Wash. Univ. 4 Oct. 2001

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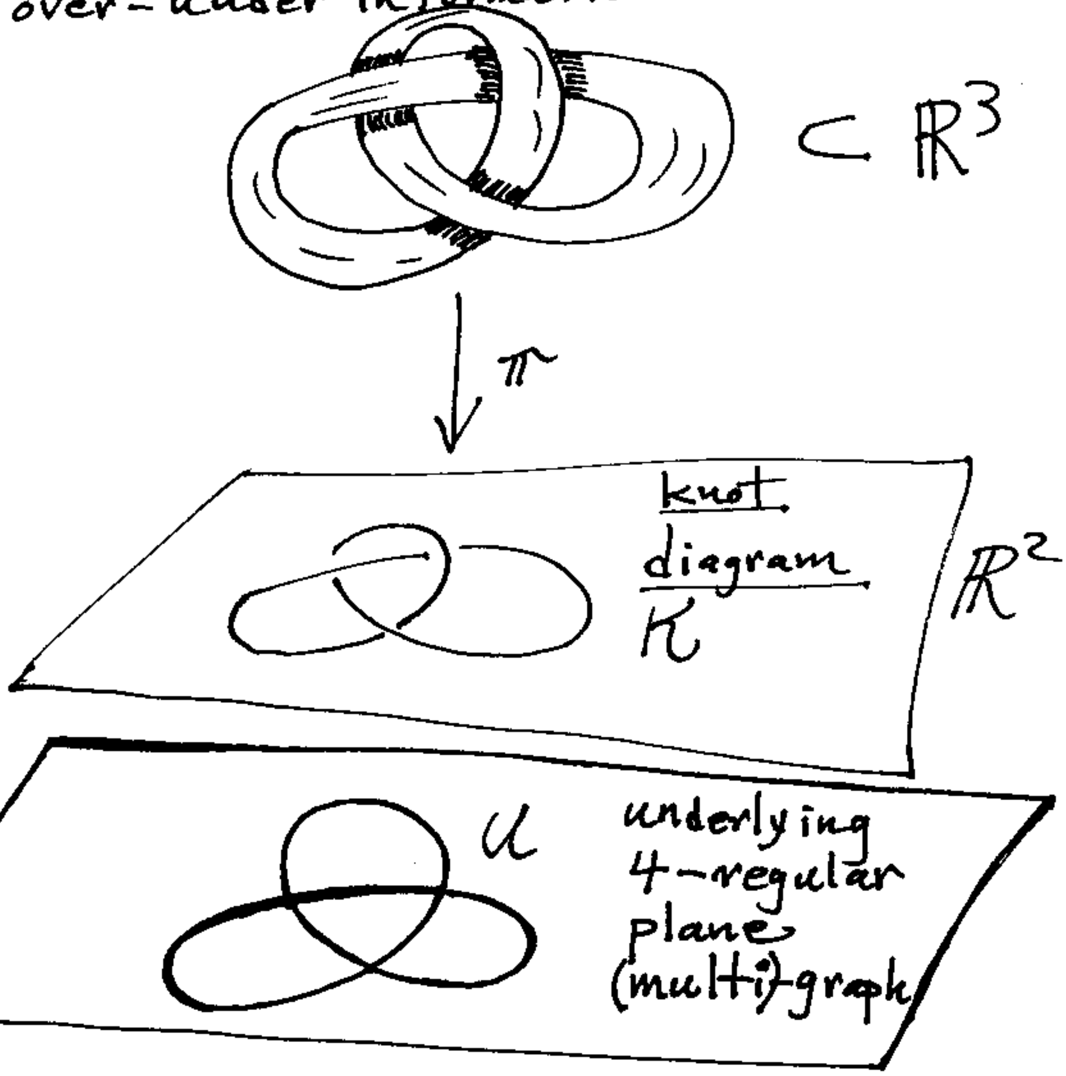
Functional Integration Without Integration

I. Background on Knot Theory and Vassiliev Invars

Knots: isotopy classes of smooth embeddings of circles in $\mathbb{R}^3$.

Links: isotopy classes of smooth embeddings of disjoint unions of circles in $\mathbb{R}^3$.

Knot and Link Diagrams: Project embedding into $\mathbb{R}^2$ with transverse double points, keep over-under information.

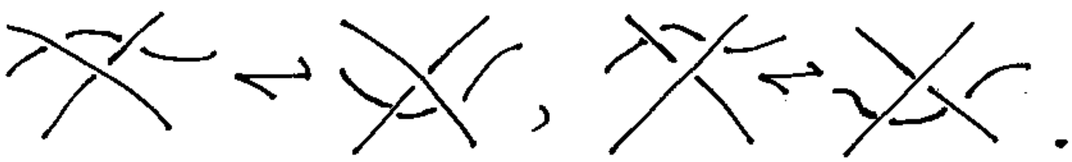


Reidemeister Moves

≂. 

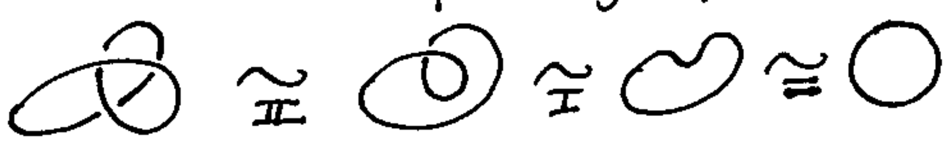
I. 

II. 

III. 

(≂ is "0" in the LK revised system of Roman Numerals.)

Reidemeister showed the the equivalence relation on diagrams generated by these moves is the same as ambient isotopy of knots and links in $\mathbb{R}^3$. Thus knot theory (i.e. theory of knots and links) can be formulated as a combinatorial theory of moves on decorated plane graphs.

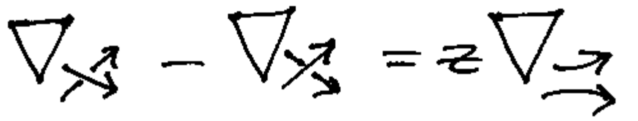
e.g. 

exercise: 

← unknot by Reidemeister moves.

exercise: 

The Alexander – Conway Polynomial (≈ 1969)

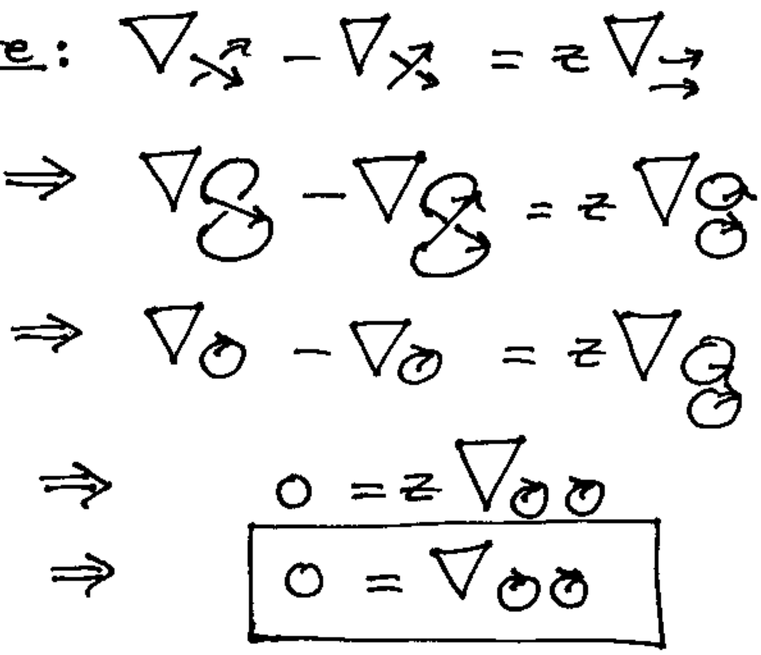


Conway Skein Identity

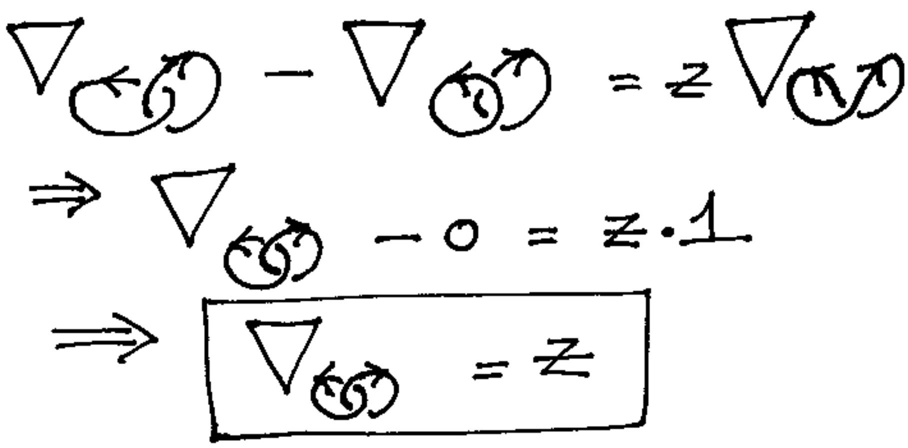
$\nabla_K = \nabla_{K'}$ <u>if</u> $K \sim K'$ (not conversely)

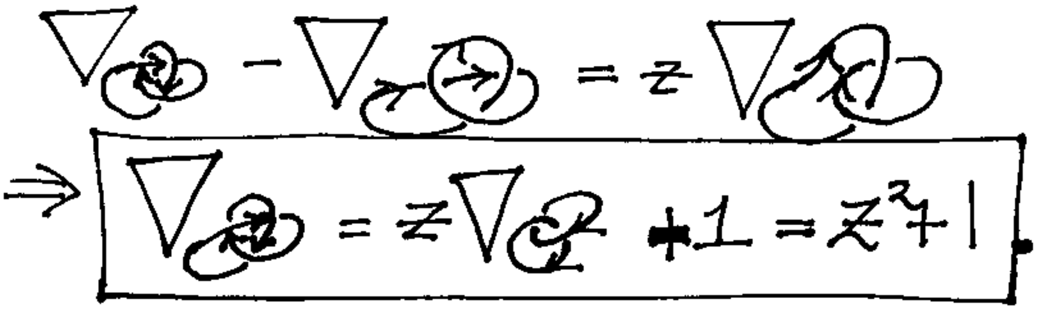
 $= 1$, $\nabla_K \in \mathbb{Z}[z]$.

<u>Note</u>:



Alexander's original polynomial was published in 1928. Conway discovered a reformulation that could be computed recursively via the diagrams.





The Jones Polynomial (1984)

(V.F.R. Jones)

$$t^{-1}V - tV = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)V$$

$$V_K = V_{K'} \text{ if } K \sim K'$$

$$V = 1, \quad V_K \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$$

Note: $t^{-1}V - tV = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)V$

$$\Rightarrow (t^{-1} - t)V = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)V$$

$$\Rightarrow \boxed{V = -\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)}$$

$$t^{-1}V - tV = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)V$$

$$\Rightarrow t^{-1}V - t\left(-\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)\right) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)$$

$$\Rightarrow \boxed{V = -\sqrt{t}\,(t^2 + 1)}$$

$$t^{-1}V - tV = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)V$$

$$t^{-1}V - t = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)\left(-\sqrt{t}\,(t^2+1)\right)$$

$$\boxed{V = -t^4 + t^3 + t^{-1}}$$

Comment on the Jones Poly.

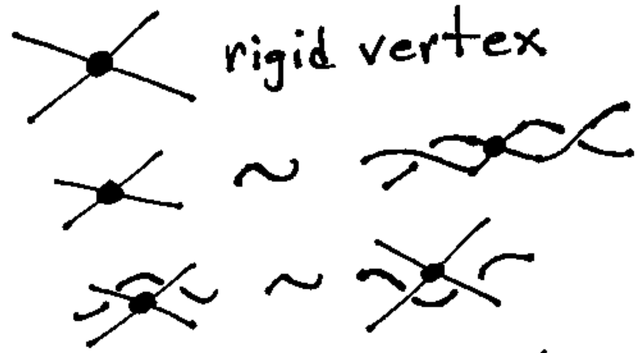
$$V_{K^*}(t) = V_K(t^{-1})$$ where K^* denotes the mirror image of K.

Here
$$\left.\begin{array}{l} V_K = -t^4 + t^3 + t^{-1} \\ V_{K^*} = -t^{-4} + t^{-3} + t \end{array}\right\} \Rightarrow K \not\approx K^*$$

Open Problem.: Does $V_K(t)$ detect knots?

Extending Invariants of Knots and Links to Invariants of Embedded (4-regular) graphs)

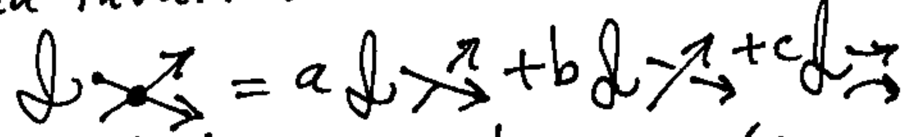
rigid vertex



Note:

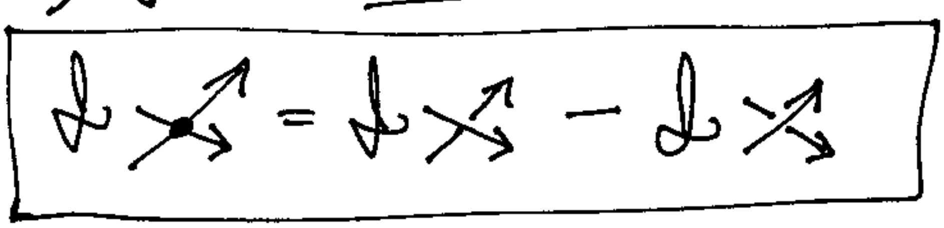
Rigid vertices can be modeled by rigid disks with flexible strings attached to their edges.

Suppose ϑ is an invariant of knots & links. Then



defines an invariant of rigid vertex graphs.

We say ϑ is a Vassiliev Invariant if



$\mathcal{V}$ is a Vassiliev invariant of <u>finite type n</u> if $\mathcal{V} = \mathcal{V} - \mathcal{V}$

<u>and</u> $\mathcal{V}\,G = 0$ for all graphs G with $\#G = \#\text{nodes of } G > n$.

<u>Example 1</u>. $\nabla_K(z) = a_0 + a_1 z + a_2 z^2 + \cdots$

$$\nabla = \nabla - \nabla = z\nabla$$

$\Rightarrow z^K \mid \nabla_G$ if $\#G \geq K$.

$\Rightarrow a_n(G) = 0$ if $\#G > n$

$\Rightarrow$ $\boxed{a_n \text{ is Vassiliev of type } n}$.

The coefficients of the Alexander–Conway Polynomial are Vassiliev invariants of finite type.

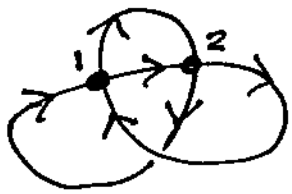
<u>Example 2</u>. $V_K(e^x) = \sum_{n=0}^{\infty} v_n(K)\, x^n/n!$

A similar argument (due originally to Birman + Lin) shows that $v_n(K)$ is Vassiliev of type n. The coefficients of the Jones polynomial are Vassiliev invariants of finite type.

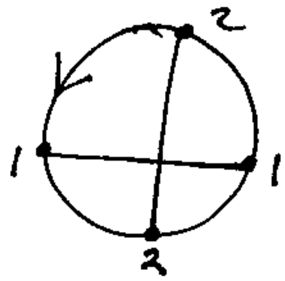
<u>Fact</u>: If $\mathcal{V}$ is Vassiliev of finite type n, then $\mathcal{V}G$ is <u>independent of the embedding of G</u> if $\#G = n$.

<u>Pf</u>: $\mathcal{V}\underbrace{\cdots}_{n} - \mathcal{V}\underbrace{\cdots}_{n} = \mathcal{V}\underbrace{\cdots}_{n+1}$

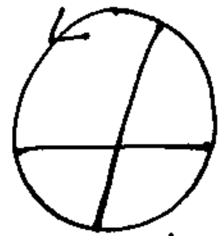
$= 0$ //



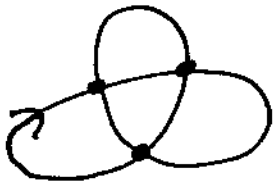
graph ①

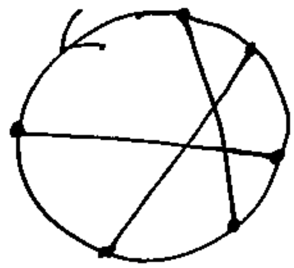


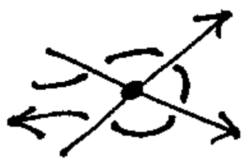
or



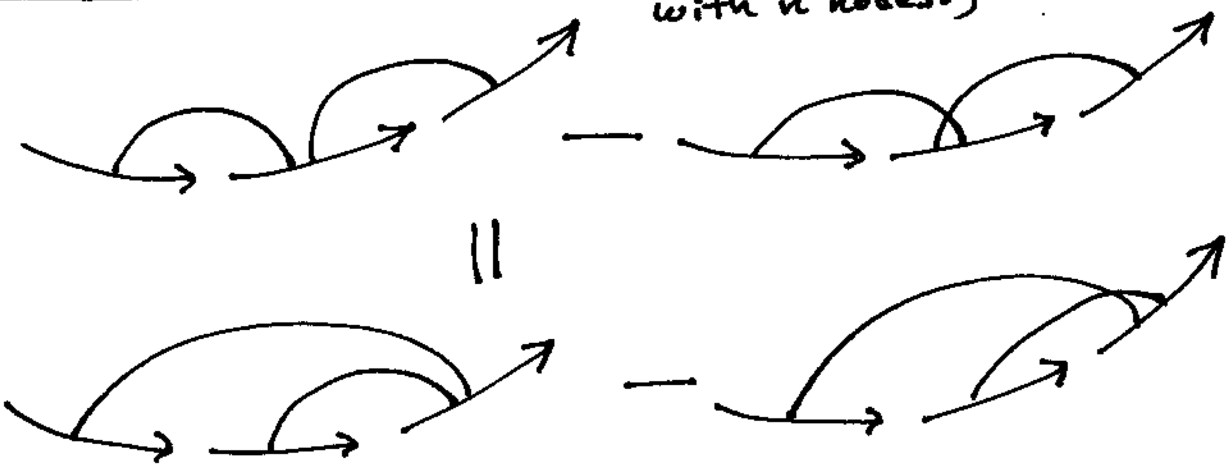
chord diagram corresponding to ①.

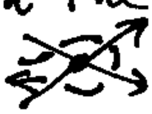
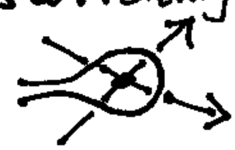




The isotopy  $\sim$ 

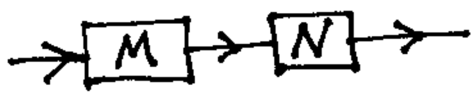
implicates the following $\boxed{\text{4-term relation}}$ on chord diagram evaluations for Vassiliev invariants: (For evaluations of type n invariants on graphs with n nodes.)



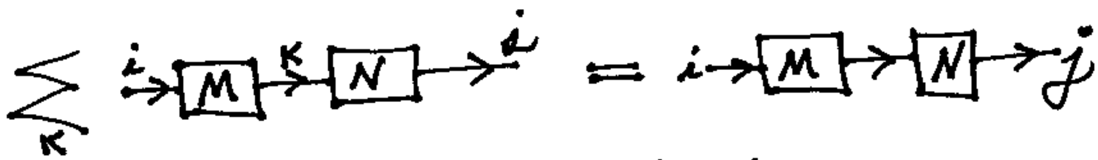
Proof: Write down the four switching equations to transform  to . Add them up! //

Diagramatic Matrix Multiplication

$M = (M_{ij})$, $N = (N_{kl})$



stands for MN

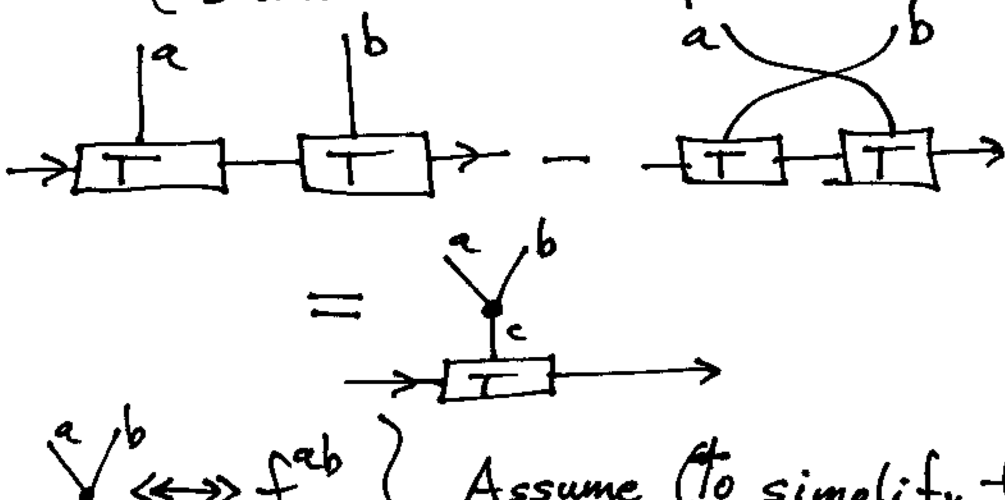


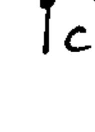
$$\sum_k M_{ik} N_{kj} = (MN)_{ij}$$

Closure in a matrix Lie algebra:

$$T^a T^b - T^b T^a = f^{ab}_{\ \ c} T^c$$

(summation on repeated indices)



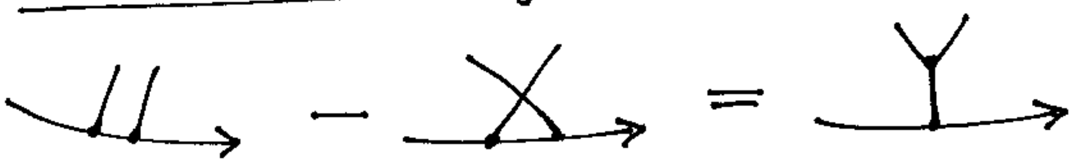
 $\Longleftrightarrow f^{ab}_c$ Assume (to simplify the discussion) that ~~f~~ f^{ab}_c invariant under cyclic permutation of a, b, c.

Example: $\underbrace{T^1 = \frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix},\ T^2 = \frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix},\ T^3 = \frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{su(2)\ \text{Lie Algebra}}$

$$f_c^{ab} = i\,\varepsilon_c^{ab} = i\,\varepsilon_{abc}$$

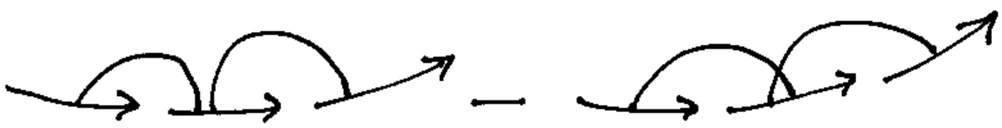
where $\varepsilon_{abc} = \begin{cases} \text{sign of } (abc) \text{ if } abc \text{ is a permutation of } 123. \\ 0 \quad \text{else.} \end{cases}$

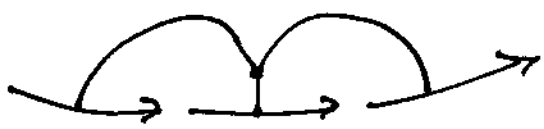
Abstract Lie Algebra Pattern



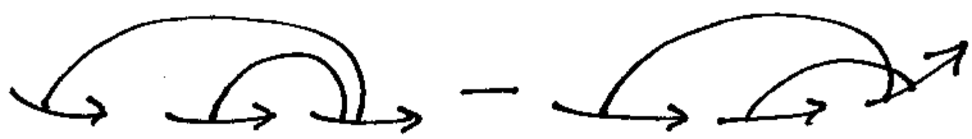
$\Downarrow$

$\boxed{\text{4 Term Relation}}$

Proof: 

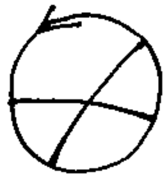
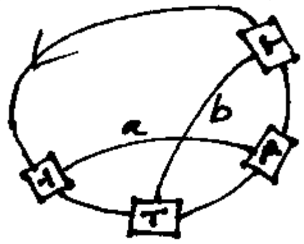
= 

=  (invar under cyclic perm)

= 

QED

Conclusion: Lie algebras can be used to create the weights for chord diagrams for Vassiliev invariants.

wt  = 

$$= \mathrm{tr} \sum_{a,b} \left(T^a T^b T^a T^b \right).$$

Theorem (Kontsevich). Each weight system satisfying the 4 Term Relation (and minor conditions not mentioned here) extends to make a Vassiliev invariant of finite type. Hence Lie algebra representations imply link invariants!

The heuristic background for Kontsevich's Theorem is the approach to link invariants via functional integration due to Edward Witten. This is the subject of the second part of this talk.

II. Integration & Functional Integration

A) Recall: If $Z = \int_{-\infty}^{\infty} e^{-x^2/2} dx$

then $$Z^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2-y^2}{2}} dx\,dy$$

$$= \int_0^{\infty}\int_0^{2\pi} e^{-R^2/2} R\,d\theta\,dR$$

$$= 2\pi \int_0^{\infty} e^{-R^2/2} d(R^2/2)$$

$$= 2\pi \int_0^{\infty} e^{-s} ds = 2\pi \left(-e^{-s}\right|_0^{\infty}$$

$$Z^2 = 2\pi$$

Hence $$\boxed{\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}}.$$

Then $$\int_{-\infty}^{\infty} e^{-x^2/2 + Jx} = Z(J)$$

$$\parallel$$

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2-2Jx+J^2)+J^2/2}$$

$$\parallel$$

$$e^{J^2/2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-J)^2}$$

$$\parallel$$

$$e^{J^2/2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}$$

$$\parallel$$

$$\boxed{Z(J) = e^{J^2/2} Z(0)}$$

We also get

$$\left.\frac{d^n Z(J)}{dJ^n}\right|_{J=0} = \left.\frac{d^n}{dJ^n}\int_{-\infty}^{\infty} e^{-x^2/2 + Jx}\,dx\right|_{J=0}$$

$$= \int_{-\infty}^{\infty} x^n e^{-x^2/2}\,dx$$

Hence $\displaystyle\int_{-\infty}^{\infty} x^n e^{-x^2/2}\,dx = \sqrt{2\pi}\,\left.\frac{d^n e^{J^2/2}}{dJ^n}\right|_{J=0}$

and so on ... $\longrightarrow$

But suppose you did not know how to integrate!

<u>Definition</u>. Let $f(x)$ and $g(x)$ and $h(x)$ be functions <u>rapidly vanishing at ∞</u>. ($f(x) \to 0$ as $x \to \infty$ and $d^n f/dx^n \to 0$ as $x \to \infty$ $\forall n = 1, 2, 3, \ldots$). We will write $f \downarrow 0$ as $x \to \infty$. Then we say that $f(x)$ is integrally equivalent to $g(x)$ ($f \sim g$) if

$$\boxed{f(x) - g(x) = dh(x)/dx}$$

for some $h \downarrow 0$ as $x \to \infty$.

We also assume that (like $e^{-x^2/2}$) the functions f, g, h have everywhere convergent Taylor series.

<u>Note</u>: $\frac{df}{dx}(x) \downarrow 0$ as $x \to \infty \Rightarrow \int_{-\infty}^{\infty} \frac{df}{dx}\,dx = f(\infty) - f(-\infty) = 0$.

Thus $f \sim g \Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dx$.

Remember, we assume no integration neccesarily exists.

<u>Definition</u>. Let $\mathcal{F}_c$ be the class of functions described in the previous definition. Let $\int f$ denote the integral equivalence class of $f \in \mathcal{F}_c$.

<u>Lemma</u>. 1) $f \sim g \Rightarrow \int f = \int g$
and if $\exists$ an actual integral $\int_{-\infty}^{\infty}$
then $f \sim g \Rightarrow \int_{-\infty}^{\infty} f = \int_{-\infty}^{\infty} g$.

2) $g(x) = f(x+J)$ for a constant J,
then $\int g = \int f$. 3) $\int Df = 0$
$(D = d/dx)$

<u>Proof</u>. 1) tautology + second part already done.

2) $$f(x+J) = f(x) + f'(x)J + f''(x)\frac{J^2}{2} + \cdots$$
$$= f(x) + D\left[f(x) + f'(x)\frac{J^2}{2} + \cdots\right]$$
$$\Rightarrow f(x+J) \sim f(x)$$
$$\Rightarrow \int f(x+J) = \int f(x) \quad //$$

<u>WE NOW SEE THAT MOST OF THE CALCULATIONS ABOUT $e^{-x^2/2}$ were about the integral classes of functions:</u>

$$e^{-\frac{x^2}{2}+Jx} = e^{-\frac{(x-J)^2}{2}} e^{J^2/2} \sim e^{J^2/2} e^{-x^2/2}$$

$$\Rightarrow \boxed{\int e^{-\frac{x^2}{2}+Jx} = e^{J^2/2} \int e^{-x^2/2}}$$

When we write $\int e^{-\frac{x^2}{2}+Jx} = e^{J^2/2}\int e^{-x^2/2}$,

we are making a statement of equivalence based on the equivalence $e^{-\frac{(x-J)^2}{2}} \sim e^{-x^2/2}$ that throws away the many derivatives of $e^{-x^2/2}$ from the Taylor series for $e^{-\frac{(x-J)^2}{2}}$. These do not contribute to any actual integral of the function from $-\infty$ to ∞.

The reason we are concerned about creating a theory of "integration without integration" is because there is a fundamental and undefined integral in back of the theory of Vassiliev invariants. It is

B) Witten's Functional Integral (Without Integration)

Witten (1989) suggested

$$Z_K = \int \mathcal{D}A \; e^{-\frac{ik}{4\pi}\int_{\mathbb{R}^3} tr(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)} \; \underbrace{tr(\mathbb{P}e^{\oint_K A})}$$

Knot $\subset \mathbb{R}^3$ (or a link)

hypothetical integral over all gauge fields $A(x) = A_i^a(x) T_a dx^i$

Chern-Simons 3-form

"Wilson Loop"

We will explain these terms and analyze Z_K in the context of integration without integration.

The main mathematical problem associated with this functional integral is that there is no known measure theory on the space of all gauge connections that will bring it into existence (There are some partial solutions to the problem.). For this talk, we will adopt a generalization of the previous "integration without integration" dictum.

<u>Namely</u>: Let $\psi(A)$ and $\psi'(A)$ be functions of a gauge connection

$A(x) = A_i^a(x) T_a dx^i$

($A_i^a(x)$ is a C^∞ fcn of $x \in \mathbb{R}^3$ with values in $\mathbb{R}^3$ for $i = 1, 2, 3$ + $a = 1, 2, \ldots, d$ where $\{T_1, T_2, \ldots, T_d\}$ is a matrix Lie algebra basis.). We shall say that $\psi(A)$ and $\psi'(A)$ are <u>functionally equivalent</u> written

$$\psi(A) \sim \psi'(A)$$

if

$$\boxed{\psi - \psi' = \mathbb{D}\Phi}$$

where $\mathbb{D}$ denotes $\dfrac{\delta}{\delta A_i^a(x)}$

a <u>functional derivative</u> with respect to some choice of $a \in \{1, \ldots, d\}$, $i \in \{1, 2, 3\}$ and $x \in \mathbb{R}^3$.

We restrict the functions ψ of gauge connections A to a space $\mathcal{F}$ of functions of gauge connections "rapidly vanishing at ∞" where this means that $\psi(A) \longrightarrow 0$ as $\sum_{i,a} |A_i^a(x)|^2 \longrightarrow \infty$ for any $x \in \mathbb{R}^3$ and the same is true for all $\mathbb{D}\psi$ and finitely iterated functional derivatives as well.

For example $e^{\int_{\mathbb{R}^3} A \wedge dA}$, being essentially quadratic in A, has this property, as does

$$\boxed{\psi(A) = e^{\frac{ik}{4\pi}\int_{\mathbb{R}^3} tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)} \, tr(\mathbb{P} e^{\oint_K A}).}$$

We the define the free integral of $\psi(A)$ for $\psi \in \mathcal{F}$ to be its functional equivalence class:

$$\int \psi(A) = \sim \text{class of } \psi(A).$$

The rest of the talk will analyze the functional class of Witten's functional: $\psi(A)$.

Various Background Remarks

1° Functional Derivatives

$$\lim_{\varepsilon\to 0} \frac{F(a(x)+\delta(x_0)\varepsilon) - F(a(x))}{\varepsilon} = \frac{\delta F}{\delta a(x_0)}$$

where $\delta(x_0)$ = Dirac delta fcn concentrated at x_0.

e.g. $F(a(x)) = \int_a^b a(x)^2 dx$, $x_0 \in [a,b]$

$$\Rightarrow \frac{\delta F}{\delta a(x_0)} = \int_a^b 2a(x)\,\delta(x_0)\,dx = 2a(x_0).$$

2° Wilson Loops

$$W_K(A) = tr\left(\mathbb{P}\, e^{\oint_K A}\right)$$

$\boxed{tr = \text{matrix trace}}$

$\mathbb{P}$ refers to path-ordering since A is a <u>matrix-valued one-form</u>.

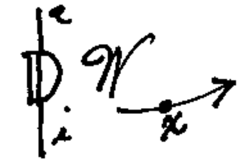
One way to define this is as:

$$W_K(A) \underset{def}{=} tr\prod_{x\in K}(\mathbb{1}+A(x))$$

$$\left(\prod_{x\in K} = \lim_{n\to\infty}\prod_{i=1}^{n} \text{ where } x_1,\dots,x_n \text{ is a partition of } K\right)$$

Then $\boxed{W_K(A) = tr\prod_{x\in K}\left(\mathbb{1}+A_i^a(x)\,T_a\,dx^i\right)}$

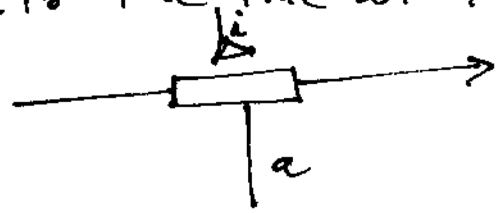
Let  $= \dfrac{\delta}{\delta A^a_i(x)}$, then

 $W_K(A) =$  $=$ 

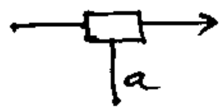
since $\dfrac{\delta}{\delta A^b_i(x_0)} \operatorname{tr} \prod_{x\in K} \left(\mathbb{1} + A^a_i(x) T_a dx^i\right)$

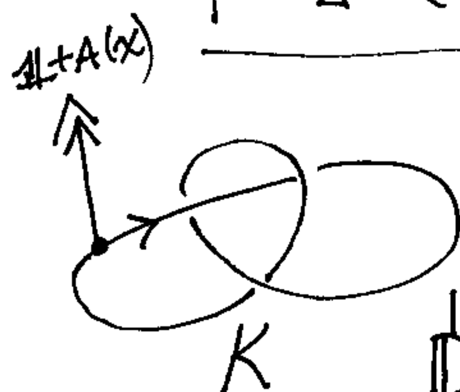
$$= \operatorname{tr} \prod_{x<x_0} \left(\mathbb{1}+A(x)\right) \underbrace{\left[T_a dx^i\right]}_{\text{at } x_0} \prod_{x>x_0} \left(\mathbb{1}+A(x)\right)$$

We indicate this insertion of a Lie algebra matrix into the line at x_0 by

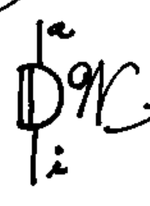
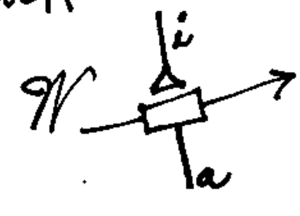


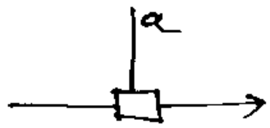
where

	$\Longleftrightarrow T_a$
	$\Longleftrightarrow dx^i$

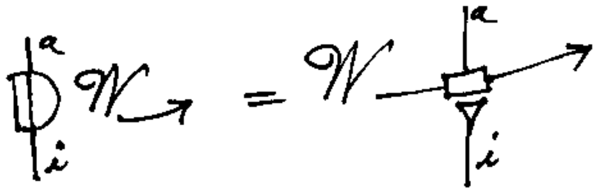


$$W_K(A) = \cancel{\operatorname{tr}}\prod_{x\in K} \left(\mathbb{1} + A(x)\right)$$

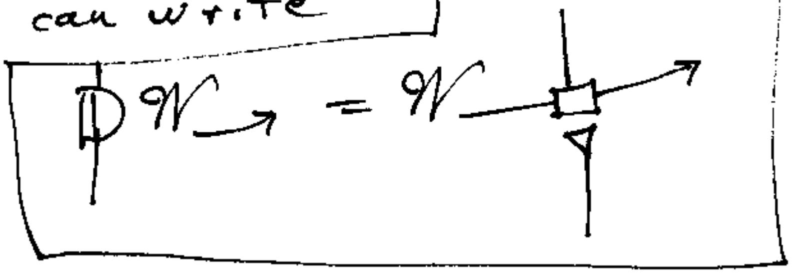
 $=$ 

Notation: We will write  for T_a

 for dx^i

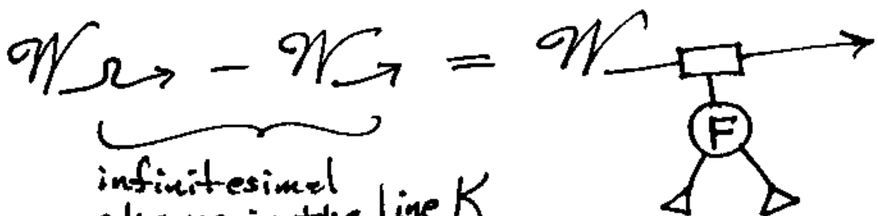
and rewrite 

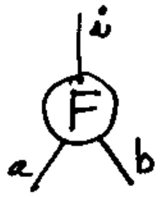
so that we can write



with up and down lines coinciding.

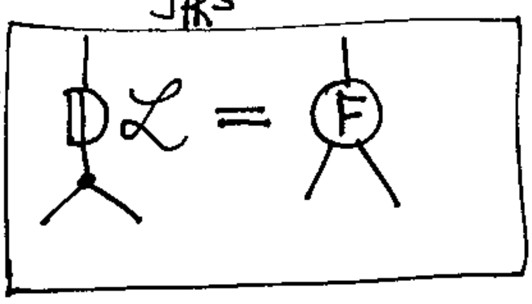
3° Curvature Tensor



 $= F_{ab}^{\;\;\;i}$ curvature tensor

4°

$$\mathcal{L} = \int_{\mathbb{R}^3} tr\left(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A\right)$$

$\Rightarrow$ 

where  $= \varepsilon_{abc}$

$$\left(\begin{array}{l} \varepsilon_{123} = +1 \\ \varepsilon_{213} = -1 \\ \varepsilon_{113} = 0 \\ \text{etc.} \end{array}\right)$$

Curvature Tensor as
Functional Deriv of $\mathcal{L}$

5° <u>Note</u>: $\mathbb{D}(\cancel{A}\psi') = (\mathbb{D}\psi)\psi' + \psi(\mathbb{D}\psi')$

$$\Rightarrow \int(\mathbb{D}\psi)\psi' + \int\psi(\mathbb{D}\psi') = 0$$

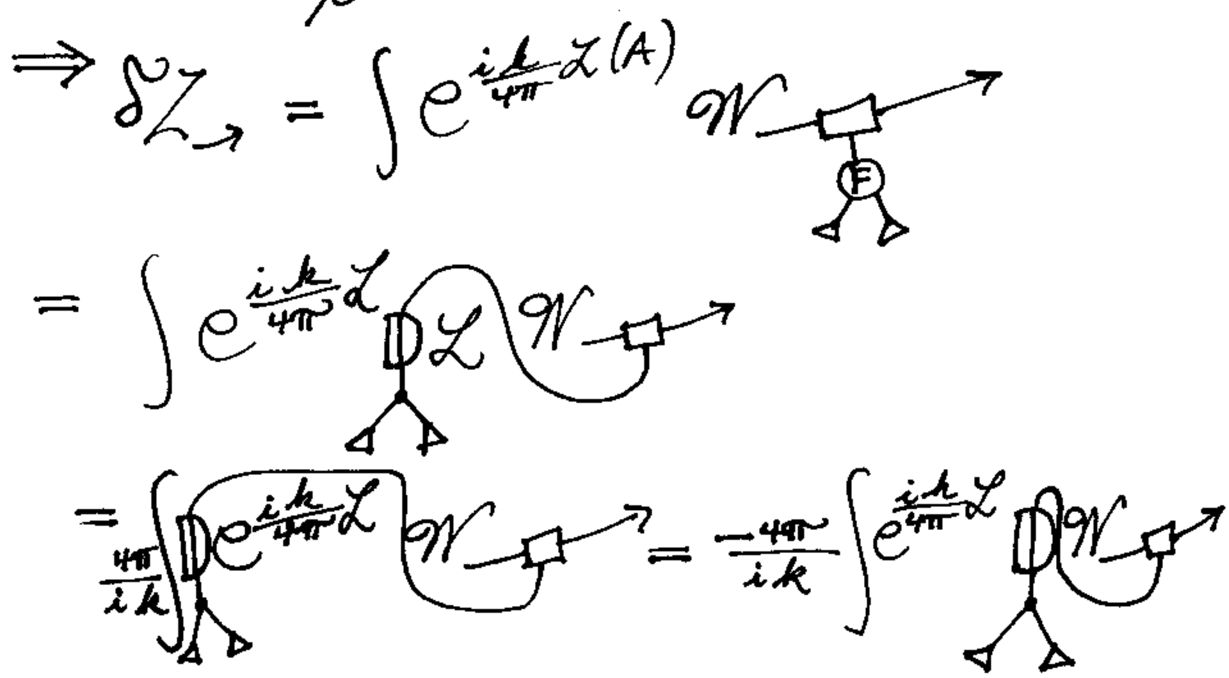
where 0 denotes the equivalence class of zero.

6° Let $Z_K = \int \psi(A)$ where

$$\psi(A) = e^{\frac{ik}{4\pi}\mathcal{L}(A)}\,\mathcal{W}_K(A)$$

$$\mathcal{L}(A) = \int_{\mathbb{R}^3} tr\left(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A\right).$$

$$\delta Z_{\nearrow} \underset{df}{=} Z_{\nearrow} - Z_{\nearrow}$$

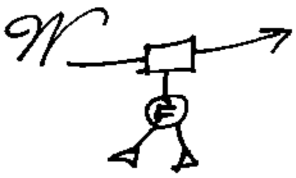


So far we have

$$\delta Z_{\to} = -\frac{4\pi}{ik}\int e^{\frac{ik}{4\pi}\mathcal{L}}$$ 

Without integration symbology, this is the statement of functional equivalence:

$$\delta\psi = \psi_{\rightsquigarrow} - \psi_{\to} = e^{\frac{ik}{4\pi}\mathcal{L}} W_{\rightsquigarrow} - e^{\frac{ik}{4\pi}\mathcal{L}} W_{\to}$$

$$= e^{\frac{ik}{4\pi}\mathcal{L}} W$$ 

$$\sim \frac{4\pi i}{k} e^{\frac{ik}{4\pi}\mathcal{L}}$$ 

Continuing, we must differentiate the Wilson loop, and this results in a Lie algebra insertion  exactly where depending upon the choice of deformation. Generally, we get:

$$\delta Z_{\to} = \frac{4\pi i}{k}\int e^{\frac{ik\mathcal{L}}{4\pi}}$$ 

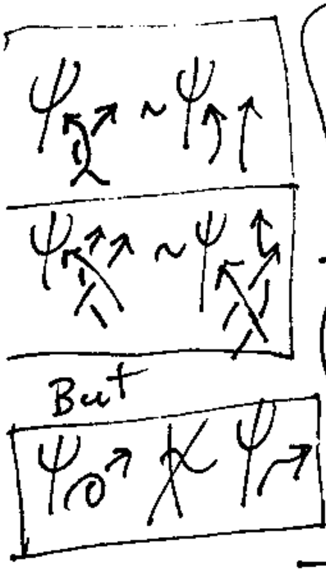
$$\delta Z_{\to} = \frac{4\pi i}{k}\int e^{\frac{ik}{4\pi}\mathcal{L}}$$ 

 $= \varepsilon_{ijk}\, dx^i dx^j dx^k$

is a volume form.

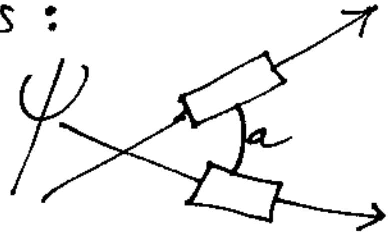
This means that if there is no volume swept by ($\leftarrow\!\!\dashrightarrow$) then $\delta Z = 0$. Whence

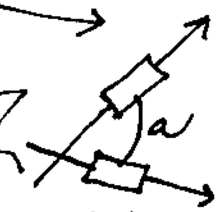
$$\psi(A) \sim \psi .$$



In other words, the equivalence class of $\psi_K(A)$ is invariant under "flat" topological deformations of K. This is <u>what we mean by</u> $\int \psi = \int \psi$ or $\delta \int \psi = 0$.

A similar analysis, attending to the local geometry yields:

$$\psi = \psi - \psi \sim \sum_a \frac{c}{k}\, \psi$$ 

So $Z = Z - Z = \sum_a \frac{c}{k} Z$ 

This is the integral analogue for the Lie algebra weights formula for Vassieliv invariants.

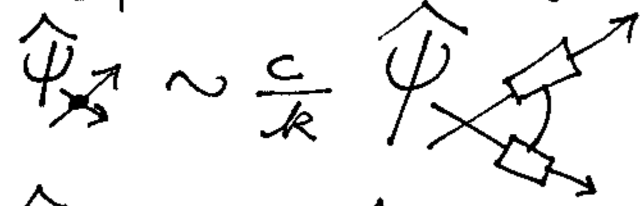
In fact, if we write

$$\psi_K(A) = e^{\frac{ik}{4\pi}\mathcal{L}(A)}\,\mathcal{N}_K(A)$$

and replace A by $A/\sqrt{k}$, then we can write

$$\hat{\psi}_K(A) = e^{\frac{i}{4\pi}\int tr(A\wedge dA)}\, e^{\frac{i}{4\pi\sqrt{k}}\int tr(A\wedge A\wedge A)}\,\mathcal{N}_K(A/\sqrt{k})$$

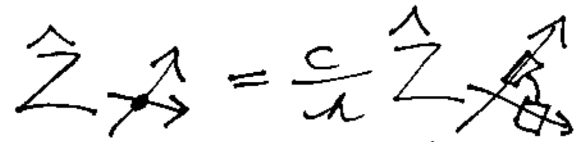
and expand this as a power series in inverse powers of k:

Then 

$$\hat{\Psi} = \sum_n \frac{1}{k^n}\hat{\Psi}_n$$

$\Longrightarrow$ $\boxed{\text{up to} \sim \hat{\Psi}_n \text{ is Vassiliev of type } n}$

<u>and</u> $\hat{Z}_n$ <u>is</u> Vassiliev of type n with values in the equivalence class.

 tells us that the weight system of this Vassiliev invariant is exactly the weight system we discussed in the first part. (Via leading terms of the $\frac{1}{k}$ expansion of the Wilson loop.)

$\boxed{\text{More in a paper soon to be completed.}}$ LK.