_3 overcrosses at v, while the segment e_3e_4 undercrosses at v. If the reader will look back at Figure 4, she will see two diagrams and their admissible labellings. The trefoil diagram T is alternating. A diagram is alternating if every edge in its shadow graph G has two labels (UO or OU), hence exactly one parenthesis. That is, a weave is alternating if one encounters an overcrossing immediately after each undercrossing and vice versa.

Notation. We shall denote by (G,L) a 4-regular plane graph G with an admissible labelling L.

Alternation Lemma 2.4. Let G be a 4-regular plane connected graph. Then there exists an admissible labelling for G such that each edge of G receives exactly one parenthesis. Hence, there is a link diagram G# with shadow G such that G# is an alternating diagram.

Proof. Call the operation of replacing a vertex in G by reassembled edges as shown in Figure 5 a *smoothing* of that vertex.

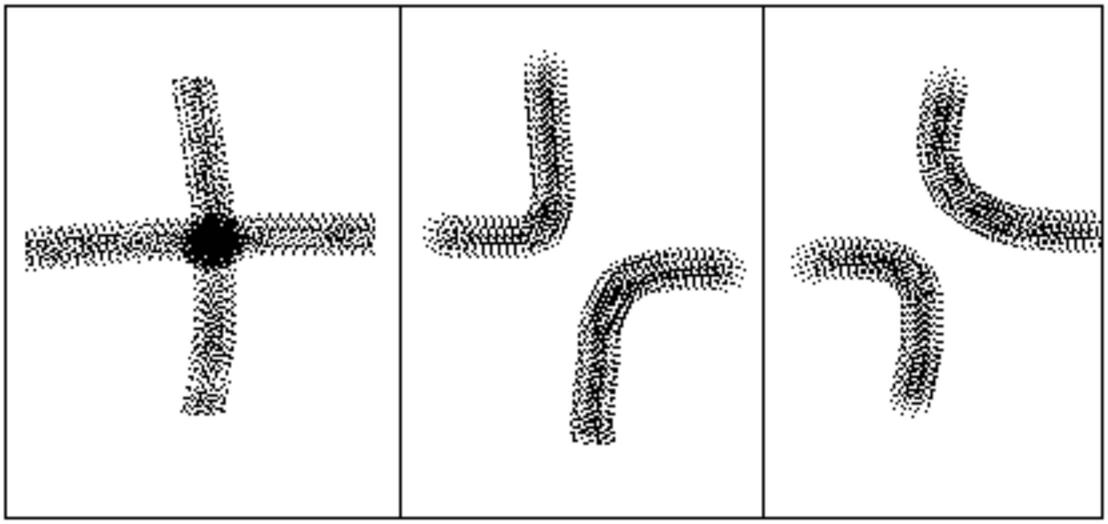

Figure 5 - Four-regular node and two smoothings

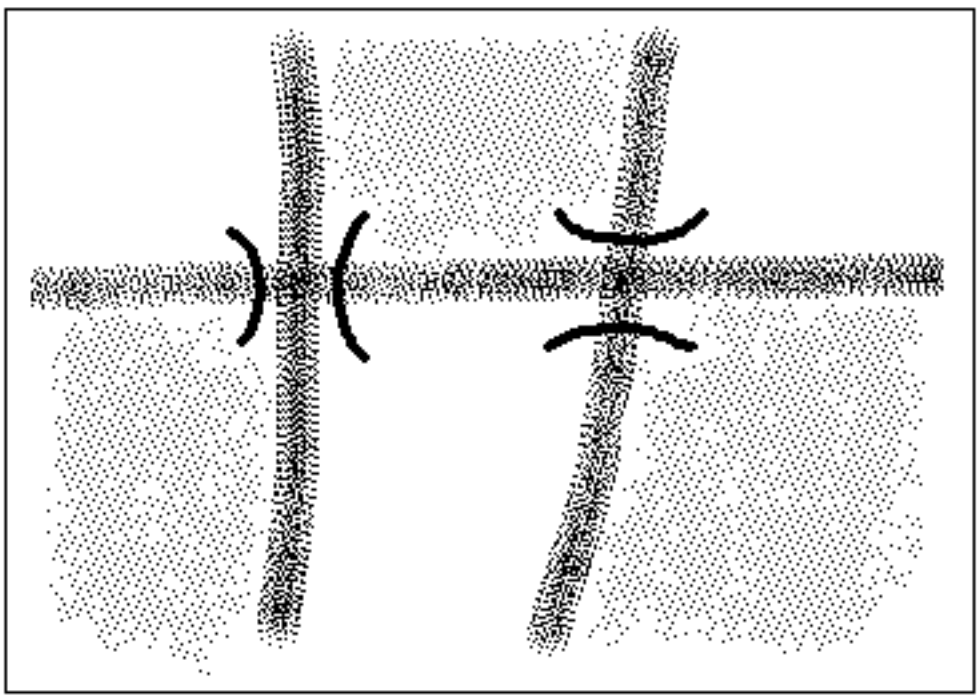

Figure 6 - Parentheses in relation to shaded regions

It is easy to see that one can choose to smooth the vertices of G, a connected 4-regular graph in the plane, in such a way that the result is a single Jordan curve (a simple closed curve) J in the plane. By theJordan Curve Theorem, J divides the plane into two connected regions. Color the unbounded region white and the bounded region black. Then each region in the plane designated by G receives the color black or white in such a way that regions sharing an edge of G

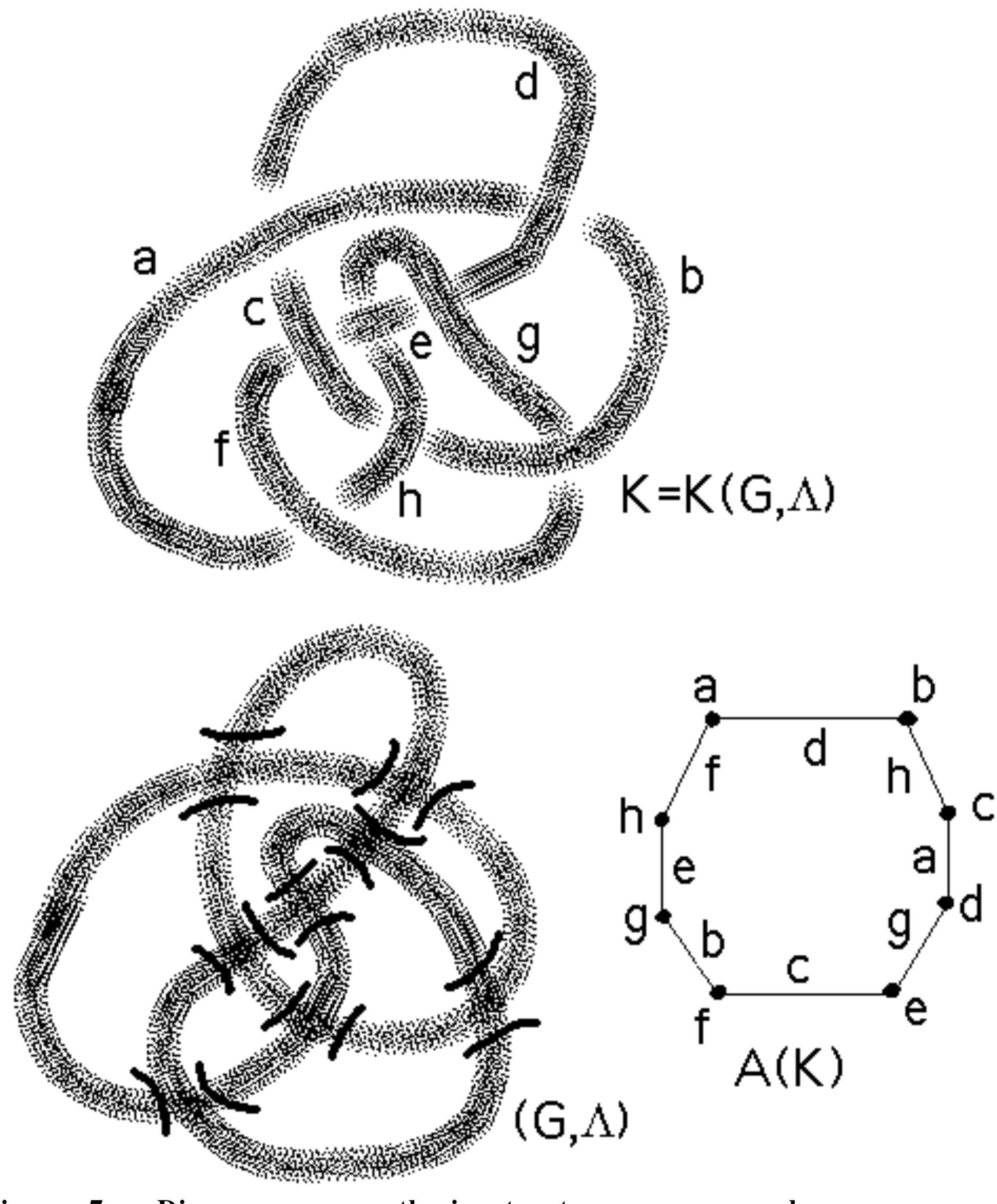

Figure 7 - Diagram - parenthesis structure - arc graph

have different colors. Now place a double parenthesis at each vertex of the shaded G according to the convention shown in Figure 6. (The parentheses appear horizontal when a black region is in the left lower quadrant.)

As shown in Figure 6, the resulting labelling assigns exactly one parenthesis to each edge of G. This completes the proof. //

Remark. Let (G,L) be a plane 4-regular graph with an admissible labelling. Note that we can form the arc-graph A(G,L) just as in the case of link diagrams. An arc in (G,L) is a sequence of edges with labels of the form UO, OO, OO, ..., OO, OU such that each adjacent pair of edges are part of the crossover structure at a vertex of G. This gives an algorithm for constructing a link diagram from (G,L). We can think of this situation as indicated in Figure 7. The labelled graph and the arc-graph are the two basic graph-theoretical structures associated with a link diagram.

Definition 2.5. We define the *(signed) checkerboard graph* CG(K) associated with a connected unoriented link diagram K as follows: Consider the 4-regular plane shadow graph of K. This graph describes a decomposition of the plane into connected regions. Color these regions with two colors (call the colors "black" and "white") so that the unbounded region is colored white. Such a coloring exists by the argument given at the beginning of the proof of the Alternation Lemma. Now construct the graph CG(K) whose vertices are in 1-1 correspondence with the black regions, such that there is an edge between two vertices of C(G) whenever the corresponding black regions meet at a crossing of K, i.e. they share a vertex of the 4-regular graph. With respect to the coloring of the regions the crossings of K fall into two types that we shall designate as +1 and -1: A crossing is said to be of *type +1* if the two shaded regions at that crossing are swept by the overcrossing line as it is turned counterclockwise; of *type -1* if the shaded regions are swept as it is turned clockwise. In CG(K), assign the sign +1 to an edge if it corresponds to a crossing of type +1 and a sign of -1 to an edge if it corresponds to a crossing of type -1. This gives CG(K) the structure of a signed graph. Figure 7.1 illustrates these ideas.

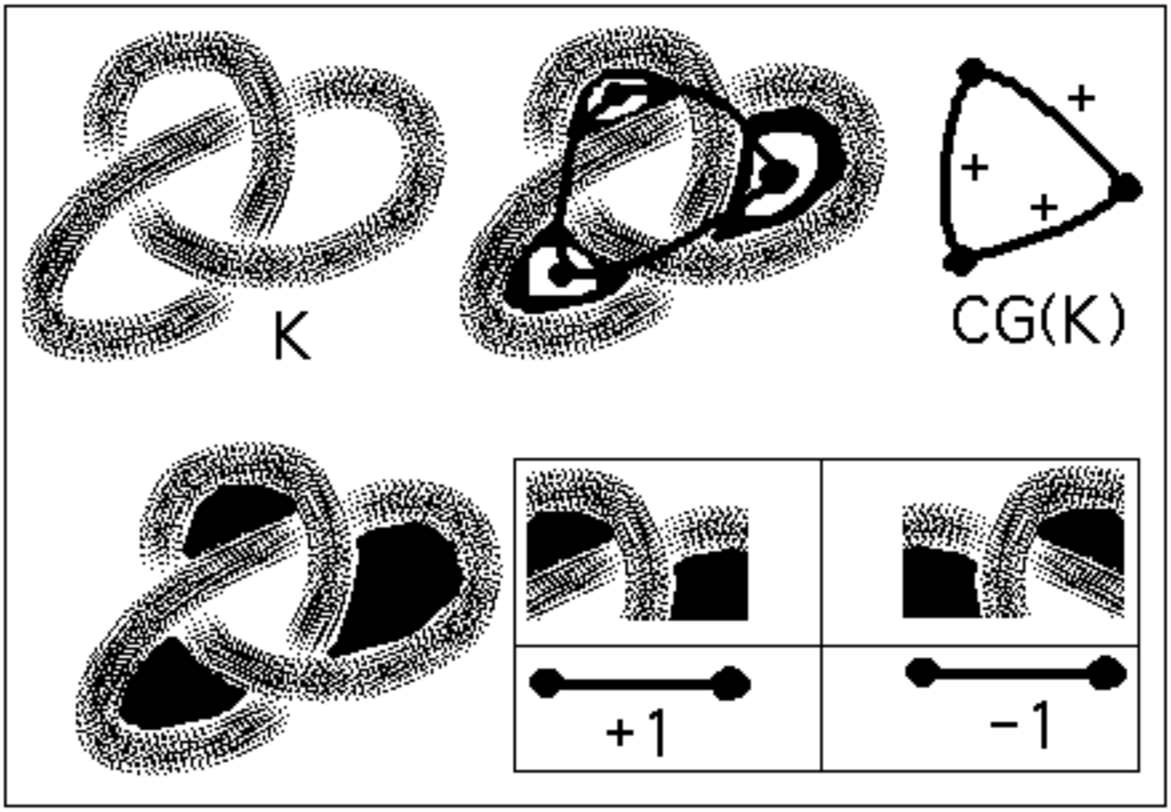

Figure 7.1 - The checkerboard graph for a knot or link diagram

The signed graph CG(K) associated with a link diagram K can be used for many graph theoretic investigations related to the topology of knots and links. We shall elaborate on this further in other papers in this series. In the present paper we give one application below, and another in the final section on coloring knots and links. Note that the signed graph for the mirror image of a link K is obtained by changing all the signs in the graph for K.

Remark. Another proof of the Alternation Lemma can be formulated by using the concept of the checkerboard graph CG(K). It is easy to see that a connected diagram K is alternating if and only if the checkerboard graph CG(K) has constant signs (all +1 or all -1). Consequently, for an arbitrary 4-regular graph H, we can take the corresponding unsigned checkerboard graph CG(H) and then choose crossing states at each vertex so that all the signs of CG(H) are (say) positive. The corresponding link diagram (call it K(H)) is necessarily alternating.

III. Reidemeister Moves

Two curves embedded in 3-space are said to be *ambient isotopic* if there is a family of embeddings parametrized on the unit interval that connects them. In [2] Reidemeister proved that a set of three basic moves, shown in Figure 8, on link diagrams suffices to study the topological properties of knots and links in 3-dimensional space. He showed that the equivalence relation generated by his moves, plus equivalence of graphical structure via homeomorphism of the plane or the two dimensional sphere, is equivalent to the relation of ambient isotopy of embedded curves in three dimensional space.

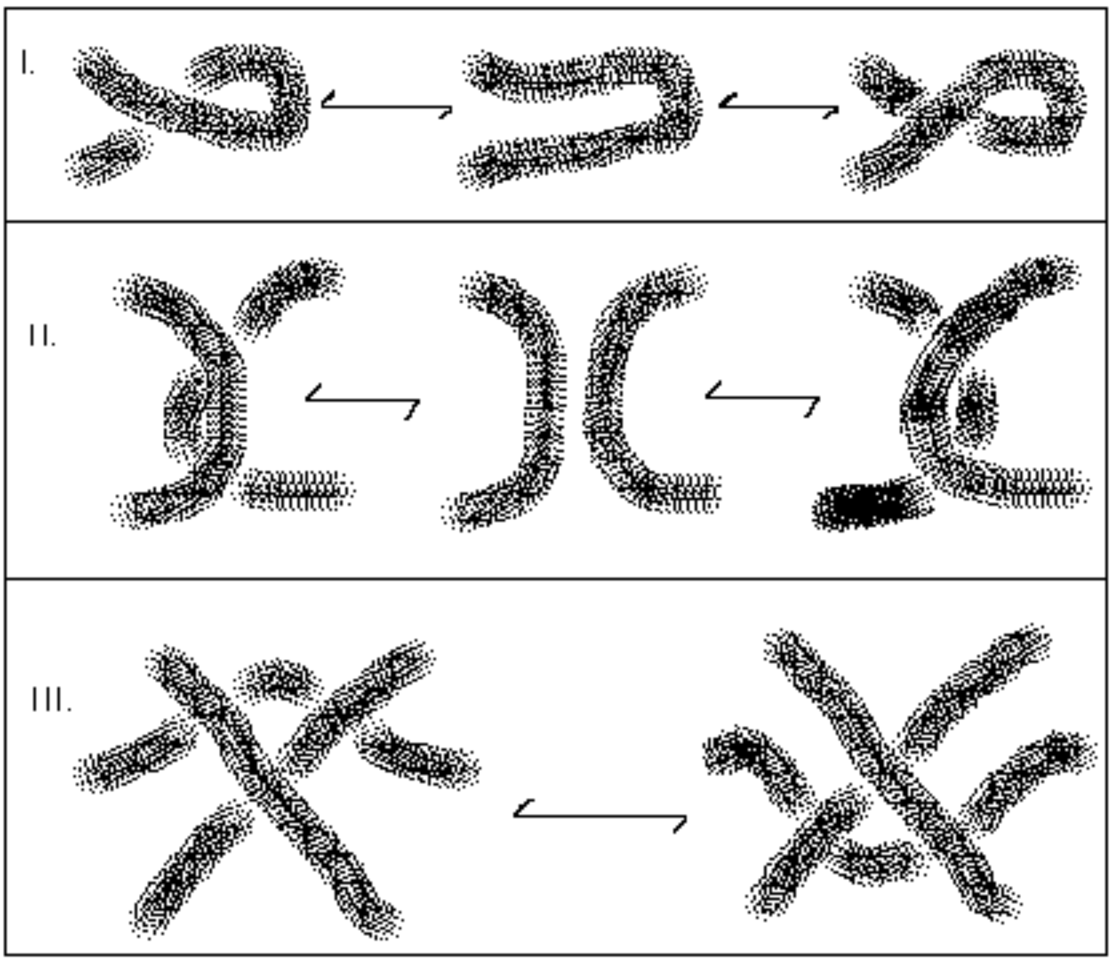

Figure 8 - The Reidemeister moves

Each of these moves is regarded as a local change in a larger diagram that is otherwise unaltered by the move. Each move is performed locally on a portion of the diagram topologically equivalent (as underlying plane graphs) to that part depicted in Figure 8. Figure 9 illustrates the use of the Reidemeister moves.

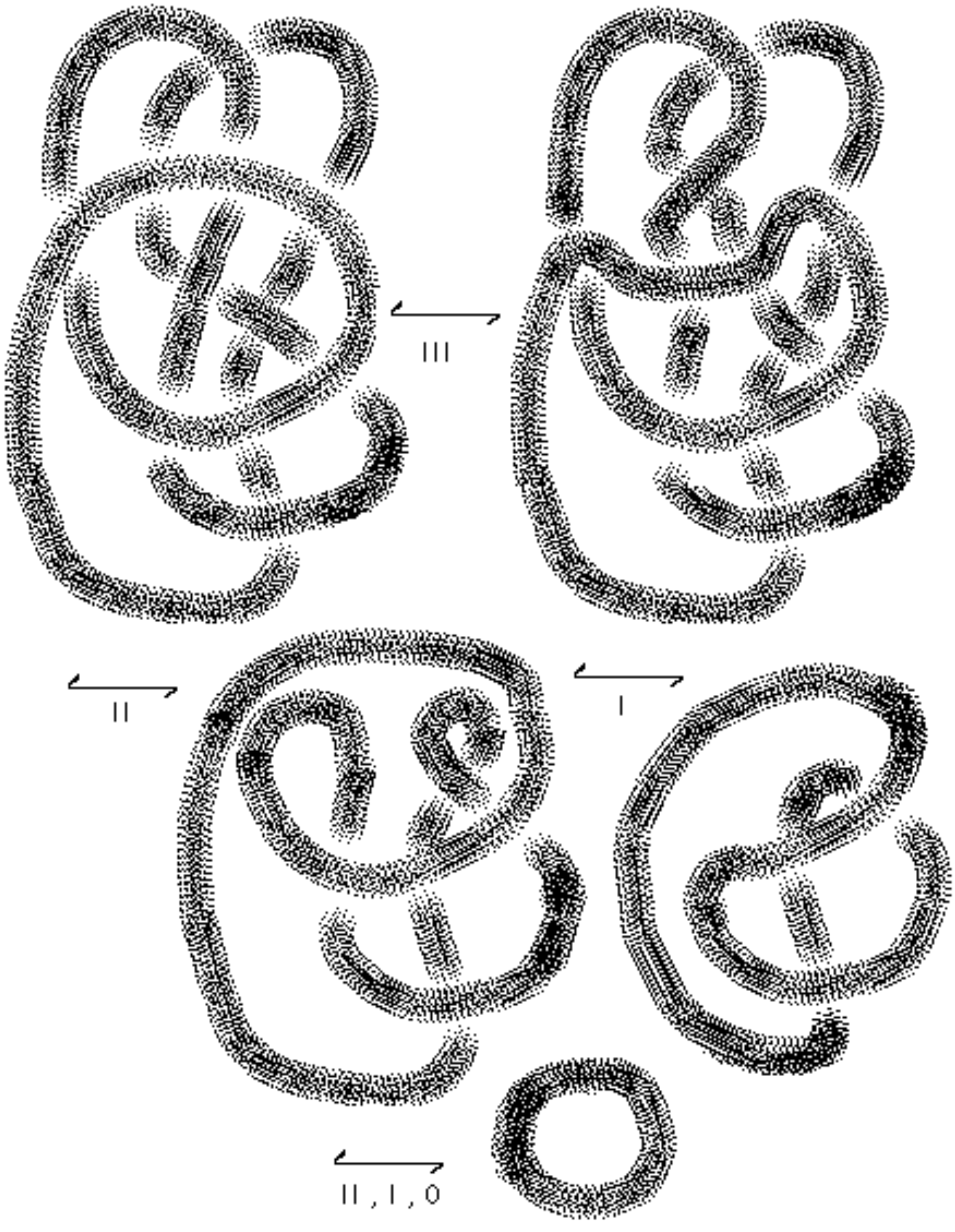

Figure 9 - Using the Reidemeister moves

The combinatorial theory of knots and links is the theory of the link diagrams under the equivalence relation generated by the Reidemeister moves. In Figure 10 we illustrate the translation of the Reidemeister moves to moves on labelled 4-regular plane graphs (G,L) as described in Section 2.

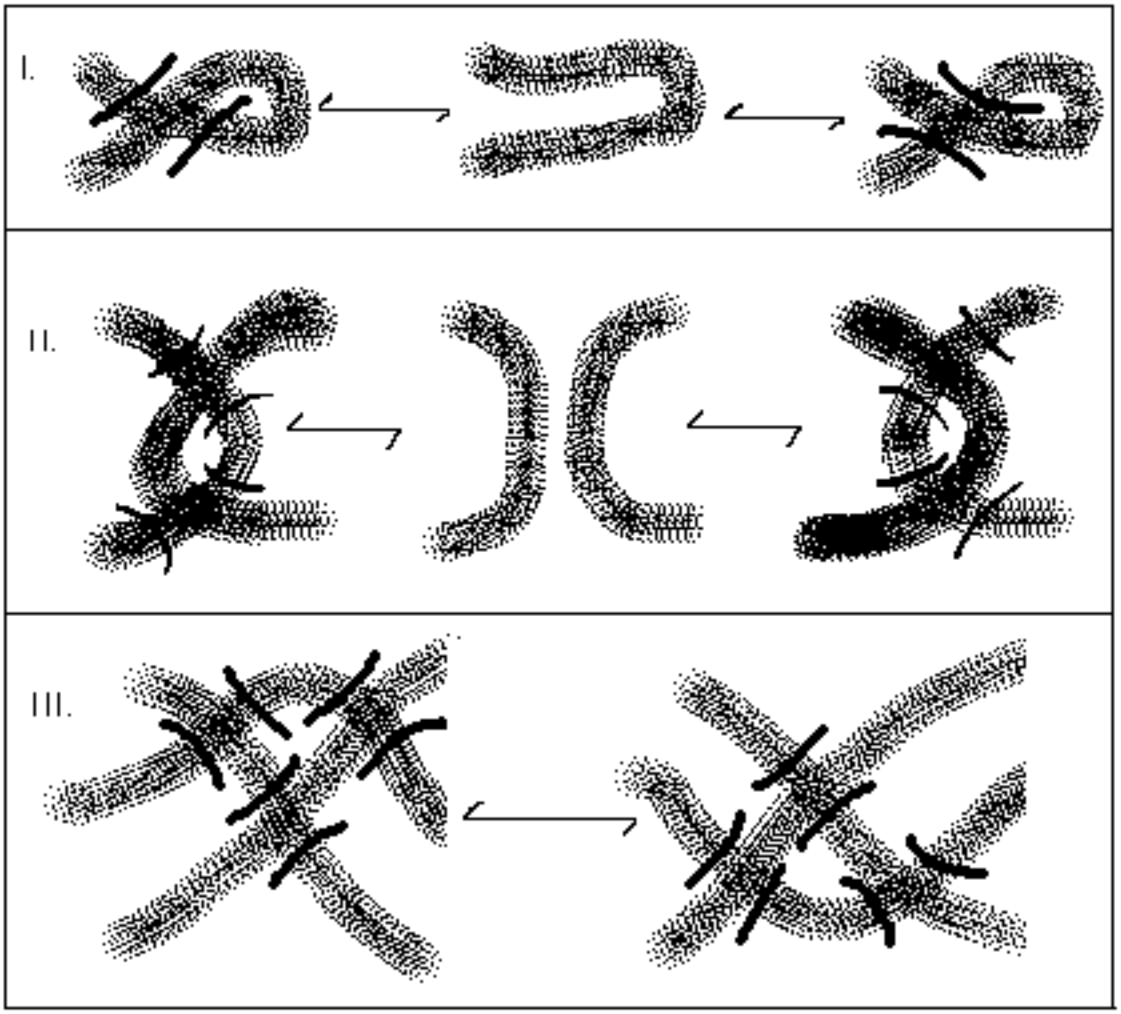

Figure 10 - Four-regular encoding of Reidemeister moves

We shall not pursue the formulation of the Reidemeister moves for admissible labelled graphs (G,L) other than to point out that this reformulation gives a complete translation of link diagrams and their associated topology into graph theory. It will be helpful to the graph theorist to work in this language. See Figure 10.1 for a parallel depiction of unknotting, first in link diagrams, then in the (G,L) language.

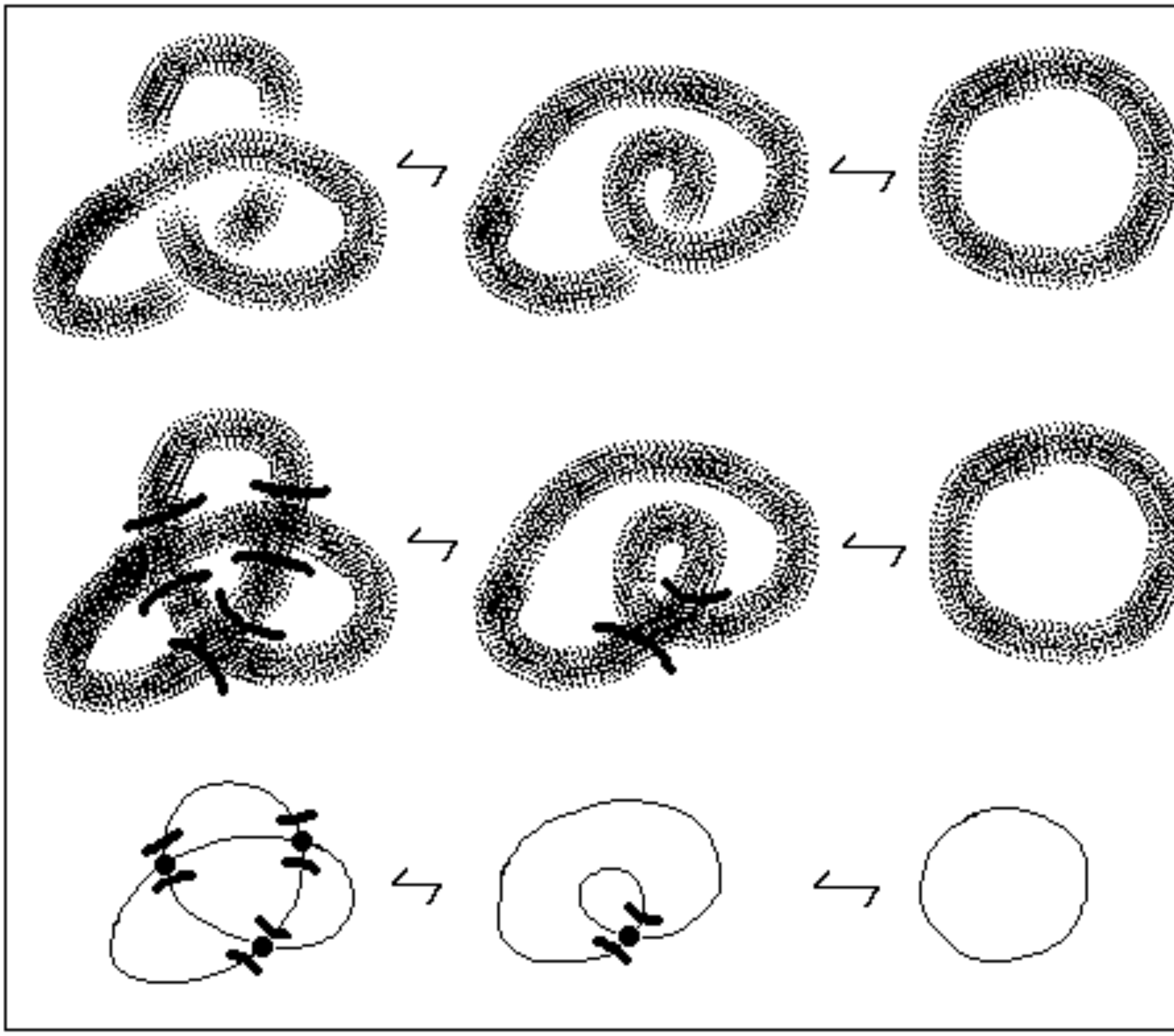

Figure 10.1 - Diagram language and graphical language

IV. The Involutory Quandle

We now show how the graph A(K) of a link diagram K is of use in studying an algebraic invariant called the involutory quandle of K, denoted IQ(K).

The IQ of a link K is an algebraic structure with one (non-associative) binary operation. We shall denote this operation a*b by the juxtaposition of symbols ab. Thus the quandle operation applied to a and b is denoted by a*b=ab. It is useful to use the convention that products without indicated parentheses are to be *left-associated*.

That is,

$$abc=(ab)c$$
$$abcd=((ab)c)d$$

and so on.

The axioms for an involutory quandle Q [3] are as follows:

1. aa = a for all a in Q.
2. (ab)b = a for all a and b in Q.
3. (ab)c =(ac)(bc) for all a,b and c in Q.

We can associate an involutory quandle IQ(K) with a link diagram K as follows [3]:

We shall write [a|b|c] when the arc b overcrosses the meeting of the arcs a and c. Thus [a|b|c] corresponds to an edge in A(K) with endpoints a and c. The quandle IQ(K) is generated by a set of elements that are in one to one correspondence with the arcs of the diagram K. For each crossing in K with the edge [a|b|c] in A(K) we have the relation ab=c in the quandle IQ(K).
See Figure 11.

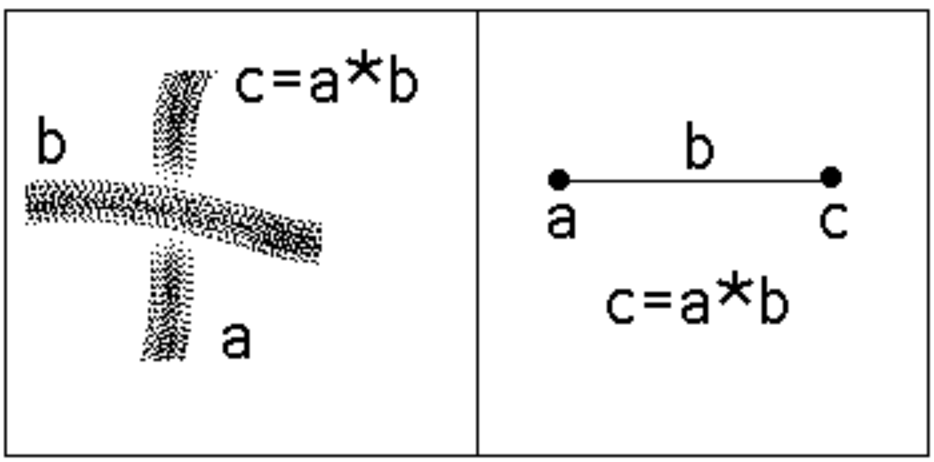

Figure 11 - Quandle operation at a crossing

Note that by axiom 2. above we have ab=c if and only if a=cb. Thus this relation does not depend upon an orientation assigned to the link diagram.

The involutory quandle IQ(K) is defined to be the algebra under the axioms 1.,2., 3. obtained by taking the generators and relations as described above.

For example, view Figure 12 for a description of IQ(T) where T is a trefoil diagram.

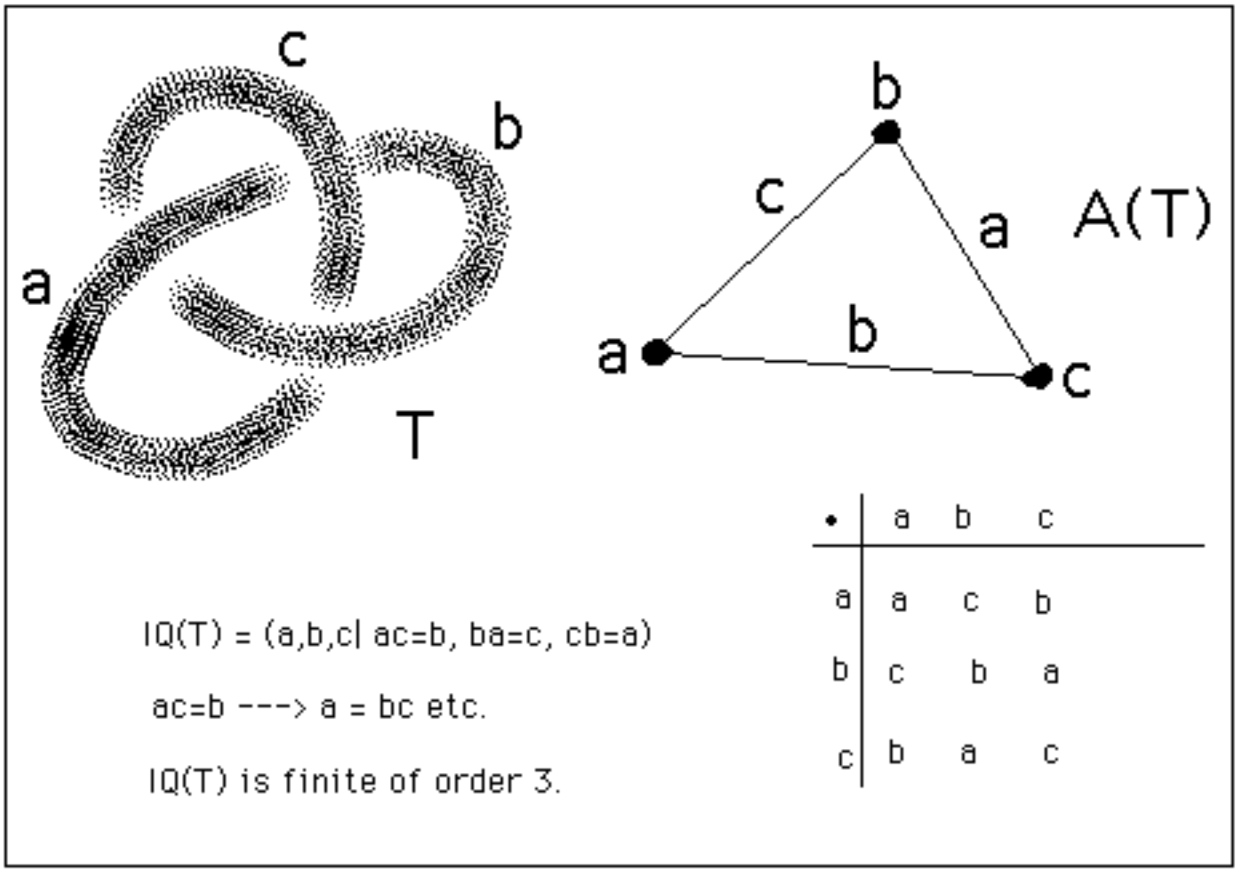

Figure 12 - The involutory quandle (IQ) of the trefoil knot

In Figure 12 a simple calculation shows that the IQ of the trefoil knot has exactly three elements. In Figure 13 we illustrate the generators and relations for the IQ of the figure eight knot E. The IQ of the figure eight knot is finite, having order five. We shall postpone the calculation of this IQ until we have discussed an important lemma due to Winker [4].

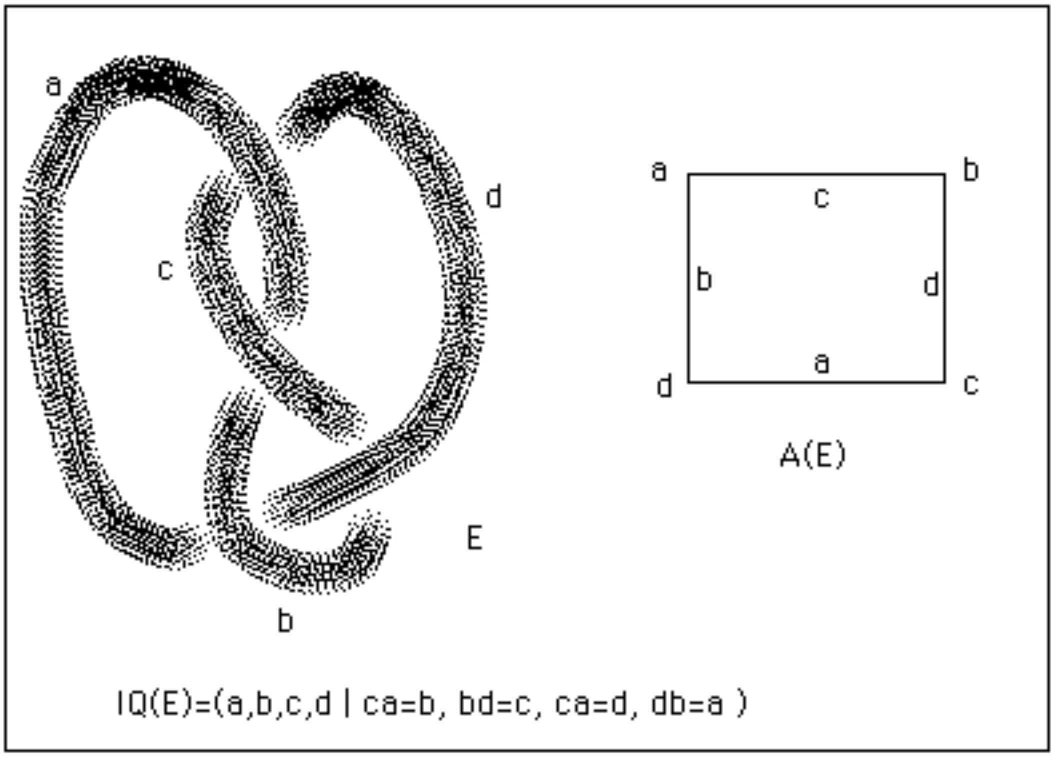

Figure 13 - The involutory quandle (IQ) of the figure eight knot

Winker's Lemma 4.1. Recall that products without parentheses are canonically left associated. If a,b,c are elements of Q, an involutory quandle, then (ab)c = acbc.

Proof. acbc = ((ac)b)c = ((ac)c)(bc) = a(bc). //

With the help of Winker's Lemma, we can efficiently rewrite the generators and relations for IQ(E), starting with the equations from Figure 13:

ac=b, bd=c, ca=d, db=a

<----->

acd=c, ca=d, d(ac) = a

<----->

acd=c, ca=d, dcac = a

<----->

ac(ca) =c, cacac =a

<----->

acaca =c, cacac = a.

Since the last pair of equations are equivalent to each other, we see that the IQ of the figure eight knot has a description with two generators and one relation.

$$IQ(E) = (a,c \mid acaca = c)$$

Using the same graphical technique for diagramming products as we had for the arc-graph, we can write the arc graph

$$G(IQ(E)) = [a|c|x]\ [x|a|y]\ [y|c|z]\ [z|a|c]$$

as shown in Figure 14 to represent the IQ of E.

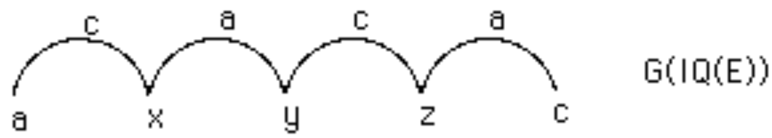

Figure 14 - Diagram for IQ(E)

In this representation x=ac, y=aca, z=acac. Note that the products obtained by traversing the edges of such a graph are naturally left-associated and Winker's Lemma tells us that every element in IQ(E) can be written a left associated product of generating elements. The relation acaca=c tells us that a, ac, aca, acac and acaca=c are the only such products. Thus IQ(E) has exactly five members forming the set {a,x,y,z,c}. It is an easy matter to use the graph G(IQ(E)) to compute products in IQ(E). For example, xy = ac(aca) = acacaca = cca = ca = z. We leave it as an exercise for the reader to determine the full multiplication table for IQ(E). Not all knots and links have finite IQ. Winker [4] shows that the first knot with an infinite involutory quandle is the knot 8_{16} (Figure 15) in the Reidemeister tables.

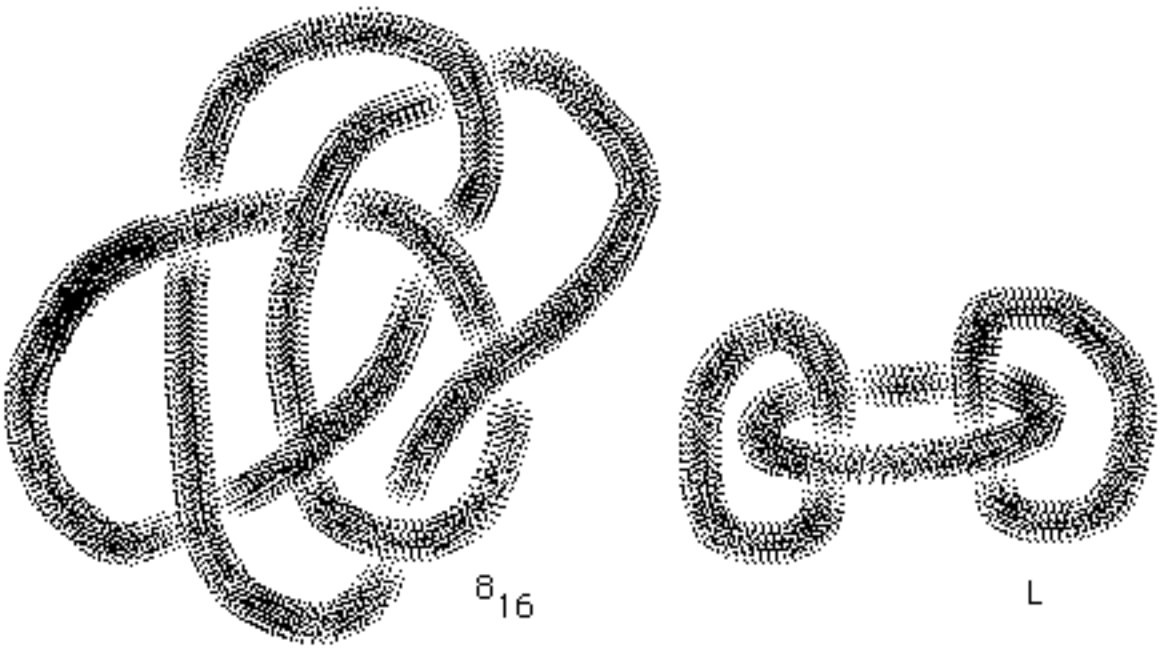

Figure 15 - Knots and links with infinite involutory quandle.

Finally, we have the basic result of Winker.

Theorem 4.2 [4]. If K is a knot and IQ(K) has one element, then K is topologically equivalent to an unknotted circle.

We shall not prove this theorem here, see[4]. It asserts that the IQ detects knottedness. Unfortunately, it is often very difficult to decide upon the triviality of IQ(K) for a given knot K.

This ends our general discussion of the involutory quandle. We now turn to its invariance under the Reidemeister moves.

Proposition 4.3. If links K and K' are equivalent via Reidemeister moves, then IQ(K) and IQ(K') are isomorphic algebraic structures.

Proof. View Figure 16. //

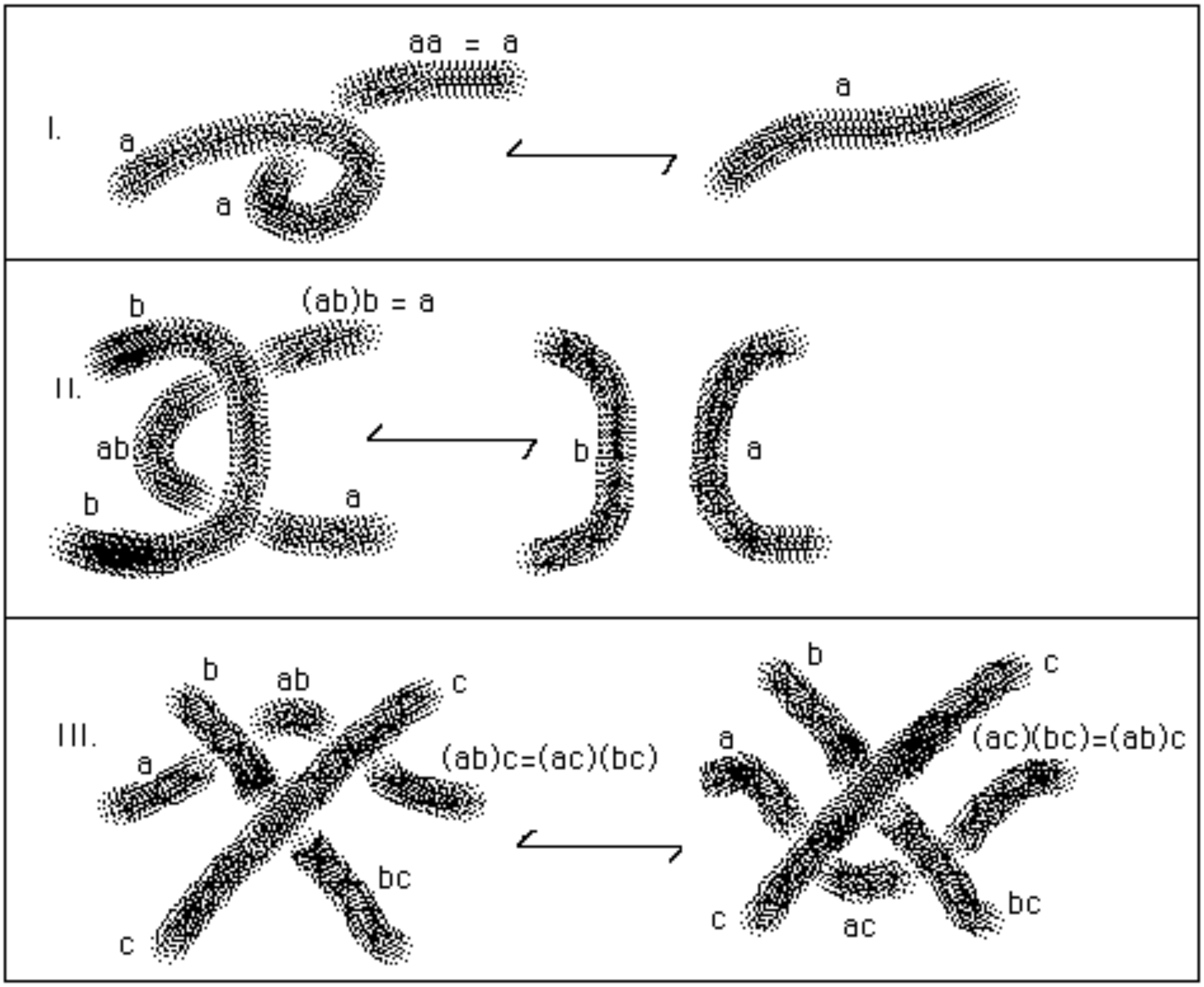

Figure 16 - Involutory quandle axioms and the Reidemeister moves

V. Coloring Link Diagrams

Let A be an abelian group, written additively. Define a binary operation on A by the formula $a*b=ra+sb$ where r and s are fixed integers.

Lemma 5.1. The above binary operation $a*b$ gives any abelian group A the structure of a (possibly trivial) involutory quandle exactly when $r=-1$ and $s=2$. This operation is given by the formula

$$a*b = 2b-a.$$

Proof. In order to have $a*a=a$ we need $a = a*a = ra + sa = (r+s)a$. Thus we require that $r+s=1$. In order to have $(a*b)*b=a$ for any a and b,

we require that $a = (a*b)*b = r(ra+sb) + sb = rra +(r+1)sb$. Hence we need $rr=1$ and $(r+1)s=0$. If $r=1$ then $s=0$ and the quandle structure is the trivial one $a*b = a$ for all a. If $r=-1$ then $s=2$ and $(a*b)*b=a$ is satisfied. Finally, taking $r=-1$ and $s=2$, we have
$(a*c)*(b*c) = 2(2c-b) - (2c-a) = 4c-2b-2c+a = 2c-2b+a = 2c-(2b-a) = (a*b)*c$. This completes the proof.//

Definition 5.2. Given a link K, we let Ab(K) be the involutory quandle defined on K by the free abelian group on the arcs of the diagram K, modulo the relation $c=2b-a$ where a and c are endpoints of an arc labelled b in the arc-graph A(K) of K. (See Figures 11 and 17) In other words we take the relation $c=2b-a$ whenever in the diagram K, b is the label of the overcrossing arc at a crossing while a and c are labels for the undercrossing arcs incident to this crossing. We call Ab(K) the *(unreduced) color quandle* of K.

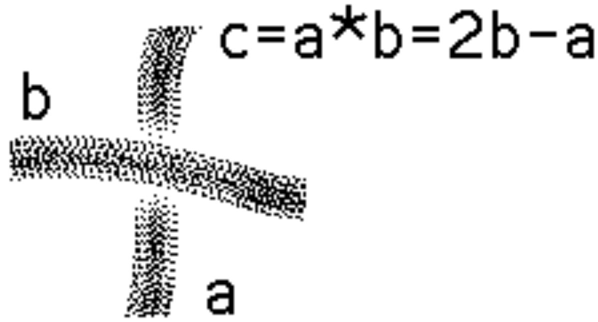

Figure 17 - The basic coloring rule

In the following examples of Ab(K), $[a_1,a_2,\dots,a_n \mid r_1,r_2,\dots,r_n]$ denotes the free (additive) abelian group with generators $a_1,a_2,\dots,a_n$ and relations $r_1,r_2,\dots,r_n$. We often write relations in the form of equations such as $3b=7a$. If there are no relations then we leave the space in the notation for relations blank and write $[a_1,a_2,\dots,a_n \mid \ ]$ to denote the free abelian group generated by
$\{a_1, a_2, \dots a_n\}$.

1. Here U is the unknot, a standard circle in the plane, and A(U) is the trivial graph with one node labelled by a; $Ab(U) = [a| \ ] = Z$ is the set of integers, i.e. the free abelian group on one generator and no relation.

2. In the example illustrated in Figure 3, the link L has two components linking each other once. So A(L) is a graph with two vertices and two loops, one loop at each vertex. The vertices are

labelled a and b and the corresponding loops are labelled b and a respectively. We have Ab(L) = [a,b| ab=a, ba=b] = [a,b| 2b-a = a, 2a-b = b] = [a,b| 2(a-b)=0] = Z + Z/2Z.

3. In this example, illustrated in Figures 3 and 12, the knot K is a trefoil and A(T) is a triangle graph with vertices a,b,c and edges labelled a,b,c such that the label of an edge and its two vertex labels always form the set {a,b,c}. We have Ab(K) = [a,b,c| 2c-a=b, 2a-b=c, 2b-c=a] = Z + Z/3Z, as can be computed by doing integer invertible row and column operations on the relation matrix.

Definition 5.3. Let Ab*(K) denote the quotient of Ab(K) obtained by setting any of the generators of Ab(K) equal to zero. We call Ab*(K) the *(reduced) color quandle of K.*

We see in these examples the general fact that Ab(K) = Z + Ab*(K). The reduced color quandle Ab*(K) is a quotient of the involutory quandle IQ(K), and is sometimes isomorphic to it (as for the trefoil T).

Notice that for the trefoil T, we are coloring the knot diagram with colors from Z/3Z = {0,1,2} so that all three colors occur not only on the standard diagram for T but also on any other diagram for T (as can be seen by doing Reidemeister moves).

Definition 5.4. It is useful to define a *relation matrix M(K)* associated with the expression of Ab(K) in terms of generators and relations. The matrix M(K) has one row for each relation of Ab(K) and one column for each generator. If each relation is written in the form $m_1 a_1 + m_2 a_2 + \dots + m_n a_n = 0$ where the m_i are integers, then the row of M(K) corresponding to this relation is the vector $(m_1, m_2, \dots, m_n)$. Let M*(K) denote any *reduced relation matrix* obtained from the relation matrix M(K) for Ab(K) by striking out any row and any column. Thus M*(K) is a cofactor submatrix of M(K).

In general Ab*(K) will be a finite abelian group whenever K is a knot diagram. The order of the group Ab*(K) is the absolute value of any cofactor of M(K), the determinant of the reduced relation matrix M*(K). This expression,

$$\mathbf{Det(K) = Abs(Det(M^*(K))),}$$

is an invariant of the knot K, called its *determinant* .

The determinant of the knot is a coloring number in the sense that the knot diagram can be colored using the color set {0,1,2, ... N-1} where N = Det(K). However this number is not necessarily the least number of colors needed to color the knot in that modulus.

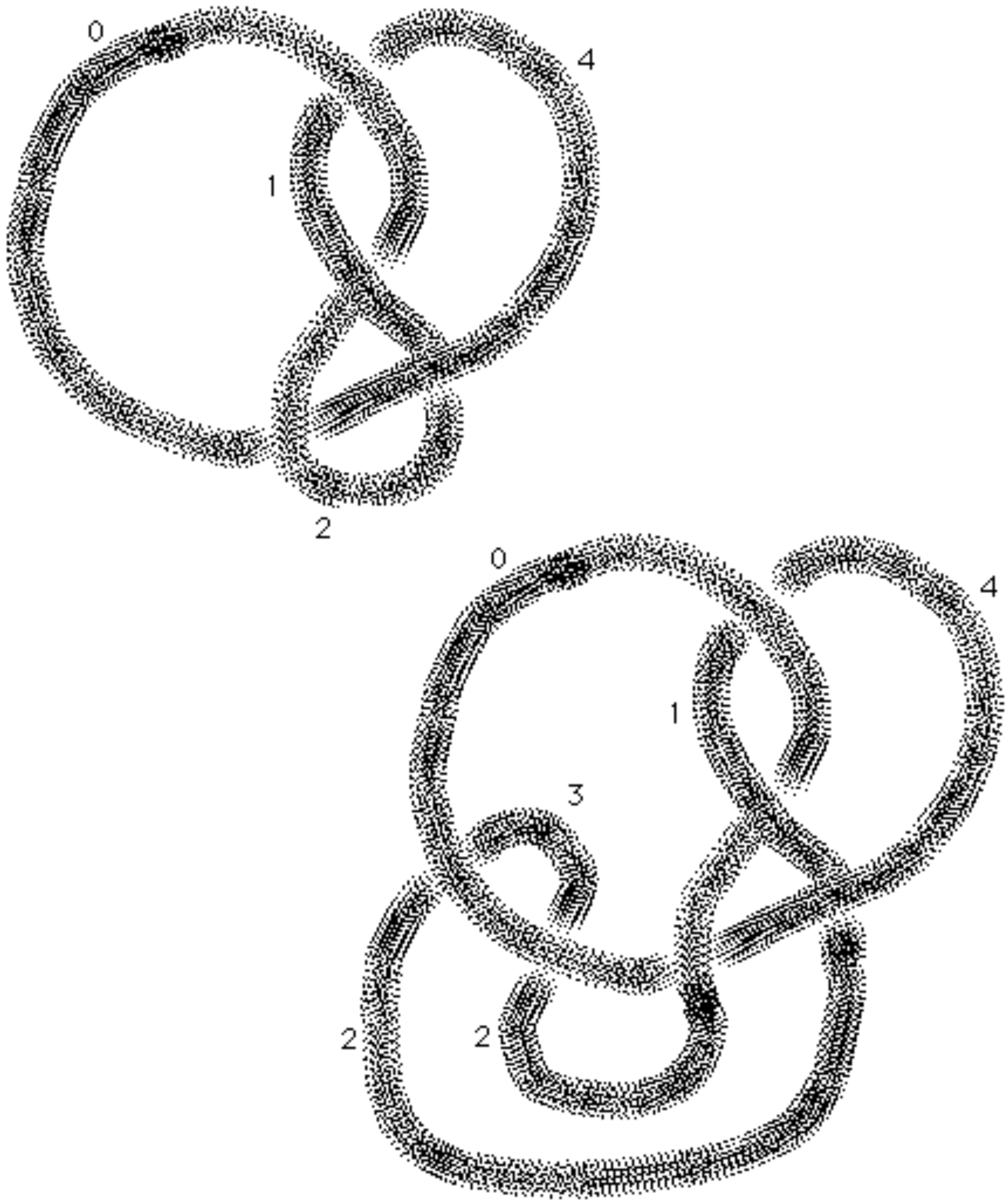

Figure 18- Coloring the Figure Eight Knot

As usual let E denote the figure eight knot. Figure 18 illustrates the fact that Det(E)=5 and exhibits a coloring of E using only four of the five colors. It also shows how the fifth color may appear in a diagram that is equivalent to the first under Reidemeister moves.

Proposition 5.5. There exist diagrams equivalent (by Reidemeister moves) to E that require five colors. Figure 19 shows such a diagram.

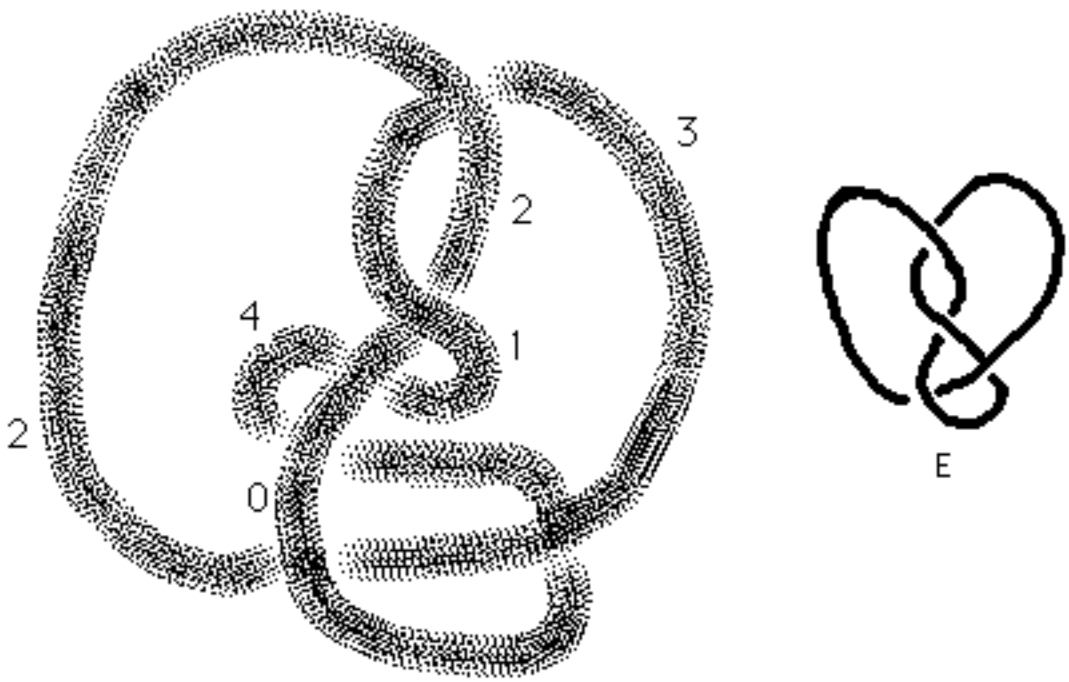

Figure 19 - A Diagram for the figure eight knot that forces five colors

Proof. One can verify that the diagram in Figure 19 requires all five colors for any coloring from the color set Z/5Z by using an enumeration of the set of all such colorings. This is done by solving the corresponding system of equations over the ring of integers. We omit the details of this tedious calculation. //

Definition 5.6. Let K be a knot or link that can be colored (in the sense of the binary operation c=a*b=2b-a) with values in Z/nZ for some n>1. Let N(K) be the least such modulus greater than 1 in which K has a nontrivial coloring. (The number N(K) is equal to the determinant of K when that determinant is a prime number. Otherwise N(K) is a divisor of the determinant that can be determined by working with the algebra of the relation matrix for Ab*(K) - as we have discussed in this section.) Define the *coloring* *number C(K)* to be the least number of colors (>1) in Z/N(K)Z that suffice to color some diagram in the Reidemeister move equivalence class of K.

Definition 5.6.1. A diagram for a link is said to be *minimal* if it has the least number of crossings among all diagrams for that link in the equivalence class generated by the Reidemeister moves.
Let K be a knot or link as in definition 5.6. Instead of considering all diagrams for K, consider only those diagrams that have a minimum number of crossings. Let CM(K) denote the least number of colors needed in modulus N(K) to color K among its minimal diagrams. This number is also an invariant of K, and can be computed in some cases where all the minimal diagrams for K are known.

In the light of our discussion for the figure eight knot it is natural to make the following surmise.

Conjecture 5.7. For a given diagram K there exists an equivalent diagram L such that L requires N(K) colors.

We can prove that the usual diagram for E requires four colors, while N(E) =5.

Proposition 5.8. For the usual figure eight knot diagram E, the coloring number is equal to 4. That is, C(E)=4 with N(E)=5.

Proof. We shall actually prove that there does not exist a knot (one component) that is colorable via Z/5Z using only three colors.
Note that a knot cannot be nontrivially colored with less than three colors since a crossing that does not support three colors supports only one color. Hence a knot that is not colored with three or more colors is necessarily single colored. Therefore consider the structure of crossings that support three colors.

Suppose that the color set is {0,1,2}. We wish to determine knots that support non-trivial colorings. Since (modulo 5) 0*1=2, 1*0=4, 0*2=4, 2*0=3,1*2=3 we see that neither 0 nor 2 can be the color of an overcrossing line color at any crossing with with three colors using the color set {0,1,2}. Therefore all overcrossing arcs are colored 1, and all undercrossing arcs are colored 0 or 2. But in a knot diagram every overcrossing arc is also an undercrossing arc at its end points. Thus no arc can be an overcrossing arc. This means that the diagram is the diagram of an unknot.

The same considerations apply to the other possible choices for 3-coloring using Z/5Z. We omit the details for these remaining cases. //

The figure eight knot is the first instance of a knot K for which C(K) is strictly less than N(K). It is quite easy to obtain knots where C(K) is less than the determinant of K. Figure 20 shows a three coloring of the torus knot K(2, 9) where Det(K(2,9))=9. However, it is easy to see that here N(K(2,9))=3. We therefore know that C(K(2,9)) = N(K(2,9)) since it is a general fact that for a knot that admits a nontrivial 3- coloring, there will be such a coloring for every diagram of that knot. Furthermore, if a knot diagram admits two colors, then it necessarily admits a third color. The only way that one of the two colors can be transformed to the other color along the arcs of the knot is if an arc crosses under a third color..

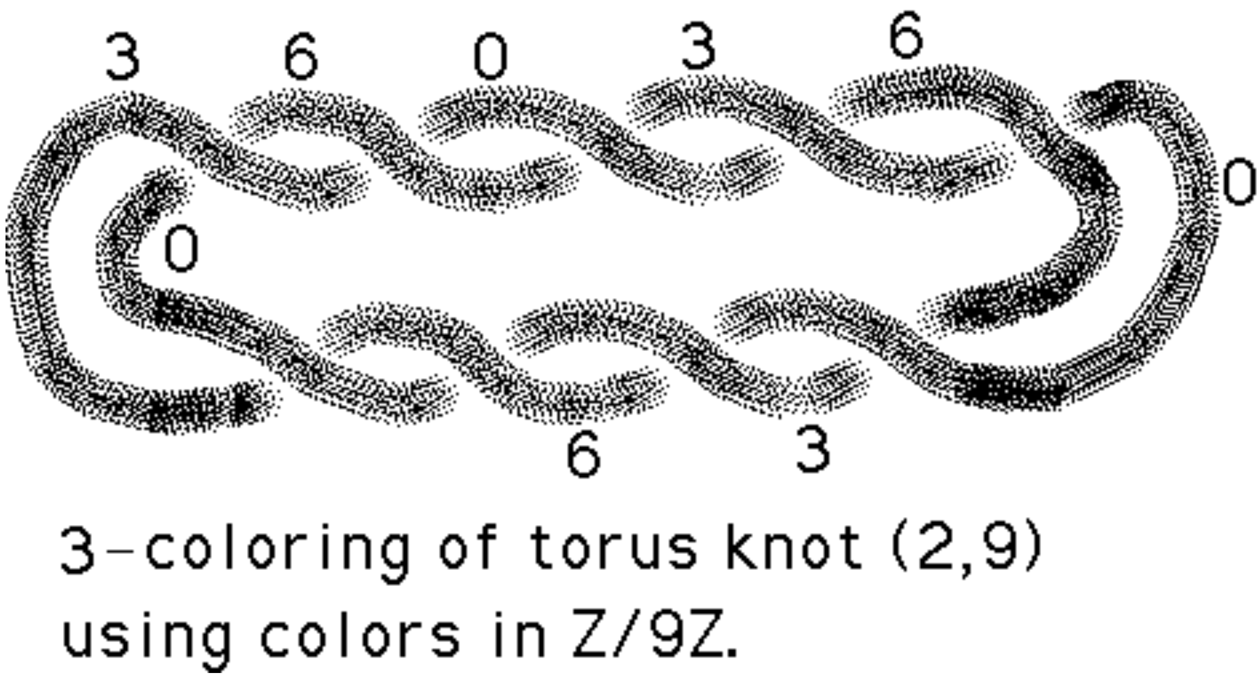

Figure 20 - Coloring an alternating knot in less than its number of vertices

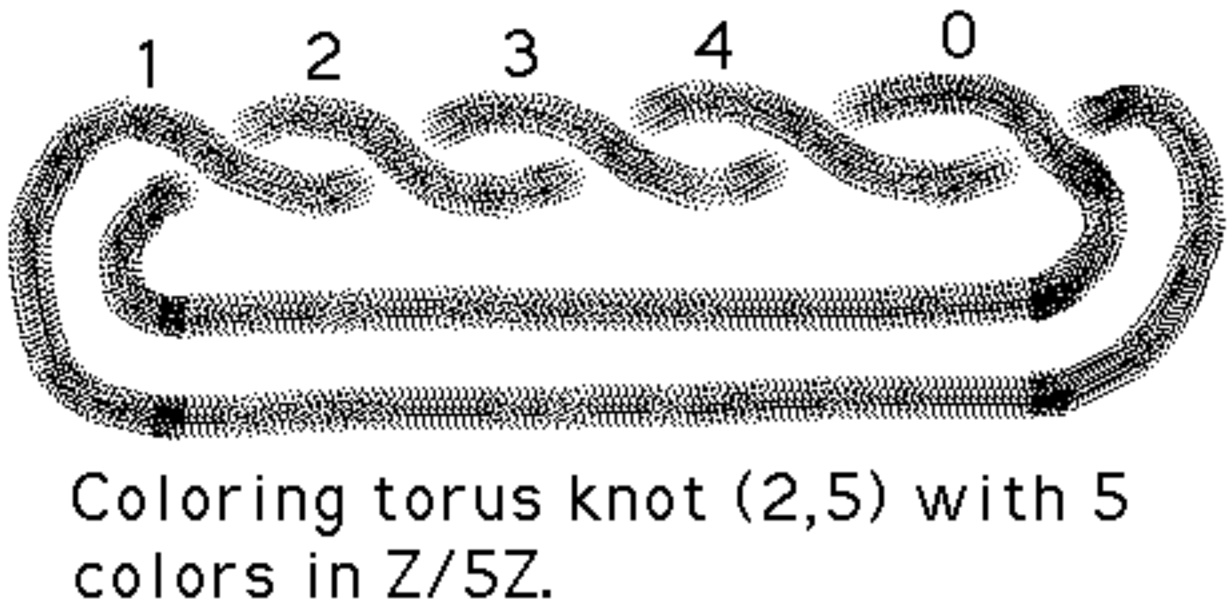

Figure 21 - Coloring a torus knot that has prime determinant

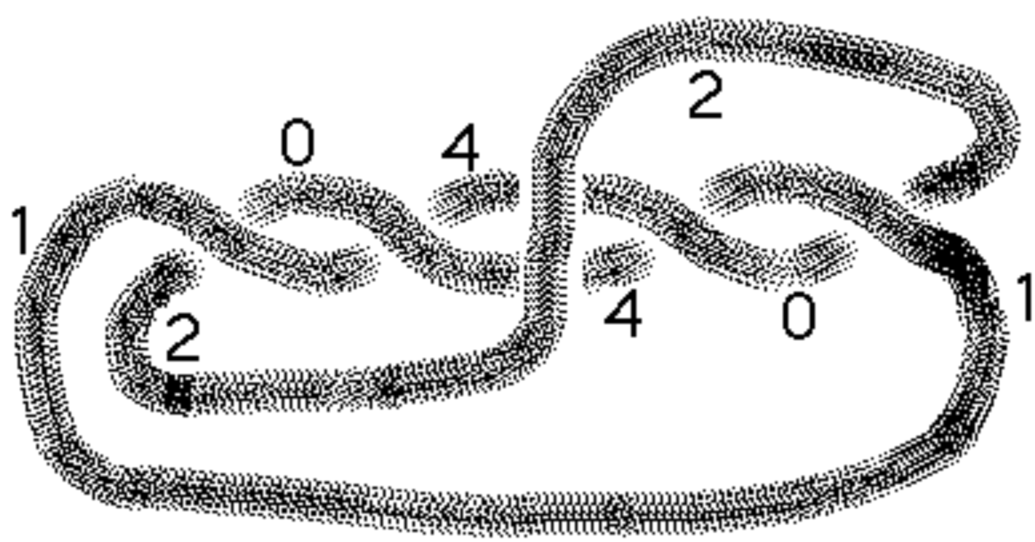

Figure 22 C(K(2,5)) < 5 : Irina Teneva's Example

Consider the example of K(2,5) in Figure 21. Here we see a coloring that uses all five colors with N(K(2,5)) =5.

Proposition 5.9. If p is a prime number then CM(K(2,p)) = p =N(K,p).

Proof. Recall that the coloring number for minimal diagrams is denoted by CM(K). We can check directly that K(2,p) can be colored with p colors just as in Figure 21 for the case p =5 and that for the diagram of type shown in Figure 21 this is the least number of colors possible. Figure 21 illustrates the essentially unique smallest diagram for the torus knot of type K(2,p). This completes the proof. / /

On the other hand C(K(2,p)) can be less than p even when p is prime! We are most grateful to Irina Teneva for the example shown in Figure 22 showing a 4-coloring of a diagram that is isotopic to K(2,5). Note that Teneva's diagram is not a minimal diagram for K(2,5).

Figure 23 gives another example of a knot K with N(K)=5 that we conjecture has C(K)=5.

VI. Graphs, Trees and a General Conjecture about CM(K)

The invariant C(K) is a new invariant of knots and links. It would be remarkable if one could find a general algorithm to compute C(K) for any knot or link. Most likely this is an NP complete problem. Nevertheless, some of the techniques of computation that we do know, [8], lead to further conjectures about C(K) and CM(K). For example, letting **Alex(t)** denote the classical Alexander polynomial and **Con(z)** the Alexander-Conway polynomial, we know that **Abs(Det(K)) = Abs(Alex(-1)) = Abs(Con(2i))** where $i^2 = -1$. It follows from properties of these polynomials that *for alternating knots, Abs(Det(K)) is equal to the number of spanning trees in the checkerboard graph CG(K) associated with K*. We defined and discussed this checkerboard graph in Section 2 above. The number of spanning trees grows exponentially in the number of crossings of the diagram K. Thus, for most alternatiing diagrams K the number of colors needed to color the diagram is much less than Abs(Det(K)).

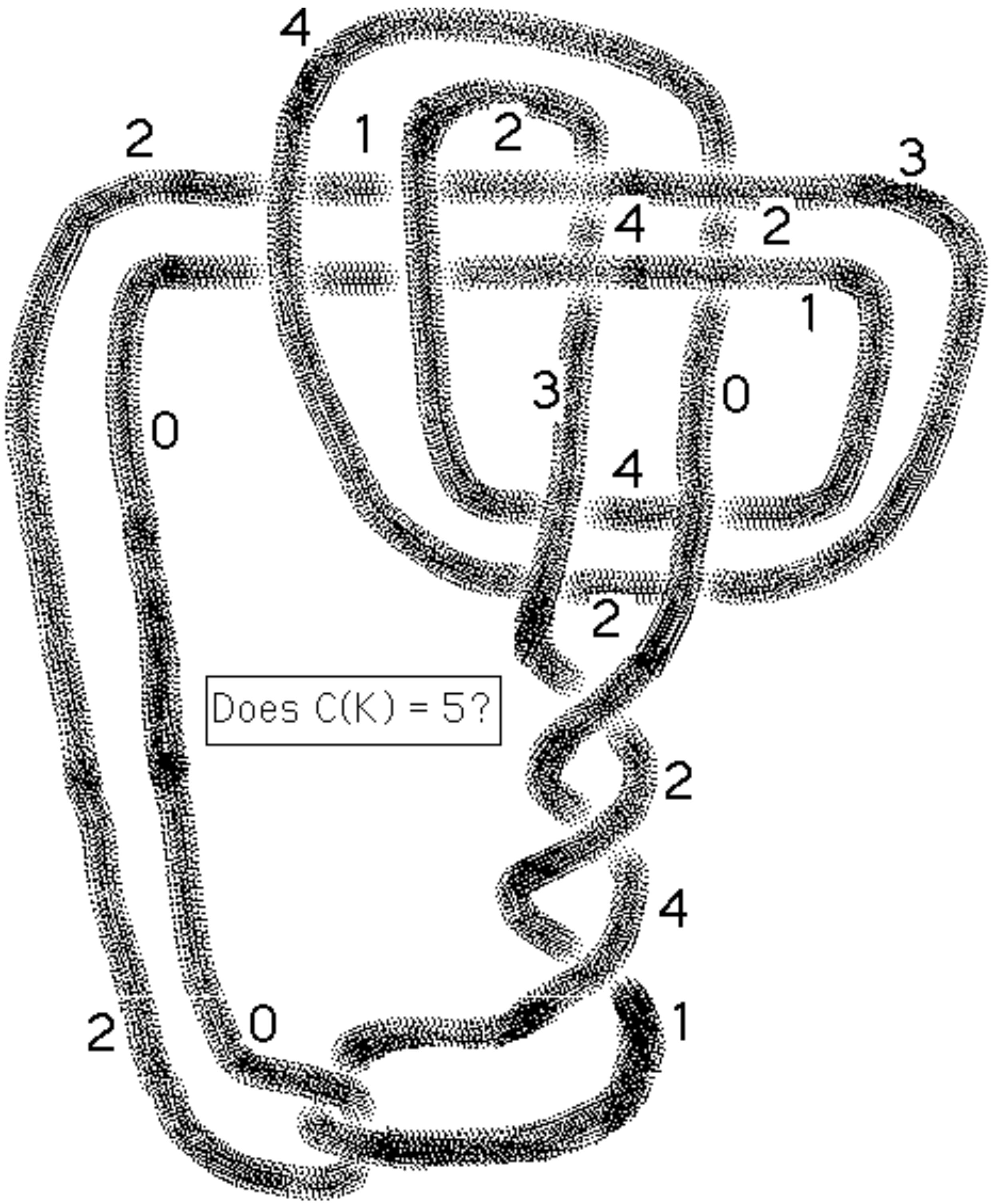

Figure 23 - A candidate for C(K)=5

Definition 6.1. Recall that a *bridge* of a graph is an edge whose deletion disconnects the graph. A connected link diagram K is said to be *reduced* if there is no bridge in its checkerboard graph CG(K). A link diagram that is not reduced can be simplified topologically by untwisting at the crossing corresponding to the bridge.

Alternation Conjecture 6.2. Let K be a connected reduced alternating knot diagram with v(K) crossings and prime p = Abs(Det(K)). We conjecture that v(K) is the coloring number of K for minimal diagrams of K: v(K) = CM(K).

In Figure 24 we show a typical instance of this conjecture: the knot 8_{17} from the Reidemeister tables. It has determinant 37 and we see that eight colors suffice for this reduced diagram.

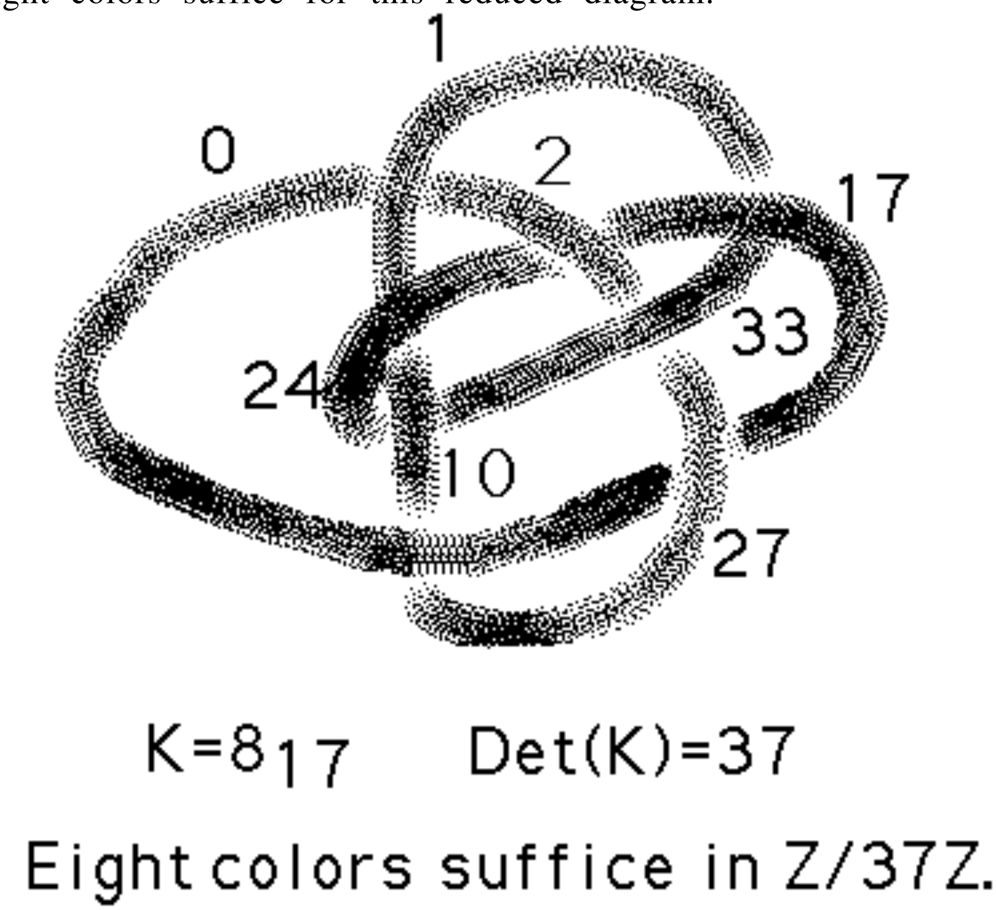

Figure 24 - Large determinant and low coloring number

This conjecture is false without the primality assumption for the determinant of K. Figure 25 shows a counterexample for the knot K = 7_7. We have Det(7_7) = 21= 3 x 7. A little work reveals that N(7_7) =7 and there is a Z/7Z coloring of the reduced alternating diagram for 7_7 shown in Figure 25 that uses only 6 colors.

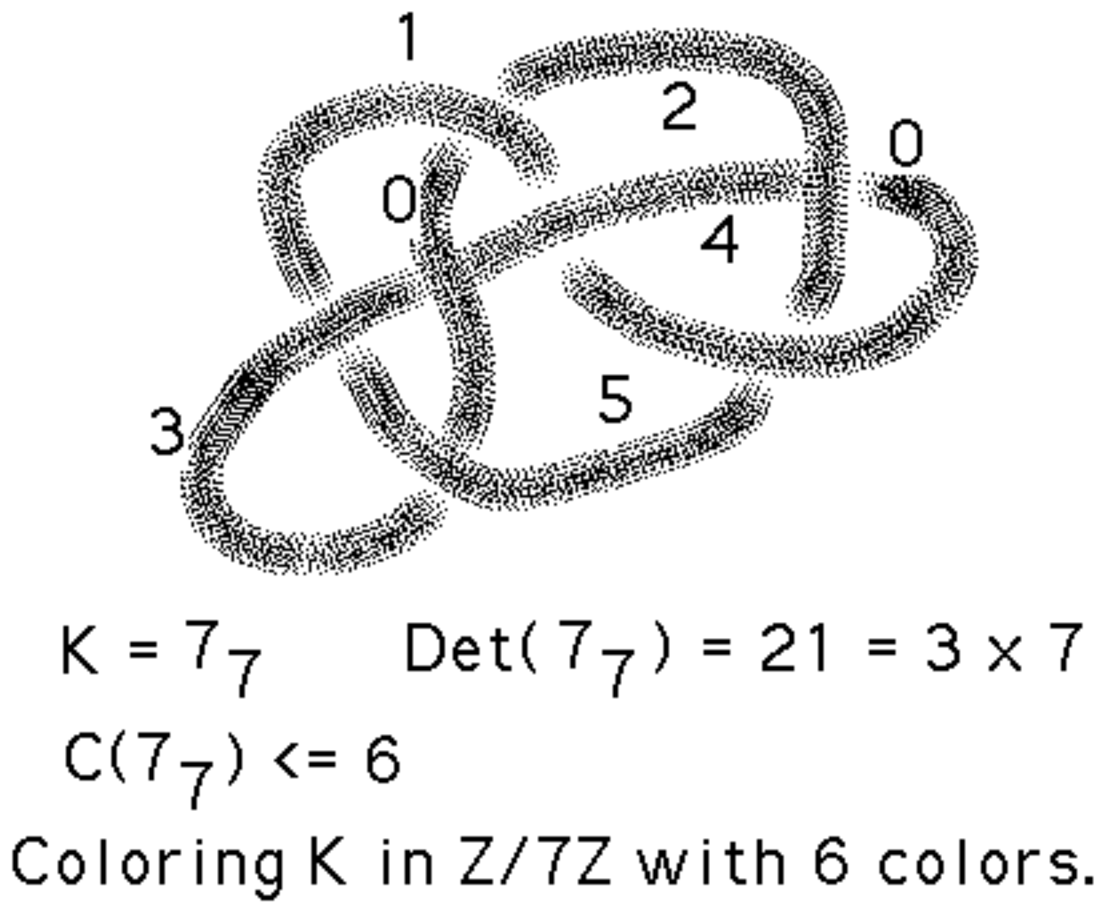

Figure 24 - Coloring an alternating knot (with non-prime determinant) with fewer colors than crossings

Remark. Since v(K) is computed directly from the span of the bracket polynomial of K, see [7], for K connected, reduced and alternating, this conjecture suggests that, in this infinity of cases, the coloring number CM(K) can be obtained from a simple algebraic calculation associated with the knot K. It may be that deeper properties of the bracket polynomial will be involved in a proof of this conjecture.

References

1. L. H. Kauffman. Gauss Codes Quantum Groups and Ribbon Hopf Algebras", Reviews in Mathematical Physics, Vol. 5, No. 4, (1993), 735-773.

2. K. Reidemeister. *Knotentheorie*, Chelsea, New York (1948), Julius Springer (1932).

3. D. Joyce. A classifying invariant of knots, the knot quandle. J. Pure Appl. Alg., Vol. 23 (1982), 37-65.

4. S. Winker. Quandles, knot invariants and the n-fold branched cover. Ph.D. Thesis, Univ. of Illinois at Chicago (1984).

5. R. Fox. A quick trip through knot theory. In *Topology of Three Manifolds*, M.K.Fort Jr. ed., Prentice Hall, Englewood Cliffs NJ (1962), 120-167.

6. F. Harary. *Graph Theory*. Addison Wesley, Reading MA (1969).

7. L. H. Kauffman. *On Knots*. Princeton University Press, Princeton NJ (1987).

8. L.H. Kauffman. *Knots and Physics*., 2nd edition, World Scientific, Singapore (1994).