PART II. KNOTS AND PHYSICS – MISCELLANY.

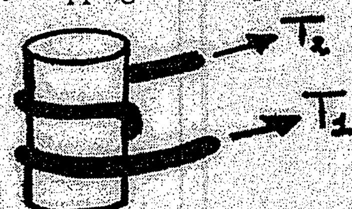
This half of the book is devoted to all manner of speculation and rambling over the subjects of knots, physics, mathematics and philosophy. If all goes well, then many tales shall unfold in these pages.

1^0. Theory of Hitches.

This section is based on the article [BAY].

We give a mathematical analysis of the properties of **hitches**. A **hitch** is a mode of wrapping a rope around a post so that, with the help of a little friction, the rope holds to the post. And your horse does not get away.

First consider simple wrapping of the rope around the post in coil-form:



Assume that there are an integral number of windings. Let tensions T_1 and T_2 be applied at the ends of the rope. Depending upon the magnitudes (and relative magnitudes) of these tensions, the rope may slip against the post.

We assume that there is some friction between rope and post. It is worth experimenting with this aspect. Take a bit of cord and a wooden or plastic rod. Wind the cord one or two times around the rod. Observe how easily it slips, and how much tension is transmitted from one end of the rope to the other. Now wind the cord ten or more times and observe how little slippage is obtained – practically no counter-tension is required to keep the rope from slipping.

In general, there will be no slippage in the T_2-direction so long as

$$T_2 \leq \kappa T_1$$

for an appropriate constant κ. This constant κ will depend on the number of windings. The more windings, the larger the constant κ.

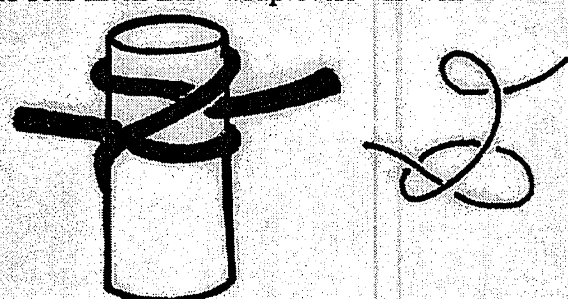
A good model is to take κ to be an exponential function of the angle (in radians) that the cord is wrapped around the rod, multiplied by the coefficient of friction between cord and rod. For simplicity, take the coefficient of friction to be unity so that

$$\kappa = e^{\theta/2\pi}$$

where θ is the total angle of rope-turn about the rod.

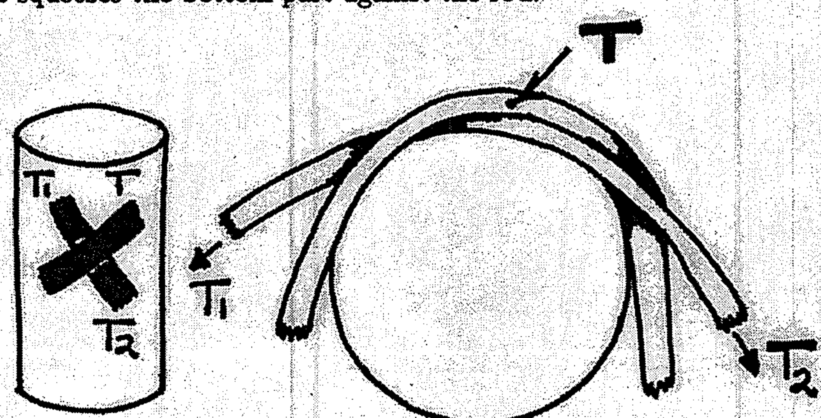
Thus, for a single revolution we need $T_2 \leq eT_1$ and for an integral number n of revolutions we need $T_2 \leq e^n T_1$ to avoid slippage.

A real hitch has "wrap-overs" as well as windings:



Clove Hitch

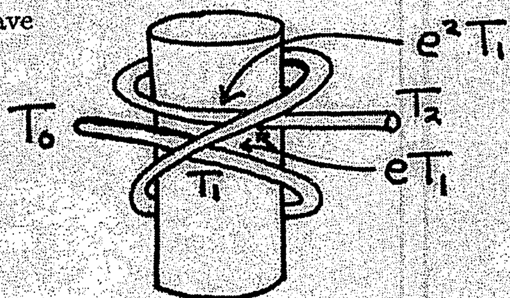
Here, for example, is the pattern of the clove hitch. In a wrap-over, under tension, the top part squeezes the bottom part against the rod.



Hold Fast
$T_2 \leq T_1 + uT$

This squeezing produces extra protection against slippage. If, at such a wrap-over point, the tension in the overcrossing cord is T, then the undercrossing cord will hold-fast so long as $T_2 \leq T_1 + uT$ where u is a certain constant involving the friction of rope-to-rope, and T_2 and T_1 are the tensions on the ends of the undercrossing rope at its ends.

With these points in mind, we can write down a series of inequalities related to the crossings and loopings of a hitch. For example, in the case of the clove hitch we have



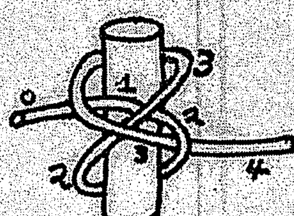
T_1	$\leq$	T_0	$+$	ueT_1
T_2	$\leq$	e^2T_1	$+$	ueT_1

that the equations necessary to avoid slippage are:

$$T_1 \leq T_0 + ueT_1$$
$$T_2 \leq e^2T_1 + ueT_1.$$

Since **the first inequality holds whenever** $ue > 1$ or $u > 1/e$, we see that the **clove hitch will not slip** no matter how much tension occurs at T_2 just so long as the rope is sufficiently rough to allow $u > 1/e$.

This explains the efficacy of this hitch. Other hitches can be analyzed in a similar fashion.



T_1	$\leq$	T_0	$+$	ueT_3
T_2	$\leq$	T_1	$+$	uT_3
T_3	$\leq$	eT_2	$+$	ueT_3
T_4	$\leq$	eT_3	$+$	uT_2

In this example, if we can solve

$$T_1 \leq ueT_3$$
$$T_2 \leq T_1 + uT_3$$
$$T_3 \leq eT_2 + ueT_3$$

then the hitch will hold.

T_1	T_2	T_3
1	0	$-ue$
-1	1	$-u$
0	$-e$	$1-ue$

In matrix form, we have

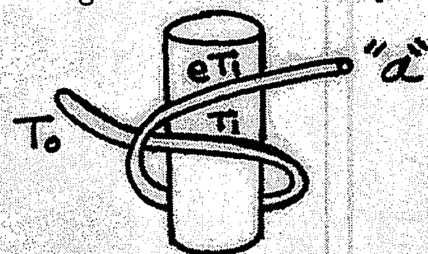
$$\begin{bmatrix} 1 & 0 & -ue \\ -1 & 1 & -u \\ 0 & -e & 1-ue \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The determinant of this matrix is

$$1 - ue(2-e).$$

Thus the critical case is $u = 1/e(2+e)$. For $u > 1/e(2+e)$, the hitch will hold.

Remark. Let's go back to the even simpler "hitch":



$$\boxed{T_1 \le T_0 + ueT_1}$$

Our abstract analysis would suggest that this will hold if $ue > 1$. However, there is no stability here. A pull at "a" will cause the loop to rotate and then the "u-factor" disappears, and slippage happens. A pull on the clove hitch actually tightens the joint.

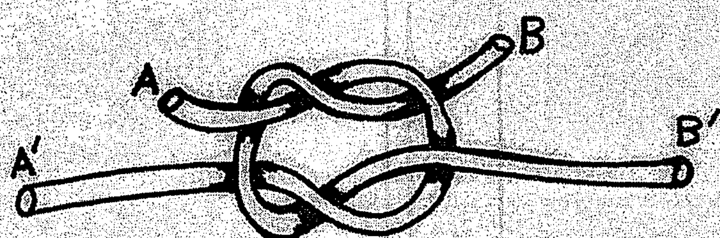
This shows that in analyzing a hitch, we are actually taking into account some properties of an already-determined-stable mechanical mechanism that happens to be made of rope. [See also Sci. Amer., Amateur Sci., Aug. 1983.]

There is obviously much to be done in understanding the frictional properties of knots and links. These properties go far beyond the hitch to the ways that ropes interplay with one another. The simplest and most fascinating examples are the **square knot** and the **granny knot**. The square knot pulls in under tension, each loop constricting itself and the other - providing good grip:

Square Knot

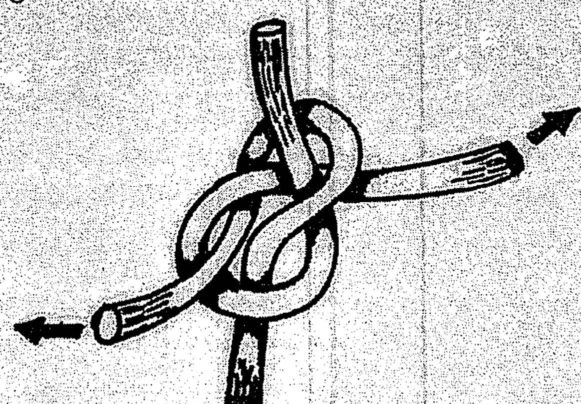
Construct this knot and watch how it grips itself.

The **granny** should probably be called the **devil**, it just won't hold under tension:



Granny Knot

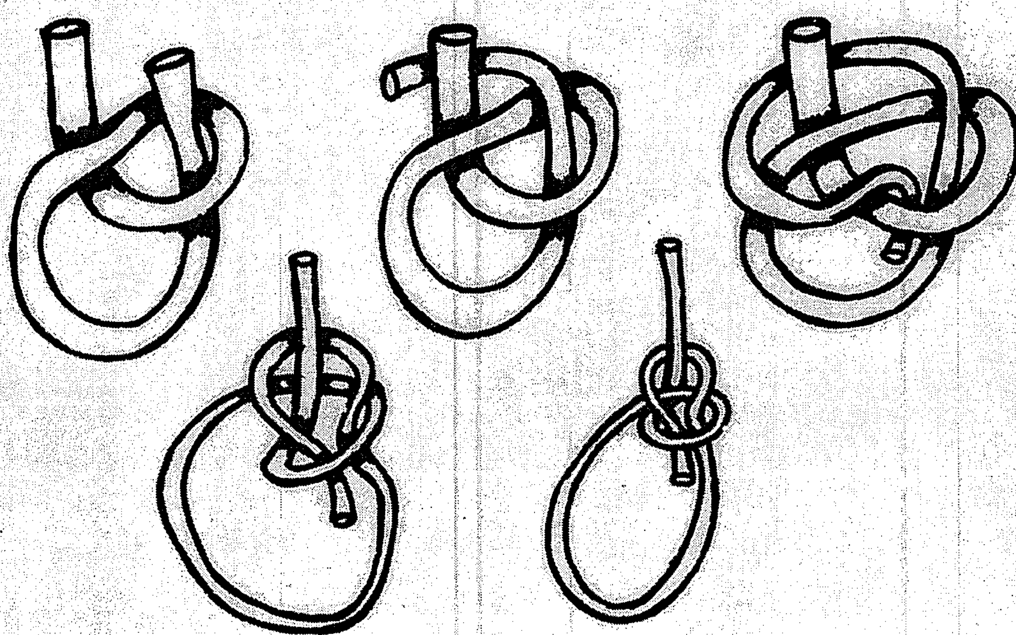
Try it! Ends A and B, are twisted perpendicular to ends A' and B' and the rope will feed through this tangle if you supply a sufficient amount of tension.



The fact of the matter is that splices and hitches are fantastic sorts of me-
hanical devices. Unlike the classical machine that is composed of well-defined
parts that interact according to well-understood rules (gears and cogs), the slid-
ng interaction of two ropes under tension is extraordinary and interactive, with
ension, topology and the system providing the form that finally results.

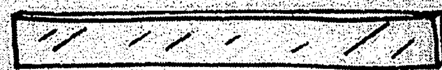
Clearly, here is an arena where a deeper understanding of **topology instan-
iated in mechanism** is to be desired. But here we are indeed upon untrodden
round. The topology that we know has been obtained at the price of initial ab-
traction from these physical grounds. Nevertheless, it is the intent of these notes
o explore this connection.

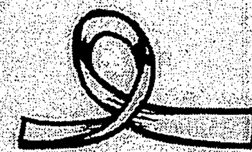
We do well to ponder the knot as whole system, decomposed into parts only
ia projection, or by an observer attempting to ferret out properties of the inter-
cting rope. Here is a potent metaphor for observation, reminding us that the
lecompositions into parts are in every way our own doing - through such explica-
ions we come to understand the whole.



Tying the Bowline

2^0. The Rubber Band and Twisted Tube.





(one ordinary supercoil)

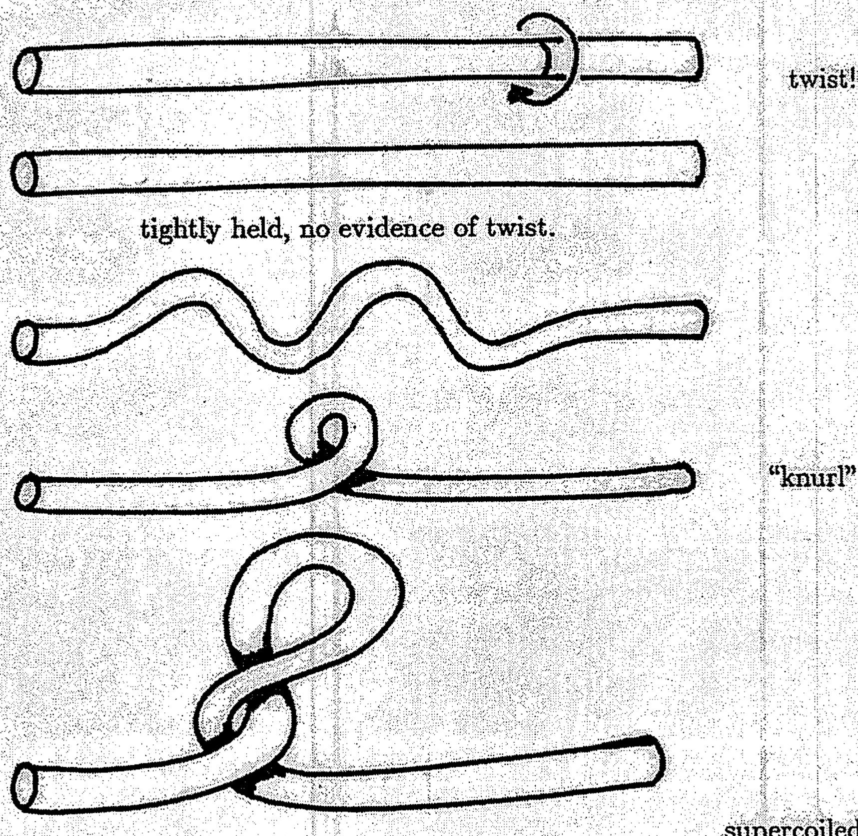
"all rolled up"

"the first "knurls" "

all "knurled up"

If you keep twisting the band it will "knurl", a term for the way the band gets in its own way after the stage of being all rolled up. A knurl is a tight super-coil and if you relax the tension ( ) on the ends of the band, knurls will "pop" in as you feel the twisted band relax into some potential energy wells. It then takes a correspondingly long pull and more energy to remove the knurls.

The corresponding phenomena on a twisted – spring-loaded tube are even easier to see:



In all these cases, it is best to do the experiment. The interesting phenomenon is that once the first knurl has formed, it takes a lot of force to undo it due to the interaction of the tube against itself.

The energetics of the situation demand much experimentation. I am indebted to Mr. Jack Armel for the following experiment.

The Armel Effect. Take a flat rubber band. Crosscut it to form a rubber strip. Paint one side black. Tape one end to the edge of a table. Take the other end in your hand and twist gently counterclockwise until the knurls hide one color. Repeat the experiment, but turn the band clockwise. Repeat the entire experiment using a strip of paper.

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KNOTS AND PHYSICS

Louis H. Kauffman
Department of Mathematics,
Statistics and Computer Science
University of Illinois at Chicago

World Scientific
Singapore • New Jersey • London • Hong Kong