

The Conway Polynomial in R^3 and in Thickened Surfaces: A New Determinant Formulation

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INTRODUCTION

The purpose of this paper is to give a self-contained account of a formulation of the classical Alexander–Conway polynomial as a new determinant and to show how this version of the polynomial naturally extends to a two-variable polynomial invariant of links in thickened oriented surfaces of arbitrary genus.

The extra variable in this extended invariant carries non-trivial information. This arises from the circumstance that simple closed curves can intersect in an odd number of points on a surface of arbitrary genus. Such an extra variable also arises in [CRR], where the Homfly polynomial is generalized. It would be interesting to explore more closely the relationships between the results of [CRR] and ours.

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The determinant formulation of the Conway polynomial that we use derives from the structure of the free-fermion model in statistical mechanics [KS, LW]. Here we give, in Section I, a brief exposition of this formalism and we show in Section II how a partition function of specified type on a four-valent graph can be evaluated as a determinant. In Section III, we review the theoretical background for constructing link invariants via partition functions and solutions to the Yang–Baxter equation. This background is applied to a particular Yang–Baxter solution (see also [J, K1, K2]) to produce a two-variable invariant of links in thickened oriented surfaces and the correct normalization for the planar case (where only one variable remains), yielding the classical Alexander–Conway polynomial.

For the sake of simplicity, we have avoided discussion of the classical multi-variable Alexander polynomial and its generalizations to thickened surfaces. Our methods apply to this case *mutatis mutandis*.

To repeat, the novelty of this approach to the Alexander–Conway polynomial resides in a new formulation as a determinant stemming from a Yang–Baxter model, and the use of this model to obtain an extra variable in the case of higher genus. The fact that this extra variable can be used in a topological application gives an affirmative answer to a question of Kulish [Ku].

I. DIMERS AND DETERMINANTS

The *close-packed dimer problem* of statistical mechanics can be stated as follows. One considers a set of sites and a set of bonds connecting certain pairs of sites. Each bond b can absorb a “dimer” (which represents a diatomic molecule) with corresponding energy E_b . It is required that each site is occupied exactly once by one of the atoms of a dimer. A state σ is an arrangement of dimers which meets this requirement, and its energy $E(\sigma)$ is $\sum E_b$ where the sum is taken over all bounds b which absorb a dimer. Then the partition function of this system is $Z = \sum_{\text{states } \sigma} \exp(-E(\sigma)/kT)$, where T is the temperature and k is Boltzmann’s constant.

We may reformulate this problem in graph theoretical terms as follows. We are given a simple undirected graph $G = (V, E)$ (V is the set of sites and E is the set of bonds). Each edge b has received a weight $w(b)$, and we want to evaluate

$$Z = \sum_M \prod_{b \in M} w(b),$$

where the sum is over all perfect matchings M of G . The correspondence with the previous formulation is clear: for an edge b , we define

$w(b) = \exp(-E_b/kT)$, and every state can be considered as a perfect matching.

In some cases this problem can be solved by the classical Pfaffian method [Ka, Fi]. Let B be an antisymmetric $(2n \times 2n)$ matrix. For each partition $P = \{\{i_1, j_1\} \cdots \{i_n, j_n\}\}$ of $\{1, \dots, 2n\}$ into pairs let

$$B_P = \text{sign} \begin{pmatrix} 1 & 2 & \cdots & 2n-1 & 2n \\ i_1 & j_1 & \cdots & i_n & j_n \end{pmatrix} B_{i_1 j_1} \cdots B_{i_n j_n}.$$

Clearly, B_P does not depend on the order in which we have listed the classes of P . Similarly, by the antisymmetry of B , it does not depend on the order which we have chosen for two elements of a class. Then, as in [LP] we define the Pfaffian of B as $PfB = \sum_P B_P$ (where the sum is taken over all partitions P of $(1, \dots, 2n)$ into pairs).

The Pfaffian method relies on two facts. First, for a large class of graphs (which includes the planar graphs) one can find an orientation such that the corresponding $(0, +1, -1)$ adjacency matrix B has the following property: all terms B_P in the expansion of PfB have the same sign. In this case the partition function Z can easily be written as a Pfaffian. Second, a well known result of linear algebra states that $\det B = (PfB)^2$ for every antisymmetric matrix B . Thus, the problem is reduced to the computation of a determinant.

A further simplification occurs when the graph G is *bipartite* (with n vertices in each color class). Indeed in this case PfB can be directly expressed as an $n \times n$ determinant.

More recently, the close-packed dimer problem, as well as many other problems, have been studied using the method of anticommuting variables [Sam]. A *set of anticommuting variables* $\{v_1, \dots, v_N\}$ is endowed with a product operation which satisfies the rules: $v_i v_j + v_j v_i = 0$ ($i, j = 1, \dots, N$) (and hence $v_i^2 = 0$ for all $i = 1, \dots, N$). In other words, we consider the (real or complex) Grassmann algebra generated by $\{v_1, \dots, v_N\}$. The *Berezin integral* $\int dv_1 \cdots dv_N$ is the unique linear form on this algebra which takes the value one on $v_1 \cdots v_N$ and the value zero on any monomial of smaller degree in the v_i .

Let us illustrate the use of the Berezin integral on a few examples [Sam]. Let A be an N by N matrix and let $\{v_1, \dots, v_N, v'_1, \dots, v'_N\}$ be a set of anticommuting variables. Then

$$\int dv_1 dv'_1 \cdots dv_N dv'_N \prod (1 + A_{ij} v_i v'_j) = \det A,$$

where the product \prod is taken over all pairs $(i, j) \in \{1, \dots, N\} \times \{1, \dots, N\}$ (the order of terms is irrelevant since even products mutually commute).

Indeed $\prod (1 + A_{ij}v_i v'_j) = \sum_{S \in \{1, \dots, N\}^2} \prod_{(i,j) \in S} A_{ij}v_i v'_j$, and the products which will contribute to the Berezin integral are of the form

$$\prod_{i=1, \dots, N} A_{i\sigma(i)} v_i v'_{\sigma(i)} = \left(\text{sign}(\sigma) \prod_{i=1, \dots, N} A_{i\sigma(i)} \right) v_1 v'_1 \cdots v_N v'_N$$

for some $\sigma \in S_N$.

Now let B be an antisymmetric $2n \times 2n$ matrix. Introduce a set of anticommuting variables $\{x_1, \dots, x_{2n}\}$. Then

$$\int dx_1 \cdots dx_{2n} \prod_{i < j} (1 + B_{ij}x_i x_j) = Pf B.$$

The proof is quite similar to the preceding one

Now, following [Sam], let us sketch a proof of the identity $(Pf B)^2 = \det B$. We write

$$\begin{aligned} (Pf B)^2 &= \left(\int dx_1 \cdots dx_{2n} \prod_{k < l} (1 + B_{kl}x_k x_l) \right) \\ &\quad \times \left(\int dy_1 \cdots dy_{2n} \prod_{k < l} (1 + B_{kl}y_k y_l) \right) \\ &= \int dx_1 \cdots dx_{2n} dy_1 \cdots dy_{2n} \prod_{k < l} (1 + B_{kl}x_k x_l)(1 + B_{kl}y_k y_l). \end{aligned}$$

Let $v_k = (1/\sqrt{2})(x_k - iy_k)$, $v'_k = (1/\sqrt{2})(x_k + iy_k)$ for $k = 1, \dots, 2n$ (with $i^2 = -1$). Then $v_k v'_l = \frac{1}{2}[x_k x_l + y_k y_l + i(x_k y_l - y_k x_l)]$. It easily follows that $v_k v'_l - v_l v'_k = x_k x_l + y_k y_l$ and $v_k v'_l v_l v'_k = -x_k x_l y_k y_l$. Hence

$$\begin{aligned} (1 + B_{kl}v_k v'_l)(1 + B_{lk}v_l v'_k) &= 1 + B_{kl}(x_k x_l + y_k y_l) + (B_{kl})^2 x_k x_l y_k y_l \\ &= (1 + B_{kl}x_k x_l)(1 + B_{kl}y_k y_l). \end{aligned}$$

Moreover $v_k v'_k = ix_k y_k$ and hence

$$v_1 v'_1 \cdots v_{2n} v'_{2n} = (-1)^n x_1 y_1 \cdots x_{2n} y_{2n} = x_1 \cdots x_{2n} y_1 \cdots y_{2n}.$$

It follows that

$$(Pf B)^2 = \int dv_1 dv'_1 \cdots dv_{2n} dv'_{2n} \prod_{k < l} (1 + B_{kl}v_k v'_l)(1 + B_{lk}v_l v'_k) = \det B.$$

Remark. In mathematical physics the kind of integrals over anticommuting variables discussed above arise quite naturally in fermionic field

theories. Corresponding, in [Sam] terms like $\prod (1 + A_{ij}v_i v_j)$ are written in the form $\exp(\sum A_{ij}v_i v_j)$. The reader may enjoy a comparison of our formalism with that of the physics literature as in [Sam].

II. DETERMINANTAL INVARIANTS OF FOUR REGULAR GRAPHS

An *eulerian orientation* of a graph is an orientation such that for every vertex v , the number of edges with initial end v equals the number of edges with terminal end v . Let $G = (V, E)$ be a four-regular graph (loops and multiple edges are allowed) provided with an eulerian orientation. We call a *labelling of G* any mapping f from E to $\{1, 2\}$ such that $f^{-1}(1)$ and $f^{-1}(2)$ define subgraphs for which the orientation inherited from G is eulerian. The set of labellings of G will be denoted by $\mathcal{L}(G)$ and, for an edge e of G , the set of labellings f such that $f(e) = i$ will be denoted by $\mathcal{L}(G, e, i)$.

It will be convenient to consider that each edge e of G is subdivided into two half-edges e^+, e^- , where e^+ is incident to the initial end of e and e^- to the terminal end. Given a vertex v , a *transition at v* is an ordered pair (e_1^-, e_2^+) , where e_1^- has terminal end v and e_2^+ has initial end v . Every half-edge will be arbitrarily assigned a *type*, left or right, in such a way that at every vertex v the two half-edges with terminal end v have different types and similarly for the two half-edges with initial end v . The type of a transition (e_1^-, e_2^+) is defined as $(\text{type}(e_1^-), \text{type}(e_2^+))$. Transitions of types (left, left) or (right, right) will be called *tangent*, and transitions of the other types will be called *crossing*.

Remark. This terminology has a natural interpretation when G is embedded on an orientable surface in such a way that for every vertex v , the two edges with terminal end v are consecutive around v (see Fig. 1). Conversely, an assignment of types to the half-edges determines, with the convention of Fig. 1, a rotation system of G (see, for instance, [BW]) which defines, provided G is connected, a two-cell embedding of G on some closed orientable surface.

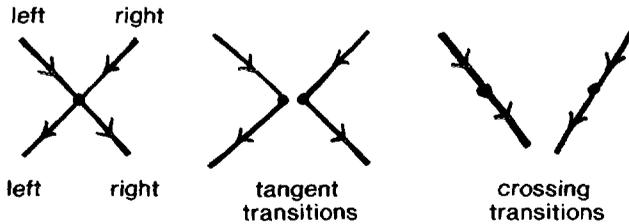


FIGURE 1

For a vertex v and $p, q \in \{l, r\}$ (where l stands for left and r for right), we denote by $\theta_{p,q}(v)$ the transition at v of type (p, q) . Let R be the polynomial ring

$$Z[\theta_{p,q}(v) \mid p, q \in \{l, r\}, v \in V].$$

For every labelling $f \in L(G)$, let $H(v, f)$ be the set of half-edges incident to v which belong to an edge e with $f(e) = 1$. Clearly $|H(v, f)|$ is even and if $|H(v, f)| = 2$ the two elements of $H(v, f)$ (taken in the appropriate order) form a transition at v . We denote by $\langle v \mid f \rangle$ the element of R which equals one if $H(v, f)$ is empty, $\theta_{l,l}(v) \theta_{r,r}(v) - \theta_{l,r}(v) \theta_{r,l}(v)$ if $|H(v, f)| = 4$, and $\theta_{p,q}(v)$ if $H(v, f)$ has exactly two elements which form the transition $\theta_{p,q}(v)$.

Let us split each vertex of G into two vertices of degree 2, each one being incident to the two half-edges of a tangent transition. In analogy with the case of planar graphs, we call *Seifert circuits* the resulting circuits and we denote by $s(G)$ the number of these circuits. Similarly $s(G, f, i)$ ($i = 1, 2$) denotes the number of Seifert circuits of the subgraph of G defined by $f^{-1}(i)$ (now, of course, only vertices of degree 4 are split).

We now consider the partition function

$$Z(G) = \sum_{f \in \mathcal{L}(G)} (-1)^{s(G, f, 2)} \prod_{v \in V} \langle v \mid f \rangle.$$

Let $n = |V|$ and $V = \{v_1, \dots, v_n\}$. We denote the left (resp. right) half-edge with initial end v_i by $h_{i,l}^+$ (resp. $h_{i,r}^+$). The half-edges with terminal end v_i are similarly denoted by $h_{i,l}^-$ (left-type) and $h_{i,r}^-$ (right-type). Thus $\theta_{p,q}(v_i) = (h_{i,p}^-, h_{i,q}^+)$ for every $p, q \in \{l, r\}$. In the sequel we write $\varepsilon_{p,q} = 2\delta_{p,q} - 1$.

Let A be the $(2n \times 2n)$ R -valued matrix with rows and columns indexed by $\{1, \dots, n\} \times \{l, r\}$, with $A = T + P$ and, for all $i, j \in \{1, \dots, n\}$ and $p, q \in \{l, r\}$:

$$\begin{aligned} T_{(i,p), (j,q)} &= \delta_{i,j} \varepsilon_{p,q} \theta_{p,q}(v_i) \\ P_{(i,p), (j,q)} &= -1 \quad \text{if } h_{i,p}^-, h_{j,q}^+ \text{ are the two halves of the same edge} \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

PROPOSITION 1. $Z(G) = \det A$.

Proof. Let us consider the half-edges as anticommuting variables (commuting with the elements of R —i.e., as formal variables the half-edges are independent of the transitions). As seen in Section I (the extension to R -valued matrices is obvious), $\det A = \int dhQ$, where dh denotes

$$dh_{1,l}^- dh_{1,l}^+ dh_{1,r}^- dh_{1,r}^+ \cdots dh_{n,l}^- dh_{n,l}^+ dh_{n,r}^- dh_{n,r}^+$$

and

$$Q = \prod_{(i,p),(j,q)} (1 + A_{(i,p),(j,q)} h_{i,p}^- h_{j,q}^+).$$

Noting that

$$1 + A_{(i,p),(j,q)} h_{i,p}^- h_{j,q}^+ = (1 + T_{(i,p),(j,q)} h_{i,p}^- h_{j,q}^+) (1 + P_{(i,p),(j,q)} h_{i,p}^- h_{j,q}^+),$$

we may write $Q = Q_V Q_E$, with $Q_V = \prod_i Q_i$,

$$Q_i = \prod_{p,q \in \{l,r\}} (1 + \varepsilon_{p,q} \theta_{p,q}(v_i) h_{i,p}^- h_{i,q}^+)$$

and

$$Q_E = \prod_{e \in E} (1 - e^- e^+) = \prod_{e \in E} (1 + e^+ e^-) = \sum_{F \subseteq E} \prod_{e \in F} e^+ e^-.$$

Then setting $Q'_F = (\prod_i Q_i) (\prod_{e \in F} e^+ e^-)$, we obtain $Q = \sum_{F \subseteq E} Q'_F$ and $\det A = \sum_{F \subseteq E} \int dh Q'_F$.

Expanding Q_i leads to

$$Q_i = 1 + \sum_{p,q \in \{l,r\}} \varepsilon_{p,q} \theta_{p,q}(v_i) h_{i,p}^- h_{i,q}^+ + (\theta_{l,l}(v_i) \theta_{r,r}(v_i) - \theta_{l,r}(v_i) \theta_{r,l}(v_i)) (h_{i,l}^- h_{i,l}^+ h_{i,r}^- h_{i,r}^+).$$

It is now easy to see that $\int dh Q'_F$ is non-zero if and only if there exists a labelling f in $\mathcal{L}(G)$ such that $F = f^{-1}(2)$. Moreover, in that case

$$\int dh Q'_F = \prod_i \langle v_i | f \rangle \int dh \left(\prod_i K(v_i, f) \right) \left(\prod_{e \in F} e^+ e^- \right),$$

where

$$\begin{aligned} K(v_i, f) &= 1 && \text{if } H(v_i, f) = \emptyset, \\ K(v_i, f) &= h_{i,l}^- h_{i,l}^+ h_{i,r}^- h_{i,r}^+ && \text{if } |H(v_i, f)| = 4, \\ K(v_i, f) &= \varepsilon_{p,q} h_{i,p}^- h_{i,q}^+ && \text{if } H(v_i, f) = \{h_{i,p}^-, h_{i,q}^+\}. \end{aligned}$$

It remains to show that $\int dh (\prod_i K(v_i, f)) (\prod_{f(e)=2} e^+ e^-) = (-1)^{s(G, f, 2)}$.

Let C_1, \dots, C_s be the Seifert circuits of the subgraph of G defined by $f^{-1}(2)$ and let V_i be the set of vertices of degree i of this subgraph. Given a Seifert circuit C_k , let e_1, \dots, e_m be its edges listed in the order of a walk around the circuit. Then $e_1^+ e_1^- \dots e_m^+ e_m^- = -(e_1^- e_2^+) \dots (e_m^- e_1^+)$, and each

factor of this second product is of the form $h_{i,p}^- h_{i,q}^+$ for some $i \in \{1, \dots, n\}$ and $p, q \in \{l, r\}$. This allows us to write

$$\prod_{f(e)=2} e^+ e^- = (-1)^{s(G, f, 2)} \prod_i K'(v_i, f), \quad \text{where } K'(v_i, f) = 1 \text{ if } v_i \in V_0,$$

$$K'(v_i, f) = h_{i,l}^- h_{i,l}^+ h_{i,r}^- h_{i,r}^+ = h_{i,r}^- h_{i,r}^+ h_{i,l}^- h_{i,l}^+ \quad \text{if } v_i \in V_4,$$

$$K'(v_i, f) = h_{i,p}^- h_{i,q}^+ \quad \text{if } v_i \in V_2 \text{ and } h_{i,p}^-, h_{i,q}^+ \text{ are the two half-edges of the subgraph defined by } f^{-1}(2) \text{ incident to } v_i.$$

Then it is easy to check that $K(v_i, f) K'(v_i, f) = h_{i,l}^- h_{i,l}^+ h_{i,r}^- h_{i,r}^+$ if $v_i \in V_0$ (and then $|H(v_i, f)| = 4$) or $v_i \in V_4$ (and then $|H(v_i, f)| = 0$). If $v_i \in V_2$ we have $K(v_i, f) K'(v_i, f) = \varepsilon_{p,q} h_{i,p}^- h_{i,q}^+ h_{i,\bar{p}}^- h_{i,\bar{q}}^+$ for some $p, q, \bar{p}, \bar{q} \in \{l, r\}$ with $p \neq \bar{p}, q \neq \bar{q}$. It is easy to check that $K(v_i, f) K'(v_i, f) = h_{i,l}^- h_{i,l}^+ h_{i,r}^- h_{i,r}^+$ also in this case. It follows that

$$\left(\prod_i K(v_i, f) \right) \left(\prod_{f(e)=2} e^+ e^- \right) = (-1)^{s(G, f, 2)} \prod_i h_{i,l}^- h_{i,l}^+ h_{i,r}^- h_{i,r}^+.$$

This completes the proof. \blacksquare

Let us now define, for a given edge e of G and $i \in \{1, 2\}$,

$$Z(G, e, i) = \sum_{f \in \mathcal{L}(G, e, i)} (-1)^{s(G, f, 2)} \prod_{v \in V} \langle v | f \rangle.$$

The following result is clear from the preceding proof.

PROPOSITION 2. *Let A'_e be the matrix obtained from A by replacing the -1 entry of P corresponding to e by 0 , and let $A[e]$ be the minor of A corresponding to this entry. Then $Z(G, e, 1) = \det A'_e$ and $Z(G, e, 2) = \varepsilon(e) \det A[e]$, where $\varepsilon(e)$ is the sign of $\det A[e]$ in a minor expansion of $\det A$ along a row or column containing the entry corresponding to e .*

Remarks. (1) The above proof could have been obtained in the “dimer” approach by constructing a graph with the half-edges as vertices, and edges representing the transitions and edges of G . The present approach avoids this construction and allows an efficient study of the signs involved.

(2) One can establish a bijective correspondence between the labellings of G and its eulerian orientations (consider the label 2 as an indication to reverse an edge). In this respect the invariant $Z(G)$ corresponds to a modified ice-type model where an extra sign factor is introduced.

III. APPLICATION TO LINKS

1. *An Invariant of Oriented Links in Thickened Surfaces*

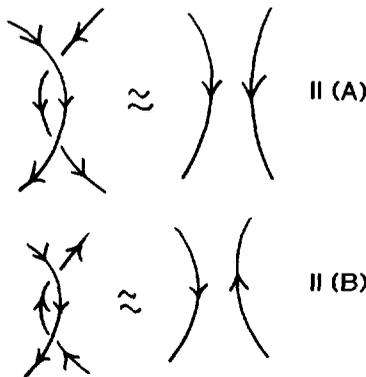
Let S be an oriented surface (possibly with boundary) and $I = [0, 1]$. An oriented link in $S \times I$ can be represented by a generic projection on S , together with an over-under information for each crossing. Such a diagram can be defined as a four-regular graph G embedded on S (we do not require that the embedding be cellular; that is, the connected components of $S - G$ need not be disks), endowed with an eulerian orientation such that the two incoming edges at any vertex are consecutive around this vertex, and where vertices are signed according to the usual convention of Fig. 2.

We also use the "left-right" convention of Fig. 1 (which is meant to represent a small disk in S around a vertex of G , endowed with the local orientation of S).

Two diagrams represent ambient isotopic links in $S \times I$ if and only if they can be transformed one into the other by a finite sequence of Reidemeister moves I, II, III indicated in Fig. 3. This result is a straightforward generalization of the classical result (see, e.g., [K2]) for links in Euclidean space.

In this figure the Reidemeister moves are indicated without orientation. These moves are performed locally on a given diagram, leaving fixed the part of the diagram not indicated by the move. The local part of the diagram that is addressed by a given move must be topologically equivalent to one of the types shown in Fig. 3.

With regard to orientation, there are four types of I-move and four types of II-move. We classify the latter in two basic types II(A) and II(B) as shown below (the other types are obtained by reversing all arrows).



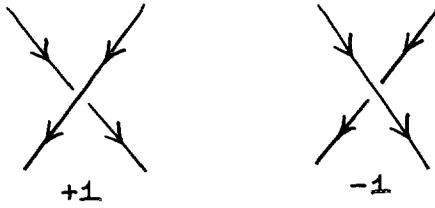
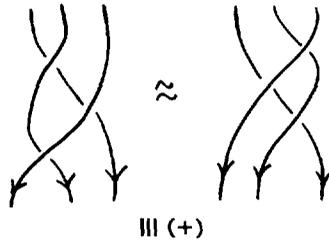


FIGURE 2

It is not hard to show that in the presence of moves II(A) and II(B), all III-moves can be generated from a unique move shown below which we shall call III(+) (see, for instance, [Tu]).



III (+)

Thus, in order to verify that a given valuation of diagrams is invariant under the Reidemeister moves it is sufficient to verify invariance under I,

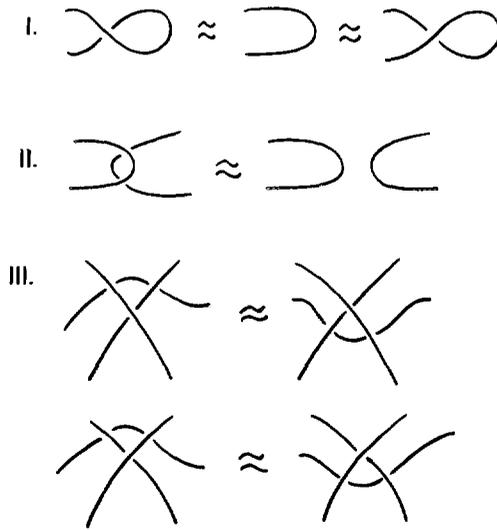


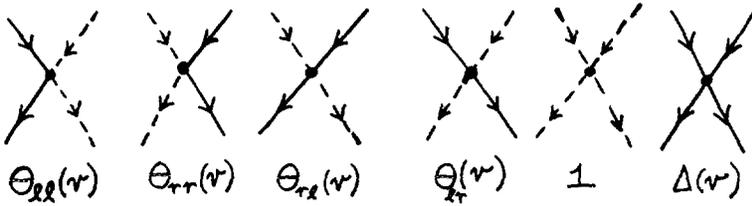
FIG. 3. Reidemeister moves.

II(A), II(B), and III(+), moves. This outlines the basic program whereby we can verify the invariance of models constructed as determinants of partition functions.

We are particularly interested in the invariance of models of the form

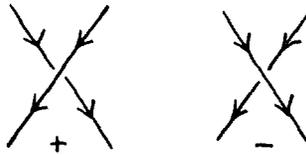
$$Z(G) = \sum_{f \in \mathcal{L}(G)} (-1)^{s(G, f, 2)} \prod_{v \in V} \langle v | f \rangle$$

as described in the previous section, where the weights $\langle v | f \rangle$ will depend upon the assignment of values (taken in a certain commutative ring) to the transitions $\theta_{p, q}(v)$. Recall that there are six possible local labelling configurations for the underlying plane graph (we use dashed lines for the label 2):

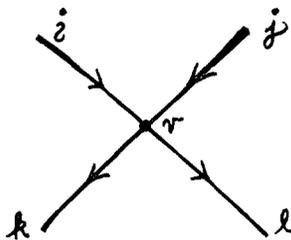


$(\Delta(v) = \theta_{ll}(v)\theta_{rr}(v) - \theta_{lr}(v)\theta_{rl}(v))$. Here the weights $\langle v | f \rangle$ are indicated for an arbitrary vertex v .

In order to construct an invariant of link diagrams on an orientable surface we will admit *two types of vertex v* , one for each crossing type:



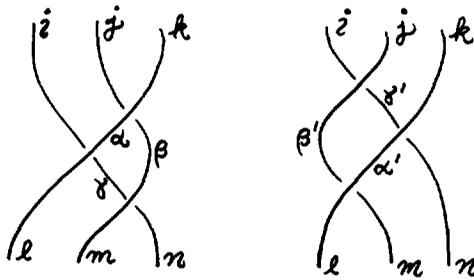
Thus for all p, q in $\{l, r\}$ and vertex v , the transition $\theta_{p, q}(v)$ will receive the value $\theta_{p, q}(+)$ if it is of positive type, and $\theta_{p, q}(-)$ otherwise. For i, j, k, l in $\{1, 2\}$, and s in $\{+, -\}$, let $R_{ij}^{kl}(s)$ be the value assigned to $\langle v | f \rangle$ for every vertex v of type s and labelling f which behaves as shown below on edges incident to v :



We shall adopt the convention that forbidden local configurations receive a weight zero. Then we can view the $R_{ij}^{kl}(s)$ as the entries of a four by four matrix $R(s)$ with rows and columns indexed by $\{1, 2\} \times \{1, 2\}$. Thus:

$$R(s) = \begin{matrix} & \begin{matrix} 11 & 12 & 21 & 22 \end{matrix} \\ \begin{matrix} 11 \\ 12 \\ 21 \\ 22 \end{matrix} & \begin{pmatrix} \Delta(s) & 0 & 0 & 0 \\ 0 & \theta_{ll}(s) & \theta_{lr}(s) & 0 \\ 0 & \theta_{rl}(s) & \theta_{rr}(s) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

(where $\Delta(s) = \theta_{ll}(s)\theta_{rr}(s) - \theta_{lr}(s)\theta_{rl}(s)$). To obtain the invariance of $Z(G)$ under the Reidemeister moves III(+), we require the following stronger property. We fix edge-values outside the two triangles appearing in the two configurations related by the move, and we want the two total contributions of the labellings taking these fixed values to be equal. The local situation can be described as shown below



where i, j, k, l, m, n are fixed and $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ can vary. It is easy to see that all labellings \mathcal{L} taking the fixed edge-values outside the triangles have the same value of $s(G, f, 2)$ in both configurations. Then our requirement reduces to

$$\sum_{\alpha, \beta, \gamma} R_{jk}^{\alpha\beta}(+) R_{\alpha\gamma}^{\beta\gamma}(+) R_{\gamma\beta}^{mn}(+) = \sum_{\alpha, \beta, \gamma} R_{ij}^{\beta'\gamma'}(+) R_{\gamma'k}^{\alpha'n}(+) R_{\beta'\alpha'}^{lm}(+).$$

This equation is a special form of the Yang–Baxter equation (see, for instance [K1, KS, Tu]). It can be written in matrix terms as follows. We introduce two eight by eight matrices S, T , with rows and columns indexed by $\{1, 2\}^3$ defined by

$$S_{ijk}^{lmn} = R_{ij}^{lm}(+) \delta_k^n, \quad T_{ijk}^{lmn} = \delta_i^l R_{jk}^{mn}(+).$$

Then the above equation becomes: $TST = STS$.

We leave it to the reader to check (combinatorially as, for instance, in [J], or using the matrix form) that the following assignment of values in $Z[x, x^{-1}, y, y^{-1}]$ satisfies the equations

$$\theta_{l,l}(+) = 1 + x, \quad \theta_{r,r}(+) = 0, \quad \theta_{l,r}(+) = y, \quad \theta_{r,l}(+) = -xy^{-1}.$$

A similar analysis of the moves of types II(A) shows that the invariance of $Z(G)$ under these moves follows from the equation $R(+)R(-) = I$, where

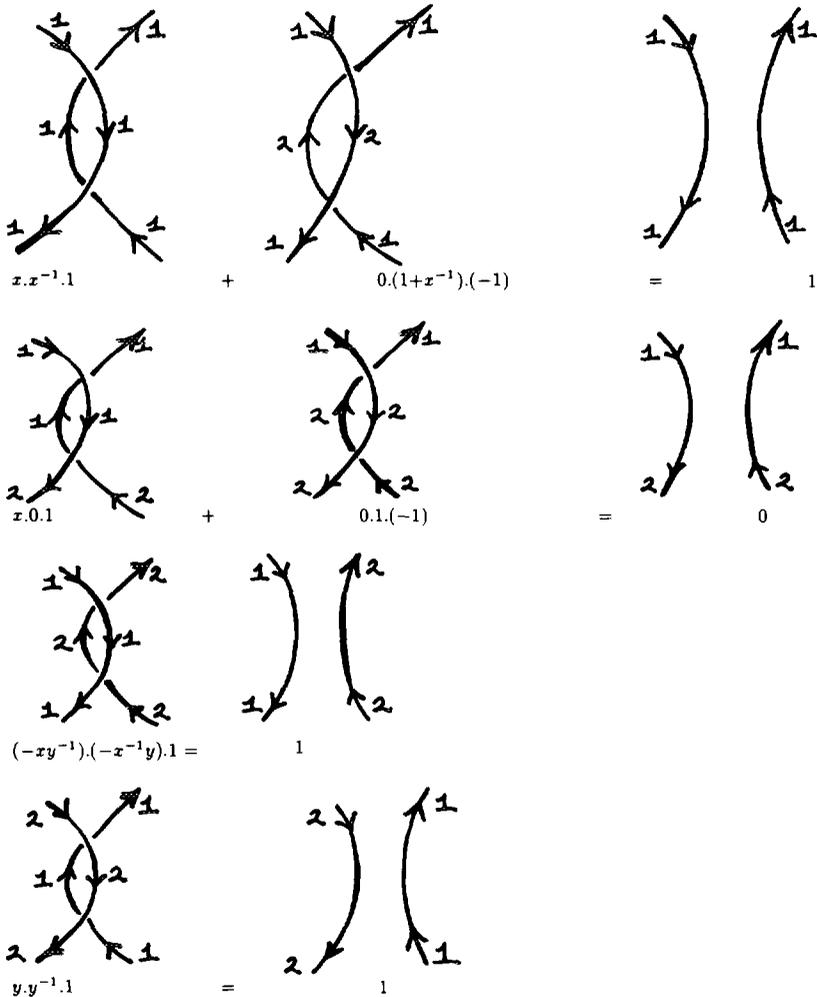


FIGURE 4 Invariance of Z under Reidemeister moves of type II(B).

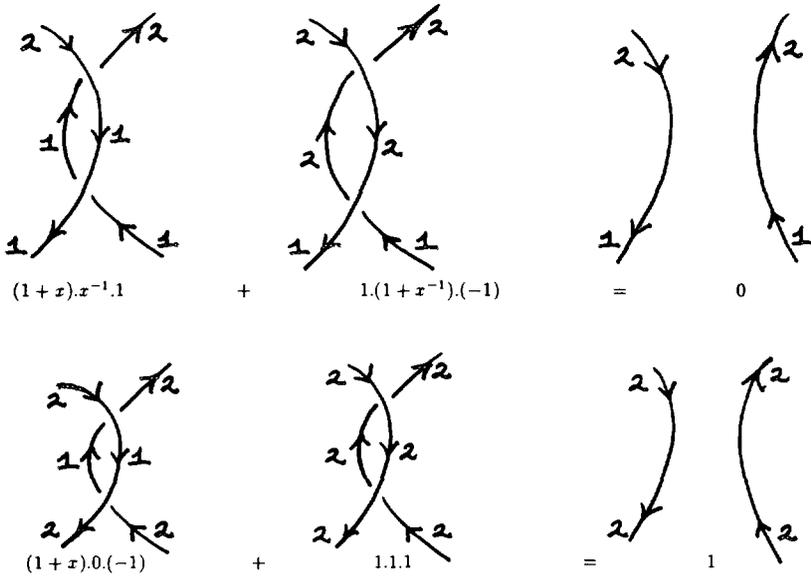


FIGURE 4—Continued

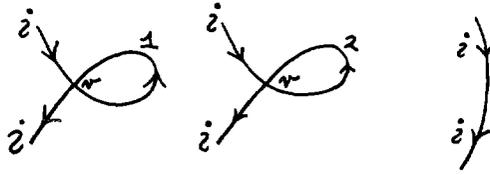
I is the unit matrix. It is easy to verify that this equation is satisfied with $\theta_{l,l}(-) = 0$, $\theta_{r,r}(-) = 1 + x^{-1}$, $\theta_{l,r}(-) = -x^{-1}y$, and $\theta_{r,l}(-) = y^{-1}$.

For moves of types II(B) we must take into account in detail the factor $(-1)^{s(G, f, 2)}$. We shall deal with only one kind of move, the other being essentially similar. There are six cases to consider, according to the possible values of the edges incident to the two vertices concerned by the move.

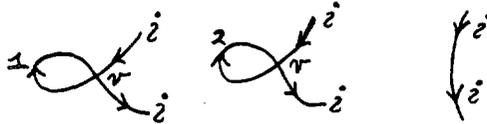
For each case, we compare the total contributions (to the two evaluations of Z) of the labellings compatible with the fixed edge-values, factoring out the vertex-weights and signs corresponding to the vertices and Seifert circuits which are not modified by the move.

Figure 4 should help the reader to check the invariance of Z under Reidemeister moves of type II(B). Here in an expression of the form $a.b.c$, a and b correspond to the weight of the upper and lower crossing, respectively, while c represents the local contribution to the sign factor $(-1)^{s(G, f, 2)}$. The only non-trivial evaluation of this local sign contribution appears in the last case, where we need the easily established fact that two diagrams related by a Reidemeister move of type II(B) have the same number of Seifert circuits modulo 2.

Finally, let us consider Reidemeister moves of type I. In the following figure, v is a vertex of type $s \in \{+, -\}$:

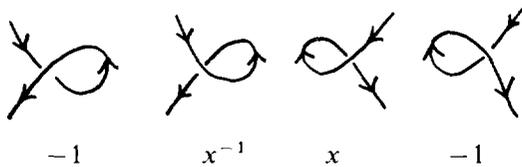


$$R_{ii}^{i1}(s) \cdot 1 + R_{i2}^{i2}(s) \cdot (-1) = (R_{ii}^{i1}(s) - R_{i2}^{i2}(s))(1);$$



$$R_{ii}^{i1}(s) \cdot 1 + R_{i2}^{i2}(s) \cdot (-1) = (R_{ii}^{i1}(s) - R_{i2}^{i2}(s))(1).$$

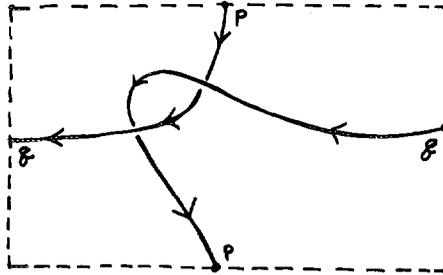
It turns out that the expressions $R_{i1}^{i1}(s) - R_{i2}^{i2}(s)$ and $R_{1i}^{1i}(s) - R_{2i}^{2i}(s)$ are independent of the choice of i in $\{1, 2\}$. Hence the above figure shows (using the same approach that for the study of moves II(B)) that the creation of a new crossing by a move of type I multiplies the value of Z by a constant factor which depends only on which kind of move is performed. We display these multiplicative factors on the following figure:



Let us denote by $Z(G, x, y)$ the evaluation of Z corresponding to the assignments made above to the $\theta_{p, q}(s)$. We have proved:

PROPOSITION 3. *For every oriented surface S , $Z(G, x, y)$, considered up to sign and powers of x , defines an invariant of oriented links in $S \times I$.*

EXAMPLE. We consider the following diagram G on the torus.



The corresponding matrix A is

$$\begin{matrix}
 & h_{1,l}^+ & h_{1,r}^+ & h_{2,l}^+ & h_{2,r}^+ \\
 h_{1,l}^- & (1+x) & -y & 0 & -1 \\
 h_{1,r}^- & xy^{-1} & 0 & -1 & 0 \\
 h_{2,l}^- & -1 & 0 & 1+x & -y \\
 h_{2,r}^- & 0 & -1 & xy^{-1} & 0
 \end{matrix}$$

Hence $Z(G, x, y) = \det A = x^2 - xy + x^2y^{-1} - y + xy^{-1} - 1$.

Note that the invariant $Z(G, x, y)$ can be computed efficiently (i.e., in polynomial time) as a two-variable determinant. However, the next two results show that this invariant is trivial when S is the plane or sphere.

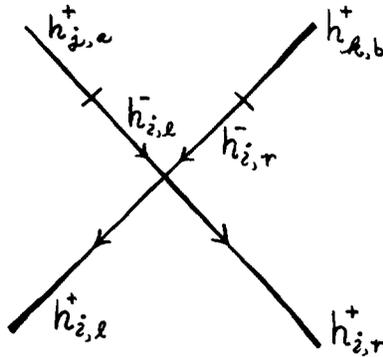
The *Seifert digraph* of G has the Seifert circuits of G as vertices, two vertices C, C' being connected by an arc from C to C' for every vertex of G where C uses the (left, left) transition and C' uses the (right, right) transition. A digraph will be said to be *balanced* if each cycle contains the same number of arcs going in each direction.

PROPOSITION 4. *If the Seifert digraph of G is balanced, then $Z(G, x, y) = 0$.*

Proof. In a balanced graph, if we assign to every edge the value 1, we obtain a tension, or equivalently a potential difference. This means that we may color the vertices with integers in such a way that whenever there exists an edge from a vertex v towards a vertex v' , $\text{color}(v') = \text{color}(v) + 1$. Color the Seifert circuits of G according to this rule, and for every edge e of G let $i(e)$, the *index* of e , be the color of the Seifert circuit which contains it. Let A be the matrix associated to G as in the previous example, so that $Z(G, x, y) = \det A$. For every edge e , let $X(e^+) = X(e^-) = (xy^{-1})^{i(e)}$. We claim that (with the notations of Section II):

$$\forall (i, p) \in \{1, \dots, n\} \times \{l, r\}, \sum_{(j, q)} A_{(i, p), (j, q)} X(h_{j, q}^+) = 0.$$

To check this, we consider a fixed vertex v_i and we denote by $h_{j,a}^+$ (resp. $h_{k,b}^+$) the other half of the edge containing $h_{i,l}^-$ (resp. $h_{i,r}^-$):



If the vertex v_i is positive, the property reduces to

$$\text{(for } p = l) \quad (1 + x) X(h_{i,l}^+) - yX(h_{i,r}^+) - X(h_{j,a}^+) = 0,$$

$$\text{(for } p = r) \quad xy^{-1}X(h_{i,l}^+) - X(h_{k,b}^+) = 0,$$

and if v_i is negative, it reduces to

$$\text{(for } p = l) \quad x^{-1}yX(h_{i,r}^+) - X(h_{j,a}^+) = 0,$$

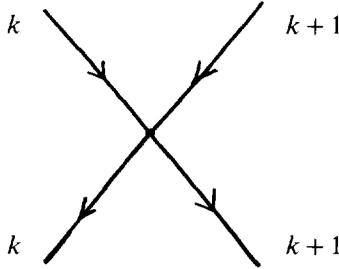
$$\text{(for } p = r) \quad (1 + x^{-1}) X(h_{i,r}^+) - y^{-1}X(h_{i,l}^+) - X(h_{k,b}^+) = 0.$$

Without loss of generality we may assume that $X(h_{j,a}^+) = X(h_{i,l}^-) = X(h_{i,l}^+) = 1$. Then $X(h_{k,b}^+) = X(h_{i,r}^-) = X(h_{i,r}^+) = xy^{-1}$ and the property follows easily. We have found a non-zero vector in the kernel of A and hence $Z(G, x, y) = \det A = 0$. ■

PROPOSITION 5. *If G is a diagram on the plane or sphere $Z(G, x, y) = 0$.*

Proof. This is corollary of Proposition 4, since the Seifert digraph of a diagram in the plane or sphere is balanced. This is easy to show directly. Another approach is to construct the index function $e \mapsto i(e)$ as follows. If we assign to every edge of the diagram the value 1, we obtain a flow, which can be viewed as a potential difference on the dual graph. This means that we may color the regions with integers in such a way that for every edge e separating a region r on its left from a region r' on its right, $\text{color}(r) = \text{color}(r') + 1$. Assuming with no loss of generality that the infinite region receives the color 0, this is the classical Alexander indexing of regions. Now

we define $i(e)$ as equal to $\text{color}(r')$. It is immediate that on edges incident to a given vertex the index function behaves as shown below:



In other words the index function $i: e \rightarrow i(e)$ derives as required from a coloring of the Seifert circuits which increases by one along each edge of the Seifert digraph (taken from the initial end to the terminal end). ■

However for diagrams with balanced Seifert digraphs we can still obtain a non-zero evaluation by using

$$Z(G, x, y, e, 1) = \sum_{f \in L(G, e, 1)} (-1)^{s(G, f, 2)} \prod_{v \in V} \langle v | f \rangle$$

multiplied by a normalization factor which will make the product independent of the choice of the edge e .

PROPOSITION 6. *Let G be a diagram with balanced Seifert digraph and index function i . The value of $(x^{-1}y)^{2i(e)} Z(G, x, y, e, 1)$ is independent of the choice of the edge e .*

Proof. Recall from Proposition 2 that $Z(G, x, y, e, 1) = \det A'_e$ and $Z(G, x, y, e, 2) = -\varepsilon(e) \det A[e]$. In the present case,

$$Z(G, x, y) = Z(G, x, y, e, 1) + Z(G, x, y, e, 2) = \det A = 0,$$

so that $Z(G, x, y, e, 1) = \varepsilon(e) \det A[e]$. In general let $\varepsilon((i, p), (j, q)) \det A[(i, p), (j, q)]$ be the signed minor corresponding to the $((i, p), (j, q))$ entry of A . Thus $\det A = 0$ yields, using row expansions,

$$\sum_{(j, q)} A_{(i, p), (j, q)} \varepsilon((i, p), (j, q)) \det A[(i, p), (j, q)] = 0$$

$$(\forall (i, p) \in \{1, \dots, n\} \times \{l, r\})$$

and similarly, using column expansions,

$$\sum_{(i, p)} A_{(i, p), (j, q)} \varepsilon((i, p), (j, q)) \det A[(i, p), (j, q)] = 0$$

$$(\forall (j, q) \in \{1, \dots, n\} \times \{l, r\}).$$

Recall from the proof of Proposition 4 that

$$\sum_{(j, q)} A_{(i, p), (j, q)} X(h_{j, q}^+) = 0 \quad (\forall (i, p) \in \{1, \dots, n\} \times \{l, r\}).$$

One can show in similar fashion that

$$\sum_{(i, p)} A_{(i, p), (j, q)} X(h_{i, p}^-) = 0 \quad (\forall (j, q) \in \{1, \dots, n\} \times \{l, r\}).$$

The result is true when the rank of A is less than $2n - 1$ since in this case $Z(G, x, y, e, 1) = 0$ for every edge e . On the other hand, if the rank of A equals $2n - 1$, its kernel is one-dimensional. Hence, the column vectors (indexed by (j, q))

$$(j, q) \varepsilon((i, p), (j, q)) \det A[(i, p), (j, q)](i, p) \text{ fixed}$$

and $X(h_{j, q}^+)$ are colinear, and similarly the row vectors (indexed by $(i, p) \in ((i, p), (j, q)) \det A[(i, p), (j, q)](j, q)$ fixed and $X(h_{i, p}^-)$ are colinear.

It follows that $\varepsilon((i, p), (j, q)) \det A[(i, p), (j, q)] [X(h_{i, p}^-) X(h_{j, q}^+)]^{-1}$ does not depend on the choice of $((i, p), (j, q))$. In particular $\varepsilon(e) \det A[e] [X(e^-) X(e^+)]^{-1}$ does not depend on the choice of e . Since $X(e^-) = X(e^+) = (xy^{-1})^{i(e)}$ the proof is complete. ■

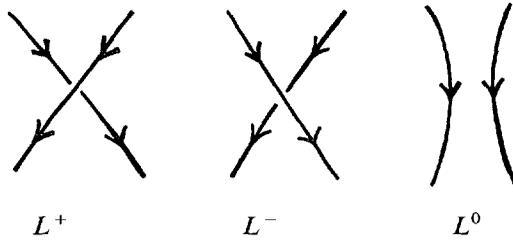
In the case of plane diagrams, the property that the Seifert digraph is balanced is invariant under Reidemeister moves. We take advantage of this in the next section.

2. A NEW CONSTRUCTION OF THE ALEXANDER-CONWAY POLYNOMIAL

In the following we consider only plane diagrams. We recall that the Alexander-Conway polynomial ∇ is an invariant of oriented links in R^3 which, considered as a valuation of plane diagrams (with values in $\mathbb{Z}[z]$) is uniquely characterized by the following properties (in addition to the invariance under Reidemeister moves; see, for instance, [K3]):

(i) ∇ takes the value 1 for the diagram with one component and no crossings;

(ii) $\nabla(L^+) - \nabla(L^-) = z\nabla(L^0)$ whenever L^+, L^-, L^0 are three diagrams which coincide outside a disk and behave as shown below inside this disk:



We present now a new combinational model for ∇ . It is an extension to arbitrary diagrams of models previously obtained for braid diagrams [J, K1] or diagrams with a distinguished edge [KS].

For any diagram L , we denote by $w(L)$ its *writhe*, that is, the sum of signs of its vertices. We denote by $r(L)$ its *rotation number* (see, for instance, [K3], that is, the sum of signs of the Seifert circuits of L , where a counterclockwise oriented circuit has sign $+1$ and a clockwise oriented circuit has sign -1 . As above, $i(e)$ denotes the index of an edge e , with the convention that the index of an edge with the infinite region on its right (resp. left) is 0 (resp. -1).

PROPOSITION 7. *For every plane diagram L and edge e of L ,*

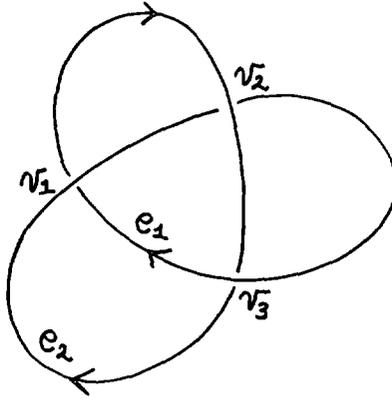
$$\nabla(L, z = t - t^{-1}) = t^{r(L)-1-2i(e)}(-t)^{-w(L)} Z(L, x = -t^2, y = -t, e, 1).$$

Proof. By Proposition 6, the right-hand side $Z'(L)$ of the above equation is independent of the choice of the edge e . Then to study its behavior under a Reidemeister move, we may assume that the special edge e is not affected by the move. Since the writhe and rotation number are invariant under Reidemeister moves of types II and III, we immediately obtain the invariance of $Z'(L)$ under these moves as in the proof of Proposition 3. Similarly, we can check that the multiplicative factors which appear in the proof of Proposition 3 for Reidemeister moves of type I are compensated by the factor $t^{r(L)}(-t)^{-w(L)}$. Hence $Z'(L)$ is also invariant under Reidemeister moves of type I. Now if L has one component and no crossing, $r(L) - 1 - 2i(e) = 0$ (for each of the two possible orientations, e being the unique edge of L). This proves property (i) of ∇ . To show property (ii), let v^+, v^- be the vertices of L^+, L^- , respectively, where these diagrams disagree, and identify $\mathcal{L}(L^+, e, 1)$ and $\mathcal{L}(L^-, e, 1)$ in the obvious way. It is easy to check that for every labelling f in this common set of labellings,

$$(-t)^{-1} \langle v^+ | f \rangle - (-t) \langle v^- | f \rangle = (t - t^{-1}) \langle v^0 | f \rangle,$$

where $\langle v^0 | f \rangle = 0$ if the edges labelled 1 at v^+ or v^- form a crossing transition, and equals 1 otherwise. Multiplying by the contributions of other vertices and summing over all labellings yields property (ii). ■

As an example let us consider the trefoil

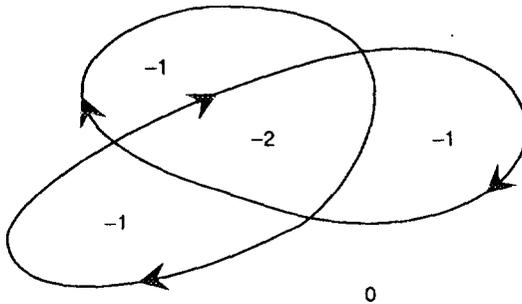


with matrix A ,

$$\begin{matrix}
 h_{1,l}^+ & h_{1,r}^+ & h_{2,l}^+ & h_{2,r}^+ & h_{3,l}^+ & h_{3,r}^+ \\
 h_{1,l}^- & \begin{pmatrix} 1-t^2 & t & 0 & 0 & -1 & 0 \\
 h_{1,r}^- & t & 0 & 0 & 0 & -1 \\
 h_{2,l}^- & -1 & 0 & 1-t^2 & t & 0 & 0 \\
 h_{2,r}^- & 0 & -1 & t & 0 & 0 & 0 \\
 h_{3,l}^- & 0 & 0 & -1 & 0 & 1-t^2 & t \\
 h_{3,r}^- & 0 & 0 & 0 & -1 & t & 0 \end{pmatrix}
 \end{matrix}$$

One finds easily, for $e_1^- = h_{1,l}^-$, $e_1^+ = h_{3,l}^+$, and $e_2^- = h_{1,r}^-$, $e_2^+ = h_{3,r}^+$,
 $\varepsilon(e_1) \det A[e_1] = -t^2(t^2 + t^{-2} - 1)$, $\varepsilon(e_2) \det A[e_2] = -t^4(t^2 + t^{-2} - 1)$.

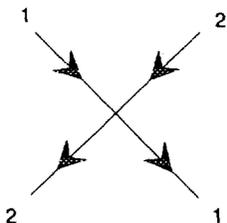
The Alexander indexing of the regions is



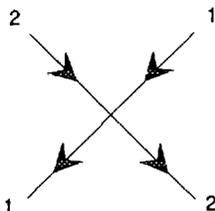
and $r(L) = -2$, $w(L) = 3$. Hence one finds in both cases that

$$\begin{aligned} \nabla &= t^{-3-2(-2)}(-t)^{-3} \varepsilon(e_1) \det A[e_1] \\ &= t^{-3-2(-1)}(-t)^{-3} \varepsilon(e_2) \det A[e_2] \\ &= 1 + (t - t^{-1})^2. \end{aligned}$$

We actually do not lose any information by using the one-variable specialization $x = -t^2$, $y = -t$ for plane diagrams. Indeed in this case any labelling has the same number of vertices of type



and of type



and the variable y does not appear in $Z(L, x, y)$.

We believe that the invariant $Z(G, x, y)$ for links in thickened surfaces should be studied in more detail. In particular, one may look for a topological interpretation in the spirit of the classical interpretation of the Alexander polynomial. We also do not know any direct connection between the combinatorial models of Proposition 7 and of [K3] for ∇ , but we feel that such a connection could be found.

ACKNOWLEDGMENT

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