Discrete physics and the derivation of electromagnetism from the formalism of quantum mechanics

By Louis H. Kauffman[1] and H. Pierre Noyes[2]
Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, 851 South Morgan Street, Chicago, IL 60607-7045, USA
Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309, USA

Freeman Dyson has recently focused attention on a remarkable derivation of electromagnetism from apparently quantum mechanical assumptions about the motion of a single particle. We present a new version of the Feynman–Dyson derivation in a discrete context. In the course of our derivation, we have uncovered a useful and elegant reformulation of the calculus of finite differences.

1. Introduction

In unpublished work circa 1948, Richard Feynman discovered that a quantum mechanical particle whose coordinates and momenta obeyed the simplest non-relativistic commutation relations will admit a description of acceleration that is compatible with Newton's second law and with the action of a classical electromagnetic field. This remarkable derivation was recently brought to the attention of the scientific community by the elegant paper of Freeman Dyson (1990). In his editorial comment on the reconstructed proof, Dyson remarks, '... here we find Galilean mechanics and Maxwell equations coexisting peacefully. Perhaps it was lucky that Einstein had not seen Feynman's proof when he started to think about relativity.' The proof has been generalized by Tanimura (1992) in a paper that embeds the Feynman argument into the contexts of gravity and gauge theories.

There are many themes to consider in the project of understanding the Feynman–Dyson derivation. In this paper, we concentrate on the following consideration: Feynman and Dyson assume commuting spatial coordinates $X_1(t), X_2(t), X_3(t)$, each a differentiable function of the time t. This occurs in the context of commutation relations of the form $[X_i, \dot{X}_j] = \kappa\delta_{ij}$ (κ a constant) giving the formalism the outward appearance of quantum mechanics. In the usual approaches to quantum mechanics, one has the corresponding equation $[q_i, p_j] = \mathrm{i}\hbar\delta_{ij}$, where q_i is the position operator and p_j is the momentum operator. These operators are not themselves functions of time in the Schrödinger representation of quantum mechanics, but they are functions of time in the Heisenberg formulation. As a consequence, the Feynman–Dyson derivation does apply directly to quantum mechanics in the Heisenberg formulation.

The derivation is *not* classical mechanics with the commutator interpreted as a Poisson bracket. As noted by Tanimura (1992), the Leibnitz rule needed in the proof holds for Poisson brackets only if the dynamical variables are derived from a

Hamiltonian or a Lagrangian. One major reason for being interested in the proof stems from the fact that this assumption is *not* made. We wish to point out that in a context of *discrete physics* the derivation can still be carried out, and that in this context there need not be any demand for simultaneous values of position and momentum operators. In fact, this idea is simply meaningless in our discrete context.

Because the variables and fields in the Feynman–Dyson derivation are non-commutative, the question of Lorentz invariance requires a special analysis that we shall not attempt in this paper, but comment on briefly in the Appendix. There is nothing paradoxical about the Feynman–Dyson derivation as it stands: it is a piece of mathematical physics asking for a good interpretation.

The purpose of this paper is to analyse the Feynman–Dyson derivation in a context of discrete physics. In this context, a spatial variable X_i has values $X_i, X_i', X_i'', \dots$ at successive values of discrete time. A measurement of velocity depends upon the difference of position values at two different (neighbouring) values of discrete time. Thus, we may (by convention) identify the value of $\dot{X}_i$ with $X_i' - X_i$ and write $\dot{X}_i := X_i' - X_i$. Since velocity depends upon two times and position on only one time, the idea of simultaneous determination of position and velocity is meaningless in the discrete context.

In order to achieve our aims, we have had to go to the roots of the calculus of discrete differences and discover an ordered version of this calculus that just fits the desired application. In this discrete ordered calculus (described in § 2 and 3 of this paper), the operation of differentiation acts also to shift a product to its left by one time step. Thus, $X\dot{X} := X'(X' - X)$, while $\dot{X}X := (X' - X)X$. In the discrete ordered calculus, $\dot{X}$ and X do not commute and a specific commutation relation such as $\dot{X}X - X\dot{X} = \kappa$ is regarded as a hypothesis about the structure of their non-commutativity.

Furthermore, the discrete ordered calculus (DOC) obeys the rule for the differentiation of the product, $(AB)^{\cdot} = A\dot{B} + \dot{A}B$, precisely without any time shifting (see § 2). This makes DOC an appropriate vehicle to support the calculus and non-commutative algebra that we need for our work.

In § 4 we work out the derivation of electromagnetism in this discrete context. We begin with the assumption of the commutation relations for X_i $(i = 1, 2, 3)$:

1. $[X_i, X_j] = 0$
2. $[X_i, \dot{X}_j] = \kappa\delta_{ij}$.

Here the dot $(\cdot)$ is the discrete derivative and κ is a commuting scalar in DOC. We discuss reformulations of these equations in § 4.

With $F_i = \ddot{X}_i$ and

$$H_\ell = \frac{1}{2\kappa}\epsilon_{jk\ell}[\dot{X}_j, \dot{X}_k], \quad F_j = E_j + \epsilon_{jk\ell}\dot{X}_k H_\ell, \quad (F = E + v \times H),$$

we show that

(i) $\operatorname{div} H = 0$,

(ii) $\partial H/\partial t + \nabla \times E = 0$.

This is the desired result. Note that ϵ_{ijk} is the alternating symbol, and that $F = E + v \times H$ defines E.

In order to interpret these equations as electromagnetism, we need the other two Maxwell equations:

$$\operatorname{div} E = 4\pi\rho, \quad \frac{\partial E}{\partial t} - \nabla \times H = 4\pi j$$

In our context, following Dyson, we take these equations as *definitions* of ρ and j. With these conventions we have a non-commutative electromagnetic formalism. It remains to be understood how this formalism is related to standard electromagnetism, and how the considerations of special relativity enter into this non-commutative context. It is our purpose, in this first paper, to put the derivation on a firm footing in order to provide a platform for consideration of these problems in subsequent work.

We have taken great care to perform this derivation in the discrete ordered calculus. This involves taking the following *definitions* for partial derivatives of a function $f(X)$:

$$\frac{\partial f}{\partial X_i} = \frac{1}{\kappa}[f, \dot{X}_i], \quad \dot{f} = \frac{\partial f}{\partial t} + \dot{X}_j \frac{\partial f}{\partial X_j}.$$

(The Einstein summation convention is in effect.) These definitions are discussed in § 4.

We wish to close this introduction with a remark about the commutativity of X and X'. X' is regarded as the indicator of X after one discrete time step. Formally, we can write both XX' and $X'X$. However, in our convention, XX' means [measure X', *then* measure X] and this would require the observer to step backwards in time! For this reason we do not assume that $XX' = X'X$, and this gives us the *formal freedom* to postulate (in § 4) a set of commutation relations among $\{X_i, X_j'\}$ that can be regarded as the basis of our derivations. In a sequel to this paper, we shall discuss actual numerical solutions to these relationships.

Obviously, much more work remains to be done in this domain. We shall discuss gravity/quantum formalism in a sequel to this paper.

2. Motivating a discrete calculus

In one-dimensional standard quantum mechanics in the Heisenberg formulation (Dirac 1947) the uncertainty principle takes the form of a commutation relation

$$QP - PQ = \hbar \mathrm{i}$$

where Q and P denote, respectively, the position and momentum operators for the quantum mechanical particle, and $\hbar$ is Planck's constant divided by 2π ($\mathrm{i}^2 = -1$).

There are many interpretations of this formalism. In the Schrödinger picture of quantum mechanics, the system is represented by a wavefunction $\psi = \psi(x, t)$, where x denotes the spatial coordinate and t denotes the temporal coordinate. The operators Q and P are defined by the equations

$$Q\psi = x\psi, \quad P\psi = \frac{\hbar}{\mathrm{i}}\frac{\partial \psi}{\partial x} .$$

Thus,

$$(QP - PQ)\psi = x\frac{\hbar}{\mathrm{i}}\frac{\partial \psi}{\partial x} - \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x}(x\psi) = (\hbar \mathrm{i})\psi.$$

Hence, $QP - PQ = \hbar \mathrm{i}$.

The Heisenberg picture is not tied to this particular interpretation. It simply asserts that the order of application of the position and momentum operators matters—and that the difference of these orders is described by the commutation relations.

We can find an almost identical commutation relation by thinking about position and momentum in a classical but discrete context. In a discrete universe, time goes forward in measured ticks, and space occurs only in discrete intervals. We can imagine position determined at an instant, but to find velocity or momentum the clock must advance one tick to allow computation of the ratio of change of position to change of time. In measuring position first and then momentum, we advance the clock *after* determining position. If momentum is measured before position, the clock advances *before* the measurement of position and the position is determined at a later time. In this way, PQ and QP differ due to the intervening time step.

Let us quantify these last remarks by working with discrete position X and discrete velocity $\dot{X}$. Let X, X', X'', ... denote the sequence of values for X at successive times $t_0, t_1, t_2, \ldots$. Define the value of $\dot{X}$ to be $X' - X$ and write $\dot{X} := X' - X$ to indicate this evaluation. We regard $\dot{X}$ as a discrete velocity with the time step normalized to unity by convention.

Let $X\dot{X}$ denote the *process*—measure $\dot{X}$ then measure X. Thus, on evaluating, we find

$$X\dot{X} := X'(X' - X)$$

since measuring $\dot{X}$ requires stepping forward in time to the position X'.

On the other hand, $\dot{X}X$ denotes the process—measure X then measure $\dot{X}$. Thus, $\dot{X}X := (X' - X)X$.

We conclude that

$$X\dot{X} - \dot{X}X := X'(X' - X) - (X' - X)X.$$

This difference is not zero, and if it turns out to be a constant (κ) then we have the equation $X\dot{X} - \dot{X}X := \kappa$: a discrete analogue to the Heisenberg commutation relation.

In order to take the derivations of Dyson (1990) and Tanimura (1992) and place them on a discrete foundation, we shall develop a time-ordered calculus that generalizes the ideas that have been presented in this section. We end this section with an informal discussion of some of the issues that are involved.

One issue that must be faced is the question of the commutativity of X and X'. We can formally write both $X'X$ and XX'. The first ($X'X$) means—measure X, take a time step, measure X after the time step. However, XX' does not have operational meaning in this same sense, since X' demands a time step while X asks for the value at a previous time. *We therefore assume that $X'X$ and XX' are distinct* without yet making any explicit assumption about the value of their difference.

The second issue involves evaluation. We have been careful to write $\dot{X} := X' - X$ rather than $\dot{X} = X' - X$, since the dot in $\dot{X}$ is a special instruction to shift time to its left in the ordered calculus. The directed equals sign (:=) is used to indicate evaluation. Thus, we can write $A\dot{B} := A'(B' - B)$ and

$$A\dot{B}\dot{C} := (A\dot{B})'(C' - C) \tag{2.1}$$

$$:= A'\dot{B}'(C' - C) \tag{2.2}$$

$$:= A''(B' - B)'(C' - C) \tag{2.3}$$

$$:= A''(B'' - B')(C' - C). \tag{2.4}$$

(We assume that $(XY)' = X'Y'$.) Each step in evaluation must perform all the time shifts for any dot that is eliminated. We shall return to this issue in the next section.

Returning to $X\dot{X}$ and $\dot{X}X$, we evaluate and find

$$X\dot{X} - \dot{X}X : = X'(X' - X) - (X' - X)X \tag{2.5}$$

$$= X'(X' - X) - X(X' - X) + X(X' - X) - (X' - X)X \quad (2.6)$$

$$= (X' - X)(X' - X) + XX' - X^2 - X'X + X^2 \quad (2.7)$$

$$= (X' - X)^2 + [X, X']. \quad (2.8)$$

Thus,

$$X\dot{X} - \dot{X}X := (X' - X)^2 + [X, X'],$$

where $[A, B] = AB - BA$. (In general, we will not assume that $[X, X'] = 0$.)

If X and X' commute, then $[X, \dot{X}] := (X' - X)^2$. In this context one might assume that $(X' - X)^2 = \kappa$ is constant and declare that $X\dot{X} - \dot{X}X = \kappa$.

If X and X' do not commute then the formula above shows how their commutator is related to $[X, \dot{X}]$.

To summarize, *the dot in $\dot{X}$ is an instruction to take a time step.* A product AB means *do B, then do A*. Therefore, $X\dot{X} := X'(X' - X)$, since measuring X after one time step yields X'.

3. A discrete ordered calculus—DOC

By a variable X we mean a collection of algebraic entities $X, X', X'', X''', \ldots$ called 'the values of X at successive steps of discrete time'. No assumptions of commutativity are made for these variables, but we do assume that multiplication is associative and that multiplication distributes over addition and that there is a unit element, 1, such that $1X = X1$ for all X. Furthermore, we assume that $1' = 1$. Similarly, there is a 0 such that $0 + X = X$ for all X and $0' = 0$.

At this point the reader will see that we are assuming that a non-commutative ring R has been given, and that $X, X', \ldots, Y, Y', \ldots$ belong to R. (Note that this means that we assume that $X' + Y = Y + X'$.) Thus, we can speak concisely by saying that we assume as a given a (non-commutative) ring R with unit (1) equipped with a unary operator $' : R \to R$, such that $1' = 1$ and $0' = 0$ *and* $(a + b)' = a' + b'$ for all a and b in R, and $(ab)' = a'b'$ for all a, b in R. In the context of the ring R, we shall define a discrete ordered calculus by first adjoining to R a special element J the sole purpose of which is to keep track of the time shifting. We assume that J has the properties:

1. $J' = J$;
2. $AJ = JA'$ for all $A \in R$; (of course $JJ' = JJ$ so this works for J as well).

We let $\widehat{R}$ be the ring obtained from R by formally adjoining J to R with these properties. Since $\widehat{R}$ is, by definition, a ring with unit, this means that

$$(X + Y)J = XJ + YJ, \quad J(X + Y) = JX + JY, \quad J0 = 0, \quad J1 = 1, \quad \text{etc.}$$

Now note that any expression in $\widehat{R}$ can be rewritten (using $AJ = JA'$) in the form of a sum of elements of the form $J^k Z$, where there is no appearance of J in Z. We can define an *evaluation map* $E : \widehat{R} \to R$ by the following equations:

(i) $E(A + B) = E(A) + E(B)$ for any $A, B \in R$;

(ii) $E(J^k Z) = Z$ whenever $Z \in R$.

E is defined on $\widehat{R}$ by writing $A \in \widehat{R}$ as a sum of elements of the form $J^k Z$ and then applying (i) and (ii) above. For example,

$$E(AJ + BJ(CJ)) = E(JA' + J^2 B''C') = A' + B''C'$$

(assuming that $A, B, C \in R$). It follows from our assumptions that $E : \widehat{R} \to R$ is well defined. Note that, by definition, $E(E(X)) = E(X)$, where we regard $R \subset \widehat{R}$ as the set of expressions in $\widehat{R}$ without any J. In fact, we note that $\widehat{R} \cong \bigoplus_{n=0}^{\infty} J^n R$, where

$$J^n R = \{J^n r \,|\, r \in R\}, \quad \text{and} \quad (J^n r)(J^m s) = J^{n+m} r^{(m)} s,$$

where $r, s, \in R$ and $r^{(m)} = r^{\prime\prime\ldots\prime}$, with m 'primes'. With this reformulation, the evaluation map is obviously well defined. Now we are prepared to define differentiation in $\widehat{R}$ and therefore initiate the discrete ordered calculus (DOC).

Definition 3.1. Define $D : \widehat{R} \to \widehat{R}$ by the equation $D(X) = J(X' - X)$. The presence of J in DX makes it a time shifter for expressions on its left. (Compare this approach with (Etter & Kauffman 1994). That approach arose from discussions about an early version of the present paper.)

Proposition 3.2. *Let $A, B \in \widehat{R}$. Then*

$$D(AB) = D(A)B + AD(B).$$

Proof.

$$\begin{aligned} D(AB) &= J((AB)' - AB) \\ &= J(A'B' - A'B + A'B - AB) \\ &= J(A'(B' - B) + (A' - A)B) \\ &= JA'(B' - B) + J(A' - A)B \\ &= AJ(B' - B) + J(A' - A)B \\ &= AD(B) + D(A)B. \end{aligned}$$

■

We see from the proof of this proposition how the ordering convention in the discrete calculus has saved the product rule for differentiation.

In a standard commutative time-discrete calculus, one of the terms in the expansion of the derivative of a product must be time shifted. The same phenomenon occurs in the infinitesimal calculus, but there an infinitesimal shift is neglected in the limit:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(fg) &= \lim_{h\to 0} \frac{f(t+h)g(t+h) - f(t)g(t)}{h} \\ &= \lim_{h\to 0} \frac{f(t+h)g(t+h) - f(t+h)g(t) + f(t+h)g(t) - f(t)g(t)}{h} \\ &= \lim_{h\to 0} f(t+h)\left(\frac{g(t+h) - g(t)}{h}\right) + \left(\frac{f(t+h) - f(t)}{h}\right) g(t) \\ &= \left[\lim_{h\to 0} f(t+h)\right] \frac{\mathrm{d}g}{\mathrm{d}t} + \left(\frac{\mathrm{d}f}{\mathrm{d}t}\right) g(t). \end{aligned}$$

It is interesting to see how the evaluations work in specific examples. In writing examples it is convenient to write $\dot{A}$ for $D(A)$. Thus, $\dot{A} = J(A' - A)$. For example,

$$\begin{aligned} (X\dot{Y})^{\cdot} &= J((X\dot{Y})' - X\dot{Y})) \\ &= J((XJ(Y' - Y))' - XJ(Y' - Y)) \\ &= J(X'J(Y'' - Y') - XJ(Y' - Y)) \\ &= J(JX''(Y'' - Y') - JX'(Y' - Y)) \\ &= J^2(X''Y'' - X''Y' - X'Y' + X'Y). \end{aligned}$$

Thus, $E((X\dot{Y})\dot{}) = X''Y'' - X''Y' - X'Y' + X'Y$. On the other hand,

$$\begin{aligned}\dot{X}\dot{Y} &= J(X' - X)J(Y' - Y)\\ &= J^2(X'' - X')(Y' - Y).\end{aligned}$$

$$\begin{aligned}X\ddot{Y} &= XJ(\dot{Y}' - \dot{Y})\\ &= XJ(J(Y' - Y)' - J(Y' - Y))\\ &= XJ^2(Y'' - Y' - Y' + Y))\\ &= J^2X''(Y'' - 2Y' + Y).\end{aligned}$$

$$\begin{aligned}\dot{X}\dot{Y} + X\ddot{Y} &= J^2[(X'' - X')(Y' - Y) + X''(Y'' - 2Y' + Y)]\\ &= J^2[X''Y' - X''Y - X'Y' + X'Y + X''Y'' - 2X''Y' + X''Y]\\ &= J^2[-X''Y' - X'Y' + X'Y + X''Y'']\\ &= (X\dot{Y})\dot{}.\end{aligned}$$

This is a working instance of our formula $D(AB) = D(A)B + AD(B)$. In the remainder of the paper it will be useful to write $A := B$ to mean that $E(A) = E(B)$. In particular, we will often use this to mean that B has been obtained from A by expanding some derivatives and throwing away some or all of the left-most J. This means that while it is true that $E(A) = E(B)$, A and B cannot be substituted for one another in larger expressions, since they contain different time-shifting instructions.

An example of this usage is

$$\dot{X}\dot{Y} := \dot{X}'(Y' - Y).$$

Note that

$$\dot{X}\dot{Y} = \dot{X}J(Y' - Y) = J\dot{X}'(Y' - Y).$$

Thus,

$$E(\dot{X}\dot{Y}) = E(\dot{X}'(Y' - Y)).$$

In calculating, the $:=$ notation allows us to 'do the Js in our heads'.

Discussion. With the DOC formalized we can return to the structure of the commutator $[X, \dot{Y}] = X\dot{Y} - \dot{Y}X$. We have

$$\begin{aligned}[X, \dot{Y}] &:= X'(Y' - Y) - (Y' - Y)X\\ &:= X'(Y' - Y) - X(Y' - Y) + X(Y' - Y) - (Y' - Y)X\\ &:= (X' - X)(Y' - Y) + XY' - XY - Y'X + YX\\ &:= (X' - X)(Y' - Y) + (XY' - Y'X) - (XY - YX)\end{aligned}$$

$$[X, \dot{Y}] := (X' - X)(Y' - Y) + [X, Y'] - [X, Y]$$

This formula will be of use to us in the next section.

Note how, in this formalism, we cannot arbitrarily substitute $\dot{X}$ for $X' - X$ since the definition of the dot ('$\cdot$') as a time shifter can change the value of an expression. Thus, $(X' - X)(Y' - Y) \neq \dot{X}\dot{Y}$. It may be useful to write $X' - X = \|\dot{X}\|$, where $\|\dot{X}\|$ is, by definition, the difference, stripped of its time-shifting properties. Then, $(X' - X)(Y' - Y) = \|\dot{X}\|\,\|\dot{Y}\|$ and we can write

$$[X, \dot{Y}] := \|\dot{X}\|\,\|\dot{Y}\| + [X, Y'] - [X, Y].$$

Since $[A, B] = AB - BA$, these commutators satisfy the Jacobi identity. That is, we have

$$[[A, B], C] + [[C, A], B] + [[B, C], A] = 0.$$

The proof is by direct calculation.

4. Electromagnetism

In this section we give a discrete version of the Feynman–Dyson (Dyson 1990) derivation of the source-free Maxwell equations from a quantum mechanical formalism. We shall work in the discrete ordered calculus (DOC) of § 3. We assume time-series variables X_1, X_2 and X_3 and the commutation relations

(i) $$[X_i, X_j] = 0, \quad \forall_{ij},$$

(ii) $$[X_i, \dot{X}_j] = \kappa\delta_{ij},$$

where κ is a constant and κ commutes with all expressions in DOC.

We further assume that there are functions $F_i(X\,\dot{X})$ $(i = 1, 2, 3)$ such that $\ddot{X}_i = F_i(X, \dot{X})$. (Here writing $F(X)$ means that F is a function of X_1, X_2, X_3.) It is the purpose of this section to show that F_i takes on the pattern of the electromagnetic field in vacuum. Our first task will be to rewrite the above relations in terms of the discrete ordered calculus.

Proposition 4.1. *Given that $[X_i, X_j] = 0$ for all $i, j = 1, 2, 3$, and letting $\Delta_i = X_i' - X_i$, the equations $[X_i, \dot{X}_j] = \kappa\delta_{ij}$ imply the equations*

$$E([X_i, X_j'] + \Delta_i\Delta_j) = E(\kappa\delta_{ij}).$$

Proof. First assume $[X_i, \dot{X}_j] = \kappa\delta_{ij}$. Then $[X_i, \dot{X}_j] := \|\dot{X}_i\|\,\|\dot{X}_j\| + [X_i, X_j'] - [X_i, X_j]$ by the calculation at the end of § 2. Here, $\|\dot{X}_i\| = X_i' - X_i = \Delta_i$ and $[X_i, X_j] = 0$. Thus,

$$[X_i, \dot{X}_j] := \Delta_i\Delta_j + [X_i, X_j'].$$

■

Discussion. This proposition shows how the Heisenberg-type relations $X_i\dot{X}_j - \dot{X}_jX_i = \kappa\delta_{ij}$ translate into the time-series commutation relations $X_iX_j' - X_j'X_i := \kappa\delta_{ij} - \Delta_i\Delta_j$. From the point of view of discrete physics, it is these relations that will implicate electromagnetism. Since there is no *a priori* reason for the elements of time series to commute with one another, we can regard the equations

$$X_iX_j = X_jX_i, \quad X_iX_j' - X_j'X_i = \kappa\delta_{ij} - \Delta_i\Delta_j, \quad (\Delta_i = X_i' - X_i)$$

as setting the *context* for the discussion of the physics of a discrete particle. It is in this context that the patterns of electromagnetism will appear.

Derivation. We shall need to interpret certain derivatives in terms of our discrete formalism. First of all, we have

$$\frac{\partial X_i}{\partial X_j} = \delta_{ij}.$$

Therefore,

$$[X_i, \dot{X}_j] = \kappa\,\frac{\partial X_i}{\partial X_j}.$$

Consequently, we make the following definition.

Definition 4.2. Let G be a function of X, then we define $\partial G/\partial X_i$ by the equation

$$\frac{\partial G}{\partial X_i} = \kappa^{-1}\left[G, \dot{X}_i\right].$$

We also wish to *define* $\partial G/\partial t$. This time derivative is distinct from $\dot{G}$. It should satisfy the usual relationship for multivariable calculus:

$$\dot{G} = \frac{\partial G}{\partial t} + \dot{X}_j \frac{\partial G}{\partial X_j}, \quad \text{(summed over } j = 1, 2, 3\text{)}.$$

Therefore, we define $\partial G/\partial t$ by the equation

$$\frac{\partial G}{\partial t} = \dot{G} - \dot{X}_j \kappa^{-1}\left[G, \dot{X}_j\right], \quad \text{(summed over } j = 1, 2, 3\text{)},$$

when $[X_m, G] = 0$ for $m = 1, 2, 3$.

The condition $[X_i, G] = 0$ for $i = 1, 2, 3$ implies that G has no dependence on $\dot{X}_j$ ($j = 1, 2, 3$) under mild hypotheses on G. For, if we assume that G is either a polynomial or a (non-commutative) power series in X_j and $\dot{X}_j$, then the equations $[X_i, \dot{X}_j] = \kappa\delta_{ij}$ and $[X_i, X_j] = 0$ show that (under these assumptions) G has no occurrence of $\dot{X}_j$. It is necessary to define $\partial/\partial t$ since our discrete theory does not carry the conventional time variable t.

With these definitions in hand, we can proceed to the consequences of the commutation relations (i) and (ii).

Lemma 4.3.

$$[\dot{X}_i, X_j] = [\dot{X}_j, X_i].$$

Proof.

$$\begin{aligned} X_i X_j &= X_j X_i \\ \Rightarrow (X_i X_j)^{\cdot} &= (X_j X_i)^{\cdot} \\ \Rightarrow \dot{X}_i X_j + X_i \dot{X}_j &= \dot{X}_j X_i + X_j \dot{X}_i \\ \Rightarrow \dot{X}_i X_j - X_j \dot{X}_i &= \dot{X}_j X_i - X_i \dot{X}_j \\ \Rightarrow [\dot{X}_i, X_j] &= [\dot{X}_j, X_i] \end{aligned}$$

■

Lemma 4.4.

$$[\dot{X}_j, \dot{X}_k] + [X_j, \ddot{X}_k] = 0.$$

Proof.

$$X_j \dot{X}_k - \dot{X}_k X_j = \kappa\delta_{jk} \tag{4.1}$$

$$\Rightarrow \dot{X}_j \dot{X}_k + X_j \ddot{X}_k - \ddot{X}_k X_j - \dot{X}_k \dot{X}_j = 0 \tag{4.2}$$

$$\Rightarrow [\dot{X}_j, \dot{X}_k] + [X_j, \ddot{X}_k] = 0 \tag{4.3}$$

■

Since $\ddot{X}_j = F_j(X, \dot{X})$ we have the following lemma.

Lemma 4.5.

$$[\dot{X}_j, \dot{X}_k] + [X_j, F_k] = 0.$$

Thus,

$$\begin{aligned} [X_\ell, [X_j, F_k]] &= -[X_\ell, [\dot{X}_j, \dot{X}_k]] \\ &= [\dot{X}_j, [\dot{X}_k, X_\ell]] + [\dot{X}_k, [X_\ell, \dot{X}_j]], \quad \textit{(by the Jacobi identity)} \\ &= [\dot{X}_j, -\delta_{k\ell}\kappa] + [\dot{X}_k, \kappa\delta_{\ell j}] \\ &= 0 + 0 = 0. \end{aligned}$$

Note that

$$[X_j, F_k] = -[\dot{X}_j, \dot{X}_k] = +[\dot{X}_k, \dot{X}_j] = -[X_k, F_j].$$

Thus,

$$[X_j, F_k] = -[X_k, F_j].$$

We now define the field H by the equation

$$-\kappa\epsilon_{jk\ell}H_\ell = [X_j, F_k],$$

where $\epsilon_{jk\ell}$ is the alternating symbol for 123. That is, $\epsilon_{123} = +1$ and $\epsilon_{abc} = \mathrm{sgn}(abc)$ if abc is a permutation of 123, where $\mathrm{sgn}(abc)$ is the sign of the permutation. Otherwise, $\epsilon_{abc} = 0$.

Note that $[X_\ell, [X_j, F_k]] = 0$ implies that $[X_\ell, -\epsilon_{jk\ell}H_\ell] = 0$, which in turn implies that $[X_\ell, H_s] = 0$. This implies that H_s has no dependence upon $\dot{X}$ since $\dot{X}$ has a non-trivial commutator with X. Under these circumstances we will regard H as a function of X and compute $\partial H/\partial t$ according to the formula

$$\frac{\partial H}{\partial t} = \dot{H} - \dot{X}_j\kappa^{-1}[H, \dot{X}_j]$$

as discussed above.

Definition 4.6.

$$E_j = F_j - \epsilon_{jk\ell}\dot{X}_k H_\ell.$$

With this definition of E, we have $F = E + v \times H$, where $v = \dot{X}$.

Lemma 4.7.

$$[X_m, E_j] = 0.$$

Proof.

$$\begin{aligned} [X_m, E_j] &= [X_m, F_j - \epsilon_{jk\ell}\dot{X}_k H_\ell] \\ &= [X_m, F_j] - [X_m, \epsilon_{jk\ell}\dot{X}_k H_\ell] \\ &= -\kappa\epsilon_{mj\ell}H_\ell - \epsilon_{jk\ell}X_m\dot{X}_k H_\ell + \epsilon_{jk\ell}\dot{X}_k H_\ell X_m \\ &= -\kappa\epsilon_{mj\ell}H_\ell - \epsilon_{jk\ell}X_m\dot{X}_k H_\ell + \epsilon_{jk\ell}\dot{X}_k X_m H_\ell \\ &= -\kappa\epsilon_{mj\ell}H_\ell - \epsilon_{jk\ell}[X_m, \dot{X}_k]H_\ell \\ &= -\kappa\epsilon_{mj\ell}H_\ell - \epsilon_{jk\ell}\kappa\delta_{mk}H_\ell \\ &= (-\kappa\epsilon_{mj\ell} - \kappa\epsilon_{jm\ell})H_\ell \\ &= 0. \end{aligned}$$

Thus, E also has no dependence upon $\dot{X}$.

Remark 4.8.

$$H_\ell = \tfrac{1}{2}\kappa^{-1}\epsilon_{jk\ell}[\dot{X}_j, \dot{X}_k].$$

Proof.

$$-\kappa\epsilon_{jk\ell}H_\ell = [X_j, F_k]$$

and by lemma4.5,

$$[X_j, F_k] = -[\dot{X}_j, \dot{X}_k].$$

Thus,

$$\epsilon_{jk\ell}H_\ell = \kappa^{-1}[\dot{X}_j, \dot{X}_k].$$

From this it follows that

$$H_\ell = \tfrac{1}{2}\kappa^{-1}\epsilon_{jk\ell}[\dot{X}_j, \dot{X}_k].$$

■

Lemma 4.9.

$$\operatorname{div} H = \sum_{i=1}^{3} \frac{\partial H_i}{\partial X_i} = 0.$$

Proof.

$$\begin{aligned}\sum_\ell \frac{\partial H_\ell}{\partial X_\ell} &= \kappa^{-1}\sum_\ell [H_\ell, \dot{X}_\ell]\\ &= \tfrac{1}{2}\kappa^{-2}\epsilon_{jk\ell}[[\dot{X}_j, \dot{X}_k], \dot{X}_\ell]\\ &= 0, \quad \text{(by the Jacobi identity).}\end{aligned}$$

Thus, $\operatorname{div} H = 0$. ■

Lemma 4.10.

$$\frac{\partial H_\ell}{\partial t} = \epsilon_{jk\ell}\frac{\partial E_j}{\partial X_k}.$$

Proof.

$$\begin{aligned}H_\ell &= \tfrac{1}{2}\kappa^{-1}\epsilon_{jk\ell}[\dot{X}_j, \dot{X}_k]\\ \frac{\partial H_\ell}{\partial t} &= \dot{H}_\ell - \dot{X}_j\kappa^{-1}[H_\ell, \dot{X}_j].\end{aligned}$$

(See the discussion of $\partial/\partial t$ given earlier in this section.)

Now

$$\begin{aligned}\dot{H}_\ell &= \tfrac{1}{2}\kappa^{-1}\epsilon_{jk\ell}[\dot{X}_j, \dot{X}_k]^{\cdot}\\ &= \tfrac{1}{2}\kappa^{-1}\ \epsilon_{jk\ell}([\ddot{X}_j, \dot{X}_k] + [\dot{X}_j, \ddot{X}_k])\\ &= \kappa^{-1}\epsilon_{jk\ell}[\ddot{X}_j, \dot{X}_k]\\ &= \kappa^{-1}\epsilon_{jk\ell}[F_j, \dot{X}_k]\\ &= \kappa^{-1}\epsilon_{jk\ell}[E_j + \epsilon_{jrs}\dot{X}_r H_s, \dot{X}_k]\\ &= \kappa^{-1}\epsilon_{jk\ell}[E_j, \dot{X}_k] + \kappa^{-1}[\dot{X}_k H_\ell, \dot{X}_k] - \kappa^{-1}[\dot{X}_\ell H_k, \dot{X}_k].\end{aligned}$$

And

$$\begin{aligned}[\dot{X}_k H_\ell, \dot{X}_k] - [\dot{X}_\ell H_k, \dot{X}_k] &= \dot{X}_k H_\ell \dot{X}_k - \dot{X}_k \dot{X}_k H_\ell - \dot{X}_\ell H_k \dot{X}_k + \dot{X}_k \dot{X}_\ell H_k \\ &= \dot{X}_k [H_\ell, \dot{X}_k] - \dot{X}_\ell [H_k, \dot{X}_k] + [\dot{X}_k, \dot{X}_\ell] H_k \\ &= \kappa \dot{X}_k \frac{\partial H_\ell}{\partial X_k} - \kappa \dot{X}_\ell \frac{\partial H_k}{\partial X_k} + [\dot{X}_k, \dot{X}_\ell] H_k.\end{aligned}$$

Thus,

$$\dot{H}_\ell = \kappa^{-1} \epsilon_{jk\ell} [E_j, \dot{X}_k] + \dot{X}_k \frac{\partial H_\ell}{\partial X_k} - \dot{X}_\ell \frac{\partial H_k}{\partial X_k} + \kappa^{-1} [\dot{X}_k, \dot{X}_\ell] H_k.$$

Hence,

$$\frac{\partial H_\ell}{\partial t} = \dot{H}_\ell - \dot{X}_j \frac{\partial H_\ell}{\partial X_j} = \epsilon_{jk\ell} \frac{\partial E_j}{\partial X_k} - \dot{X}_\ell \frac{\partial H_k}{\partial X_k} + \kappa^{-1} [\dot{X}_k, \dot{X}_\ell] H_k.$$

However, the second term on the right-hand side vanishes because $\operatorname{div} H = 0$, and the third term vanishes by symmetry. To see this, note that

$$\begin{aligned}H_1 &= \kappa^{-1} [\dot{X}_2, \dot{X}_2] \\ H_2 &= -\kappa^{-1} [\dot{X}_1, \dot{X}_3] \\ H_3 &= \kappa^{-1} [\dot{X}_1, \dot{X}_2].\end{aligned}$$

Thus,

$$\kappa [\dot{X}_k, \dot{X}_\ell] H_k = [\dot{X}_1, X_\ell][\dot{X}_2, \dot{X}_3] - [\dot{X}_2, \dot{X}_\ell][\dot{X}_1, \dot{X}_3] + [\dot{X}_3, \dot{X}_\ell][\dot{X}_2, \dot{X}_2].$$

This vanishes for $\ell = 1, 2, 3$. Therefore,

$$\frac{\partial H_\ell}{\partial t} = \epsilon_{jk\ell} \frac{\partial E_j}{\partial X_k}.$$

■

This lemma completes the derivation of Maxwell's equations. We have shown that

$$\operatorname{div} H = 0$$

and

$$\frac{\partial H}{\partial t} + \nabla \times E = 0.$$

As Dyson (1990) remarks, the other two Maxwell equations

$$\operatorname{div} E = 4\pi\rho, \quad \frac{\partial E}{\partial t} - \nabla \times H = 4\pi \boldsymbol{j}$$

can be taken to define the external charge and current densities ρ and j. *However*, it is important to realize that our entire theory has applied only to a single trajectory. We can regard this trajectory (and its 'particle') as defining an electromagnetic field, *or* we can regard this particle as moving in an external field with these properties. We cannot have it both ways. The analysis so far in no way takes into account the self-interaction of this particle or its interactions with other particles and fields. Of course, our talk at this stage about the 'trajectory' of a particle is an *analogue* of a physical trajectory. The trajectory we talk about is in the space of $A \times A \times A$, where A denotes the non-commutative operator algebra that underlies the theory. An eventual interpretation of this theory in terms of trajectories in physical space is a possible consequence of further analysis of our formalism. It is beyond the scope of this preliminary paper.

We feel that the foregoing analysis of the Feynman–Dyson derivation in a discrete context lays bare much of the beautiful structure of the electromagnetic formalism and its relation to a condition of discrete time. We hope to probe this structure more deeply in subsequent papers.

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Appendix A. Historical remarks

One of us (HPN) has already claimed that the Feynman proof is *not* paradoxical (Noyes 1991) in the context of the finite and discrete reconciliation between quantum mechanics and relativity (Noyes 1987) achieved by a new fundamental theory (Noyes 1989, 1992, 1994*a*; McGoveran 1989, 1991). Noting that the Feynman postulates,

$$\mathcal{F}_k(x,\dot{x};t) = m\ddot{x}_k, \quad [x_i, x_j] = 0, \quad m[x_i, \dot{x}_j] = \mathrm{i}\hbar\delta_{ij}$$

are independent of or linear in m, we can replace them by the *scale-invariant* postulates

$$f_k(x,\dot{x};t) = \ddot{x}_k, \quad [x_i, x_j] = 0, \quad [x_i, \dot{x}_j] = \kappa\delta_{ij},$$

where κ is any fixed constant with dimensions of area over time $[L^2/T]$ and f_k has the dimensions of acceleration $[L/T^2]$. This step is suggested by Mach's conclusion (Mach 1875) that it is Newton's third law which allows mass *ratios* to be measured, while Newton's second law is simply a *definition* of force. Hence, in a theory which contains only 'mass points', the Newtonian scale invariance of classical MLT physics reduces to the Galilean scale invariance of a purely kinematical LT theory. Breaking scale invariance in such a theory requires not only some unique specification of a particulate mass standard, but also the requirement that this particle have some *absolute* significance.

As has been remarked recently (Noyes 1994*b*), this aspect of scale invariance had already been introduced into the subject by Bohr & Rosenfeld (1933). In their classic paper, they point out that because QED depends only on the universal constants $\hbar$ and c, the discussion of the measurability of the fields can to a large extent be separated from any discussion of the atomic structure of matter (involving the mass and charge of the electron). Consequently, they are able to derive from the *non-relativistic* uncertainty relations the same restrictions on measurability (over finite space-time volumes) of the electromagnetic fields that one obtains directly from the second-quantized commutation relations of the fields themselves. Hence, to the extent that one could 'reverse engineer' their argument, one might be able to get back to the classical field equations and provide an alternative to the Feynman derivation based on the same *physical* ideas.

Turning to the commutation relations themselves, we note that a velocity measurement requires a knowledge of the space interval *and* the time interval between two events in two well separated spacetime volumes. Further, to embed these two positions in laboratory space, we must (in a relativistic theory) know the time it takes a light signal to go to one of these two positions and back to the other via a third reference position with a standard clock. Thus, we need *three* rather than two reference events to discuss the *connection* between position and velocity measurements.

We can then distinguish a measurement of position followed by a measurement of velocity from a measurement of velocity followed by a measurement of position. The minimum value of the difference between the product of position and velocity for measurements performed in the two distinct orders then specifies the constant in the basic 'commutation relation' needed in the Feynman derivation. So long as this value is finite and *fixed*, we need not know its metric value. This specifies what we mean by *discrete physics* in the main body of the text.

Relativity need not change this situation. Specify c in a scale-invariant way as both the maximum speed at which information can be transferred (limiting group velocity) and the maximum distance for supraluminal correlation *without* information transfer (maximum coherence length). If the unit of length is ΔL and the unit of time is ΔT, then the equation $(\Delta L/c\Delta T) = 1$ has a *scale-invariant* significance. Further, the interval I, specified by the equation $c^2\Delta T^2 - \Delta L^2 = I^2$, can be given a *Lorentz-invariant* significance. We can extend this analysis to includes the scale-invariant definition $\Delta E/c\Delta P = 1$ and the Lorentz-invariant interval in energy-momentum space $(\Delta E^2/c^2) - \Delta P^2 = \Delta m^2$ provided we require that $\Delta P \Delta L/\Delta m = \Delta E \Delta T/\Delta m$. Then, given any arbitrary particulate mass standard Δm, mass ratios can be measured using a Lorentz-invariant *and* scale-invariant LT theory. We trust that this dimensional analysis of the postulates used in the Feynman proof already removes part of the mystery about why it works, and suggests how it can be made 'Lorentz invariant' in a finite and discrete sense.

Appendix B. On the form of the derivative

In our discrete ordered calculus (see § 3) we have defined the derivative $\dot{X}$ by the formula $\dot{X} = J(X' - X)$, where J is a formal element satisfying $J' = J$ and $XJ = JX'$ for all X. Thus, we have the equation

$$\dot{X} = JX' - JX = XJ - JX = [X, J].$$

This suggests that the equation $\dot{X} = [X, J]$ is an analogue of the corresponding equation in the Heisenberg formulation of quantum mechanics:

$$\dot{X} = [X, U],$$

where U is the time-evolution operator.

Michael Peskin has pointed out to us (personal communication) that we could accomplish the discretization of time in our theory by taking $U = \mathrm{e}^{-\mathrm{i}H\Delta t}$ (formally), where Δt denotes a discrete timestep. Then $\dot{X} = [X, U]$ and $X' = U^{-1}XU$ serves to define the time step from X to X'. Our approach and Peskin's meet if we identify J and U! The physical interpretation of this identification deserves further investigation.

In any case, it is interesting to note that the differentiation formula $(XY)^{\cdot} = \dot{X}Y + X\dot{Y}$ follows directly from the formula $\dot{X} = [X, J]$ without the necessity of introducing the step X'. This relationship between the discrete ordered calculus and the algebra of commutators will be used in the next installment of our work.

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