CYCLIC BRANCHED COVERS
O(n)-ACTIONS
AND HYPERSURFACE SINGULARITIES

by

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INTRODUCTION

This research was motivated by a number of specific problems and by some fascinating phenomena. One problem was a feeling of dissatisfaction with the Pham calculation for the intersection matrix of a Brieskorn variety; the method was simple enough, but unilluminating. Then there were a number of periodicity phenomena discovered by Allan Durfee. These were periodicities, in k, in the lists of Brieskorn varieties of the form $\Sigma(k,a_1,a_2,\ldots,a_m)$. One wished for contexts in which to better understand them. The intersection form and its patterns are closely related to any periodicity in the corresponding varieties and, hence, these two problems are intertwined.

One of the most interesting examples of periodicity occurs for the varieties $\Sigma(k,2,2,\ldots,2)$ and these have natural orthogonal group actions. They are $O(n)$-manifolds in the sense of Jänich and have orbit space D^4 with the fixed point set corresponding to a $(2,k)$ torus link in $S^3 = \partial D^4$. On noting this, one might think that the periodicity would find its explanation in some results about actions of the orthogonal group. In fact, there was a theorem due to Hirzebruch which related the signature of a knot in S^3 with the corresponding $O(n)$-manifold. This led to the natural question: How can one classify $O(n)$-manifolds whose fixed points correspond to links in S^3? It then turned out that Hirzebruch's theorem did generalize to the link case. Using Hirzebruch's techniques in combination with Durfee's results about BP_n (BP_n is the set of diffeomorphism classes of (n-2)-connected

(2n-1)-manifolds bounding (n-1) connected parallelizable manifolds), one obtains a classification of these link-manifolds in terms of invariants of links. Letting B_{2n} denote the set of link manifolds of dimension 2n-1, we find that $B_{2n} \subset BP_{2n}$ for $n > 2$ and that $BP_{4k} \equiv B_{4k}$ for $k > 1$. Thus, in the signature case every manifold in BP_{4k} comes from a link.

One does gain some insight upon viewing the periodicity from this standpoint, since it then corresponds to periodic behavior in the invariants of (2,k) torus links. It is possible to see how this arises from the geometry of the link itself.

Other aspects of three dimensional geometry are reflected in the structure of the link manifolds. For example, a link manifold depends upon a link in S^3 with a specified orientation, that is, an orientation for each of its components. Changing the orientations of one or more components may change the link manifold radically. There is an example of a two-component link such that one orientation gives $S^{n-1} \times S^n$ as the corresponding link manifold, while reversing the orientation of a single component yields $\Sigma \# (S^{n-1} \times S^n)$, where Σ is an exotic sphere. Equivariant classification of link manifolds is directly connected with the symmetries of links. There are many interesting relationships between the three dimensional topology of links and the structure of link manifolds.

Another way of studying the periodicity is to regard $\Sigma(k,a_1,\ldots,a_n)$ as a k-fold branched cover of a sphere, with branch set $\Sigma(a_1,\ldots,a_n)$. The structure of branched covers should illuminate the periodicity. In part, this was Durfee's approach.

Let $K^{2n-1} \subset S^{2n+1}$ be a submanifold of S^{2n+1}. If K is the boundary of $F^{2n} \subset S^{2n+1}$, then, classically, one constructs a cyclic branched cover by splitting S along F, taking k copies of the split manifold, and cyclically pasting them together to form the cover M_k. Now the covers which arise from algebraic hypersurfaces are themselves boundaries in a natural way. One's feeling was that this property was independent of the algebraic nature of these coverings. In fact, this is the case and one can construct N_k such that $\partial N_k = M_k$. The intersection form for N_k is computable. It depends entirely on the Seifert pairing $\theta: H_n(F) \times H_n(F) \to Z(\theta(x,y) = \ell(i_*x,y)$ where ℓ denotes linking numbers in S and i_*x is x pushed into the complement of F). When the N_k are parallelizable, there is a periodicity theorem in the general context of branched covers.

This observation about the structure of N_k led back to the original problem about the intersection matrix for a variety. A Brieskorn variety can be considered as an iterated sequence of branched covers. Each cover is embedded in a sphere so that the next cover can be formed. If the intersection form corresponding to a given cover depends on embedding information about the branch set, why not proceed inductively? This can be done and we can determine the intersection form for $x_1^{a_1} + \ldots + x_n^{a_n} + f(z)$. The result tallies with the approach via link manifolds when $z = (z_1, z_2)$, and suggests a generalization of the results about O(n)-actions to O(n)-manifolds with higher dimensional orbit spaces.

Beyond periodicity, I think that one theme stands out from all of this. Whether considering actions of $G = Z_k$ or $G = O(n)$ we found that the structure of the G-manifolds depended strongly on embedding invariants of the fixed point set in the orbit space. One feels compelled to conjecture that everything we have said is contained in an elegant general theory of G-manifolds.

CHAPTER 0

LINKING NUMBERS AND QUADRATIC FORMS

This chapter compiles some background information which will be used throughout the rest of the paper.

1. Linking Numbers and Intersection Numbers

Suppose that $a, b \subset S^{2n+1}$ are disjoint unknotted embedded n-spheres. Assume orientations are chosen for a, b, and S^{2n+1}. We define the linking number of a and b to be $\ell(a,b) = \langle ca, cb \rangle$ where ca and cb are the radial cones in D^{2n+2} with apex the origin $o \in D^{2n+2}$. $\langle \; , \; \rangle$ denotes intersection number in D^{2n+2}. Note that this is equivalent to defining $\ell(a,b) = \langle A, b \rangle$ where A is an (n+1) chain on S, $\partial A = a$, and $\langle \; , \; \rangle$ denotes intersection number in S^{2n+1}. Note also that $\ell(a,b) = (-1)^{n+1}\ell(b,a)$.

2. Linking Invariant

Given an odd dimensional oriented manifold K^{2n+1} there is a pairing $\Lambda: \tau H_n(K) \times \tau H_n(K) \to Q/\mathbb{Z}$ where $\tau H_n(K)$ = the torsion subgroup of $H_n(K)$. Λ is defined classically [see 35] as follows: Let $u, v \in \tau H_n(K)$. Say that $x, y \in C_n(K)$ represent u and v respectively. Then for some $r \in \mathbb{Z}$, $ru = 0$, hence, $rx = \partial Z$, $Z \in C_{n+1}(K)$. Then $\Lambda(u,v) = \frac{1}{r}\langle Z, Y \rangle$ in $Q/\mathbb{Z}$. Here $\langle \; , \; \rangle$ denotes intersection number in K.

Notation. $\langle \; , \; \rangle$ always denotes intersection numbers somewhere.

Where should always be clear from context.

I) Suppose $K^{2n+1} = \partial M^{2n+2}$ where M is an n-connected manifold, oriented. K is its oriented boundary and K is (n-1)-connected. Then the exact squence for the pair (M, ∂M) becomes

$$0 \to H_{n+1}(K) \to H_{n+1}(M) \xrightarrow{g} H_{n+1}(M,\partial M) \xrightarrow{\partial} H_n(K) \to 0.$$

Poincare duality implies that the intersection pairing $H_{n+1}(M) \times H_{n+1}(M,\partial M) \xrightarrow{<,>} \mathbb{Z}$ is non-singular. Define an intersection pairing

$$f: H_{n+1}(M) \times H_{n+1}(M) \to \mathbb{Z}$$

by $f(x,y) = <x,g(y)>$. Since we may identify $H_{n+1}(M,\partial M) = H^{n+1}(M) \cong \mathrm{Hom}(H_{n+1}(M),Z)$, g may be interpreted as the map

$$Af: H_{n+1}(M) \to \mathrm{Hom}(H_{n+1}(M),Z),\ Af(y)(x) = f(x,y).$$

Lemma 0.1. Let $u,v \in \tau H_n(K)$, $x,y \in C_{n+1}(M,\partial M)$ such that $\overline{\partial x} = u$, $\overline{\partial y} = v$. For some $r,s \in \mathbb{Z}$, $r\bar{x} = g(\overline{X})$, $s\bar{y} = g(\overline{Y})$, $\overline{X},\overline{Y} \in H_{n+1}(M)$. Then $\Lambda(u,v) = \frac{-1}{rs} f(\overline{X},\overline{Y})$ in $Q/\mathbb{Z}$.

Proof. Regard all chains as chains in $C_*(M) \supset C_*(\partial M)$. Then $rx - X = \partial Z + W$, $Z \in C_{n+2}(M)$, $W \in C_{n+1}(\partial M)$. Hence, $r\partial x = \partial W$. Thus, $\Lambda(u,v) = \frac{1}{r} <W,\partial y>$.

$$\begin{aligned}
\Lambda(u,v) &= \frac{1}{r} <W,\partial y> \quad \text{intersection in } K \\
&= \frac{1}{r} <W,y> \quad \text{intersection in } M \\
&= \frac{1}{r} <rx - X,y> \\
&= \frac{-1}{r} <X,y> \text{ in } Q/Z \\
&= \frac{-1}{r} <\overline{X},\bar{y}> \\
&= \frac{-1}{rs} <\overline{X},s\bar{y}> \\
&= \frac{-1}{rs} <\overline{X},g(\overline{Y})> \\
&= \frac{-1}{rs} f(\overline{X},\overline{Y}).
\end{aligned}$$

Hence, $\Lambda(u,v) \equiv \frac{-1}{rs} f(\overline{X},\overline{Y})$ in Q/Z.

Since it will be convenient to consistently ignore this minus sign, we give the following abstract definition: Given a free module V over Z and a bilinear form $f: V \times V \to \mathbb{Z}$, form the sequence $V \xrightarrow{Af} \mathrm{Hom}(V,Z) \xrightarrow{P} G \to 0$. Define $\Lambda: \tau G \times \tau G \to Q/\mathbb{Z}$ as follows: Given $\bar{g}_1, \bar{g}_2 \in \tau G$ represented by $g_1, g_2 \in \mathrm{Hom}(V,Z)$, then $rg_1 = Af(h_1)$, $sg_2 = Af(h_2)$, $h_1, h_2 \in V$, $\Lambda(\bar{g}_1,\bar{g}_2) = \frac{1}{rs} f(h_1,h_2)$ in $Q/\mathbb{Z}$.

The main point is that the intersection form on M determines the linking form on its boundary.

2) Here is an algebraic reformulation of the above definition. Assume V is free and finitely generated over Z. Let $V^+ = \{y \in V \otimes Q \mid f(x,y) \in \mathbb{Z}, \forall x \in V\}$. Assume that the bilinear form f is non-degenerate, i.e., that $Af: V \to \mathrm{Hom}(V,\mathbb{Z})$ is injective. Let $H: V^+ \to \mathrm{Hom}(V,Z)$ be the extension of Af, $H(y)(x) = f(x,y)$. Then

Lemma 0.2. The following diagram commutes; H is an isomorphism and extends to an isomorphism $V^+/V \xrightarrow{\cong} G$.

$$\begin{array}{ccccccc} V & \xrightarrow{Af} & \mathrm{Hom}(V,\mathbb{Z}) & \longrightarrow & G & \longrightarrow & 0 \\ \| & & \uparrow H & & \uparrow & & \\ V & \hookrightarrow & V^+ & \longrightarrow & V^+/V & \longrightarrow & 0 \end{array}$$

Proof. Easy.

Define: $b: (V^+/V) \times (V^+/V) \to Q/\mathbb{Z}$, $b(\bar{x},\bar{y}) = \pi f(x,y)$ where x and $y \in V^+$ represent $\bar{x},\bar{y} \in V^+/V$ and $\pi: Q \to Q/\mathbb{Z}$. Note that $V^+/V \cong G$ is a finite group.

Claim. b is identical with the linking form $\Lambda: G \times G \to Q/\mathbb{Z}$. This follows immediately.

Thus, given $f: V \times V \to \mathbb{Z}$ non-degenerate, there is a canonically defined pairing $b: (V^+/V) \times (V^+/V) \to Q/\mathbb{Z}$ and in the topological case this is just the linking invariant.

If f is degenerate, then $f \simeq f' \oplus 0$, $V = V' \oplus N$, $f|N = 0$, $f\colon V' \times V' \to \mathbb{Z}$ non-degenerate. $V'^{+}/V' \simeq \tau G$. Define $b(f) = b(f')\colon \tau G \times \tau G \to Q/\mathbb{Z}$.

3) If f is an <u>even</u> form, then one can define a quadratic form

$$q\colon \tau G \to Q/\mathbb{Z},\ q(\bar{x}) = \pi(\tfrac{1}{2} f(x,x)).$$

b is the bilinear form associated with q.

4) Durfee [6] proves the following

<u>Theorem 0.3</u>. Let f_1 and f_2 be even non-degenerate bilinear forms with $q(f_1) \simeq q(f_2)$ (and assume $\sigma(f_1) \geq \sigma(f_2)$, σ = signature).

$$f_1 \oplus U \oplus \ldots \oplus U \simeq f_2 \oplus U \oplus \ldots \oplus U \oplus \underbrace{V \oplus \ldots \oplus V}_{\frac{1}{8}(\sigma(f_1) - \sigma(f_2))}$$

where $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, V = Hirzebruch matrix with signature 8.

<u>Theorem 0.3'</u>. On a group with no summands of order 2 or 4, the equivalence class of a quadratic form is determined by the equivalence class of its associated bilinear form.

3. <u>Quadratic Forms Over Z_2</u>

A Z_2-quadratic form ψ on a Z_2-vector space V is a map $\psi\colon V \to Z_2$ such that $\ell(x,y) = \psi(x+y) - \psi(x) - \psi(y)$ is bilinear. ℓ is the associated Z_2-bilinear form of ψ. Let rad ψ W (the <u>radical</u> of ψ) be the subspace where ψ is singular (ψ non-singular <u>means</u> ℓ non-singular). Thus, rad ψ = $\{x \in V | \psi \text{ linear}\}$. ψ is non-singular on W/rad ψ. If $\psi|\text{rad}\,\psi \equiv 0$, then we may define the Arf invariant $c(\psi) \in Z_2$. Here is the well known classification theorem [see 3].

<u>Proposition</u>. Let $\psi_1\colon W_1 \to Z_2$, $\psi_2\colon W_2 \to Z_2$ be Z_2-quadratic forms. Then $\psi_1 \simeq \psi_2$ iff

i) $\dim W_1 = \dim W_2$

ii) dim rad ψ_1 = dim rad ψ_2

iii) $\psi_i | \text{rad } \psi_i \not\equiv 0$, $i = 1,2$, <u>or</u>

$\psi_i | \text{rad } \psi_i \equiv 0$ and $c(\psi_1) = c(\psi_2)$.

Define Z_2-quadratic forms ϕ_0, ϕ_1 on $Z_2 \oplus Z_2$ with associated Z_2-bilinear form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ by

$$\phi_0(1,0) = 0 \qquad \phi_1(1,0) = 1$$
$$\phi_0(0,1) = 0 \qquad \phi_1(0,1) = 1$$

Each has radial $\{0\}$. $c(\phi_0) = 0$, $c(\phi_1) = 1$. Two singular forms are defined on Z_2 with associated bilinear form $[0]$.

$$\eta_0(1) = 0 \qquad \eta_1(1) = 1$$

Any Z_2-quadratic form is equivalent to a sum of the above. The following relations may be verified: $\phi_0 \oplus \phi_0 \simeq \phi_1 \oplus \phi_1$, $\phi_0 \oplus \eta_1 \simeq \phi_1 \oplus \eta_1$, $\eta_0 \oplus \eta_1 \simeq \eta_1 \oplus \eta_1$.

A <u>skew form with associated Z_2-quadratic form</u> is a pair (f,ψ) consisting of a skew form f on a Z-module V and a Z_2-quadratic form on $V/2V$ with associated Z_2-bilinear form ℓ such that $[f(x,y)] = \ell([x],[y])$ ($[\ ]$ = class mod 2). Durfee observes the following lemma.

<u>Lemma</u>. Let (f_1,ψ_1) and (f_2,ψ_2) be skew forms with associated Z_2-quadratic form. Suppose the torsion subgroups of cok f_1 and cok f_2 (cok f = cokernel(Af)) are of odd order. Then $(f_1,\psi_1) \simeq (f_2,\psi_2) \Longleftrightarrow f_1 \simeq f_2$ and $\psi_1 \simeq \psi_2$.

4. Diffeomorphism Classification

For $n \geq 1$ let BP_{2n} denote the set of diffeomorphism classes of oriented $(n-2)$-connected $(2n-1)$ manifolds that bound parallelizable manifolds. (Note that, by surgery, we may assume that $K \in BP_{2n}$ bounds on $(n-1)$-connected parallelizable manifold.)

For M a smooth oriented $2n$-manifold with boundary such that $H_n(M)$ is free there is the intersection form $f: H_n(M) \times H_n(M) \to \mathbb{Z}$. It is symmetric for n even and skew for n odd.

Given M parallelizable, $(n-1)$-connected, with $(n-2)$-connected boundary K and n odd, define a Z_2-quadratic form $\psi: H_n(M)/2H_n(M) \to Z_2$. For $x \in H_n(M)$ represented by an embedded sphere, $\psi([x])$ = characteristic element of the normal bundle to x. This lies in Z_2 for $n \neq 1, 3, 7$ and is zero otherwise. The pair (f,ψ) is a skew form with associated Z_2-quadratic form.

Theorem 0.4. (Wall, Smale) For $n \geq 3$, diffeomorphism classes of oriented parallelizable $(n-1)$-connected, $2n$-manifolds M with $(n-2)$-connected boundary are in one-to-one correspondence with

1) A free Z-module V of finite rank. $(H_n(M))$

2) For n even, the equivalence class of an even symmetric bilinear form on V (intersection form).

3) For n odd, the equivalence class of a skew form on V, and, for $n \neq 1, 3, 7$, an associated Z_2-quadratic form on $V/2V$ (as above).

Examples. 1) $S^n \times S^n$ minus small $2n$-disk. The boundary is S^{2n-1}. Intersection form $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for n even, $S(1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ for n odd and associated Z_2-form ϕ_0.

2) $D^n \times S^n$. Boundary $S^{n-1} \times S^n$. Intersection form $\langle 0 \rangle$. For odd n, the associated Z_2-form is η_0.

3) Tangent disk bundle to S^n. If n even, then intersection form is $\langle 2 \rangle$. If n odd, the intersection form is $\langle 0 \rangle$ and associated Z_2-form is η_1 for $n \neq 1, 3, 7$.

4) P, n even, $n \geq 4$. This is a $2n$-manifold with boundary the (Milnor) generator Σ_8 of bP_{2n}. P has intersection form V

$$V = \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & \bigcirc & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & \bigcirc & & & 1 & 2 & 1 & 0 \\ & & & & & 1 & 2 & 0 \\ & & & & 1 & 0 & 0 & 2 \end{bmatrix}$$

5) Q, n odd, $n \geq 3$. This is a 2n-manifold with boundary the Kervaire sphere Σ_1, generator of $bP_{2n} = 0$ or Z_2. Q has intersection form S(1) and associated Z_2-form ϕ_1.

If K is the boundary of an (n-1)-connected 2n-manifold with intersection form f and n is even, define the <u>quadratic form</u> of K, $q: \tau H_{n-1}(K) \to Q/\mathbb{Z}$ by $q = q(f)$, the construction described in section B.

Durfee proves two theorems, classifying elements of BP_{2n}, by appealing to his results on quadratic forms and applying the previous theorem.

<u>Theorem 0.5</u>. $K_1, K_2 \in BP_{2n}$, n even, $n \neq 2, 4, 8$. Suppose

i) $H_{n-1}(K_1) \approx H_{n-1}(K_2)$

ii) $q_1 \approx q_2$ where q_i = quadratic form for K_i.

Further suppose that K_i is boundary of a parallelizable manifold with intersection form f_i. Then $\sigma(f_1) - \sigma(f_2)$ is divisible by 8 and $K_1 \overset{\sim}{\approx} K_2 \# \frac{1}{8}(\sigma(f_1) - \sigma(f_2)\Sigma_8$.

If $H_{n-1}(K_i)$ has no summands of order 2 or 4, then we may replace (ii) by (ii)' $b_1 \approx b_2$ where b_i is the linking form of K_i.

<u>Theorem 0.6</u>. $K_1, K_2 \in BP_{2n}$, n odd ≥ 3, boundaries of (n-1)-connected 2n-manifolds M_1 and M_2 with Z_2-quadratic forms ψ_1 and ψ_2. Suppose $H_{n-1}(K_1) \approx H_{n-1}(K_2)$.

If n = 3 or 7, then $K_1 \overset{\sim}{\approx} K_2$. Suppose that the torsion subgroups of $H_{n-1}(K_i)$ have odd order.

i) If $\psi_i | (\text{rad } \psi_i) \equiv 0$ for $i = 1,2$, then

$$K_1 \approx K_2 \# (c(\psi_1) + c(\psi_2))\Sigma_1.$$

ii) If $\psi_i | (\text{rad } \psi_i) \not\equiv 0$ for $i = 1,2$, then $K_1 \approx K_2 \approx K_2 \# \Sigma_1$.

We shall apply these results to branched covers and to certain manifolds admitting O(n)-actions.

CHAPTER I

CYCLIC BRANCHED COVERS

Let $K^{2n-1} \subset S^{2n+1}$ be an oriented submanifold of the 2n+1-sphere. Suppose K is the boundary of $F^{2n} \subset S^{2n+1}$, another oriented submanifold. One usually describes the k-fold cyclic branched cover of S^{2n+1} with branch locus K by splitting S along F and then cyclically pasting k copies of the split manifold together. We extend this description by constructing a manifold whose boundary is this branched cover and compute the middle dimensional intersection form of the manifold in terms of embedding information about F. Then we consider the case where the complement of K fibers over the circle. Under these assumptions we discuss periodicity of homology and differential structure for the branched covers.

The final section discusses branched covers which arise from algebraic hypersurfaces and uses these techniques to examine the orbit space of an O(n)-action on a neighborhood boundary.

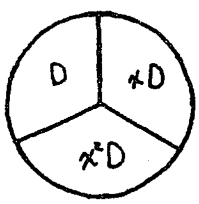
1. Construction of Branched Covers

All manifolds will be smooth. Given $K^{2n-1} \subset S^{2n+1}$, an oriented submanifold, we say that K is simple if $K = \partial F^{2n}$, $F^{2n} \subset S^{2n+1}$, an oriented submanifold which is (n-1)-connected. Thus, F has the homotopy type of a wedge of n-spheres. We will further assume that each element of $\pi_n(F)$ may be represented by an embedded sphere.

Note that by pushing F up and down via a normal vector field, it follows that there is an embedding $g: \mathring{F} \times [-1,1] \to S^{2n+1}$ such that if $F_t = \overline{g(\mathring{F} \times t)}$, then $\partial F_t = K$ and $F_o = F$. By this construction we may assume there are diffeomorphisms $\alpha_t: F_o \to F_t$ for $t \in [-1,1]$ such that $\alpha_o = 1_F$. Letting $\tilde{F} = \text{Closure}(g(\mathring{F} \times [-1,1]))$, $\tilde{F} = \bigsqcup_{t\in[-1,1]} F_t$, there is an involution $T: \tilde{F} \to \tilde{F}$ given by $T(x) = \alpha_{-t} \circ \alpha_t^{-1}(x)$ for $x \in F_t$. Whence $\tilde{F} = \tilde{F}_+ \cup \tilde{F}_-$, $\tilde{F}_+ \cap \tilde{F}_- = F$, $\tilde{F}_+ = \bigsqcup_{t\in[0,1]} F_t$, $\tilde{F}_- = \bigsqcup_{t\in[-1,0]} F_t$. Thus, $T(\tilde{F}_\pm) = \tilde{F}_\mp$.

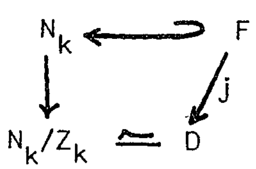
Let $D = D^{2n+2}$ and $D, xD, x^2D, \ldots, x^{k-1}D$ be k copies. (Regard x as a generator for Z/kZ so that $x^i x^j = x^{i+j}$, $x^k = 1$.) Let

$$N_k = \bigcup_{i=0}^{k-1} x^i D/[x^i\tilde{F}_+ \overset{T}{\longleftrightarrow} x^{i+1}\tilde{F}_-].$$



Then N_k is a $2n+2$ manifold with boundary. Let M_k denote ∂N_k.

Clearly, M_k fits the usual description of the k-fold cyclic cover of S^{2n+1} branching along K. In fact, there is a natural action of Z_k on N_k induced via $x^iD \to x^{i+1}D$. Clearly, $N_k/Z_k = D/[\tilde{V}_+ \overset{T}{\longleftrightarrow} \tilde{V}_-] \simeq D$ and $M_k/Z_k \simeq S$ similarly. Note that $F \subset N_k$ is the fixed point set of the Z_k-action and that $\partial F = K \subset M_k = \partial N_k$ is the fixed point set of the Z_k-action on M_k.



$j: F \to D$ is obtained by pushing the interior of F into the interior of D by a normal vector field. Thus, N_k = k-fold cyclic cover of D branching along $j(F)$, $M_k = \partial N_k$ = k-fold cyclic cover of $\partial D = S$, branching along $K = \partial F = \partial(j(F))$.

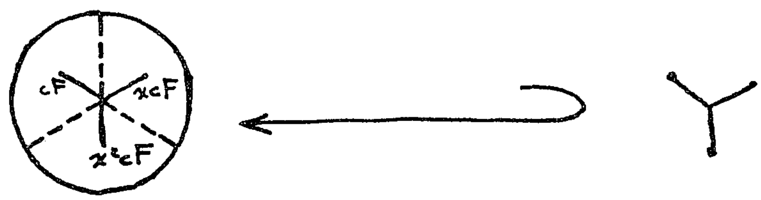
<u>Lemma 1.1.</u> K simple $\Longrightarrow H_i(K) = 0$ for $i \neq 0, n-1, n$. K simple and simply connected $\Longrightarrow$ K is $(n-2)$-connected. In either case, there

is an exact sequence $0 \to H_n(K) \to H_n(F) \to H_n(F,K) \to H_{n-1}(K) \to 0$.

Proof. Immediate from the homology sequence of the pair (F,K) and the Whitehead theorem.

Let $cF \to D$ be the join, in D, of $F \subset S = \partial D$ with the center of D. Similarly, $x^i cF \hookrightarrow x^i D$ and, in N_k, $x^i cF \cap x^j cF = F$ for $i \neq j$.

$N_k \supset N_k' = cF \cup xcF \cup \ldots \cup x^{k-1}cF$



Proposition 1.2. N_k' has the same homotopy type as N_k and the inclusion $N_k' \to N_k$ is a homotopy equivalence.

Proof. Let N_k'' be the "wedge" of k copies of D along F. Thus, $N_k'' \supset N_k'$ and N_k'' may be viewed as included in N_k in such a way that $N_k' \subset N_k'' \subset N_k$ induces the usual inclusion $N_k' \subset N_k$. To do this simply choose $D'' \subset D$ a slightly smaller disk such that $D'' \cap \partial D = F$. Then $N_k'' = \bigcup x^i D'' \subset \bigcup x^i D = N_k$. It is easily checked that N_k'' is a deformation retract of N_k. Since N_k' and N_k'' have the same homotopy type, this proves the proposition.

Since K is simple, we know that $H_n(K)$ has a basis represented by embedded spheres. Let these be $\{a_1, a_2, \ldots, a_r\}$. Let $ca_i \subset cF$ denote the radial cone over a_i in D with apex at the center of D. Similarly, we have $(x^i ca_j) \subset x^i D$. These cones fit together to form many $(n+1)$-spheres embedded in N_k. Let $\Sigma a_j = ca_j \cup xca_j$ and, as an element of $C_{n+1}(N_k)$, $\Sigma a_j \equiv ca_j - xca_j$. Similarly, $x^i \Sigma a_j = x^i ca_j \cup x^{i+1} ca_j \equiv x^i ca_j - x^{i+1} ca_j$.

Lemma 1.3. $H_{n+1}(N_k)$ is free of rank $(k-1)\cdot r$. A basis is given by $\mathcal{B} = \{x^i \Sigma a_j \mid 0 \le i \le k-2,\ j = 1,2,\ldots,r\}$.

Proof. $H_{n+1}(N_k) = H_{n+1}(N_k')$ and a simple Mayer-Vietoris sequence applied to the latter yields the lemma.

Next, we wish to determine the intersection pairing $\langle\ ,\ \rangle: H_{n+1}(N_k) \times H_{n+1}(N_k) \to \mathbb{Z}$. This requires a discussion of the embedding of F in S.

2. The Seifert Pairing

Given $K^{2n-1} \subset S^{2n+1}$ simple, $K = \partial F$, we will define the Seifert pairing $\theta: H_n(F) \times H_n(F) \to \mathbb{Z}$ by $\theta(a,b) = \ell(i_*a,b)$ where i_* = push into S-F in the direction of the positive normal to F. (Similarly, i^* = push in direction of the negative normal.) $\ell(\ ,\ )$ denotes linking number for n-cycles in S^{2n+1}.

To make this more precise note that we have diffeomorphisms $\alpha_t: F = F_0 \to F_t$ for $-1 \leq t \leq +1$. Let $i_* = \alpha_{1/2}$, $i^* = \alpha_{-1/2}$. Then $i^*i_* = i_*i^* = 1_F$. Hence, $\ell(i_*a,b) = \ell(i^*(i_*a),i^*b) = \ell(a,i^*b)$. Thus, we could also define $\theta(a,b) = \ell(a,i^*b)$.

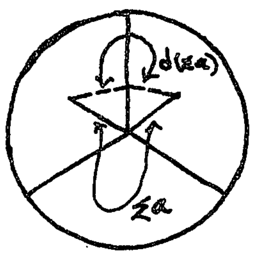
Lemma 1.4. Let $\langle\ ,\ \rangle: H_n(F) \times H_n(F) \to \mathbb{Z}$ denote the intersection pairing. Then $\langle a,b\rangle = \theta(a,b) + (-1)^n\theta(b,a)$.

Proof. First note that if $B = \bigcup_{t\in[-1/2,1/2]} \alpha_t(b)$, then B is an (n+1) chain on S^{2n+1} and, up to sign $\langle a,b\rangle = \pm\langle a,B\rangle$ where the second intersection refers to intersection numbers in S^{2n+1}. Choosing the orientation for B so that $\langle a,b\rangle = \langle a,B\rangle$ means that we regard $B = b \times [-1/2,1/2]$. Then $\partial B = (-1)^n(b \times 1/2 - b \times (-1/2))$. Thus, $\partial B = (-1)^{n+1}(i^*b - i_*b)$. But by our definition of linking numbers, $\langle a,B\rangle = (-1)^{n+1}\ell(a,\partial B)$. Hence,

$$\langle a,b\rangle = \langle a,B\rangle = \ell(a, i^*b - i_*b)$$
$$= \ell(a,i^*b) - (-1)^{n+1}\ell(i_*b,a)$$
$$= \theta(a,b) + (-1)^n\theta(b,a).$$

3. Intersection Pairing for N_k

To determine intersections of cycles on N_k we define deformations so that intersection numbers may be interpreted as certain linking numbers in S^{2n+1}. Given Σa as in 2., define $d(\Sigma a) = ci^*a - xci_*a$. Note that since $Ti^* = i_*$, where T is the involution on F used in constructing N_k, $\partial d(\Sigma a) = 0$ and, in fact, $ci^*a \cup ci_*a$ is an (n+1)-sphere in N_k.



Similarly, $d(x^j\Sigma a_i) = x^j d(\Sigma a_i)$. Clearly, $d(x^j\Sigma a_i)$ is homologous to $x^j\Sigma a_i$. Since cones are taken radially, $x^j\Sigma a$ and $d(x^j\Sigma b)$ can intersect only at the apex of cones and, hence, in at most two points.

Proposition 1.5.

$$\langle x^j\Sigma a_i, x^{j'}\Sigma a_{i'}\rangle = \begin{cases} \theta(a_i,a_{i'}) + (-1)^{n+1}\theta(a_{i'},a_i) & j = j' \\ (-1)^n\cdot\theta(a_{i'},a_i) & j = j'+1 \\ (-1)\cdot\theta(a_i,a_{i'}) & j+1 = j' \\ 0 & \text{otherwise.} \end{cases}$$

Proof. $\langle x^j\Sigma a_i, x^{j'}\Sigma a_{i'}\rangle = \langle x^j\Sigma a_i, x^{j'}d(\Sigma a_{i'})\rangle$

$$= \langle x^j ca_i - x^{j+1}ca_i, x^{j'}ci^*a_{i'} - x^{j'+1}ci_*a_{i'}\rangle$$
$$= \delta_{jj'}\langle ca_i, ci^*a_{i'}\rangle - \delta_{jj'+1}\langle ca_i, ci_*a_{i'}\rangle$$
$$- \delta_{j+1\,j'}\langle ca_i, ci^*a_{i'}\rangle + \delta_{j+1\,j'+1}\langle ca_i, ci_*a_{i'}\rangle$$

$$= \begin{cases} \langle ca_i, ci^*a_{i'}\rangle + \langle ca_i, ci_*a_{i'}\rangle & j = j' \\ -\langle ca_i, ci_*a_{i'}\rangle & j = j'+1 \\ -\langle ca_i, ci^*a_{i'}\rangle & j+1 = j' \end{cases}$$

$$= \begin{cases} \ell(a_i, i^*a_{i'}) + (-1)^{n+1}\ell(a_{i'}, i^*a_i) & j = j' \\ (-1)^n \ell(a_{i'}, i^*a_i) & j = j'+1 \\ -\ell(a_i, i^*a_{i'}) & j+1 = j'. \end{cases}$$

Corollary 1.6. With respect to the basis $\mathfrak{B}$, $\langle\ ,\ \rangle$ has matrix

$$N = \begin{bmatrix} V + (-1)^{n+1}V' & -V & & \\ (-1)^n V' & V + (-1)^{n+1}V' & -V & \\ & (-1)^n V' & \ddots & \\ & & & V + (-1)^{n+1}V' \end{bmatrix}$$

($(k-1) \times (k-1)$ blocks, each block $r \times r$) where V = Seifert matrix, $V_{ij} = \theta(a_i, a_j)$. Since $0 \to H_{n+1}(M_k) \to H_{n+1}(N_k) \xrightarrow{N} H_{n+1}(N_k, M_k) \to H_k(M_k) \to 0$, N may be regarded as a relation matrix for $H_n(M_k)$.

Example. If K is a knot or link in S^3, then $N = V + V'$ is a relation matrix for the first homology of the double branched cover.

4. The Homology of M_k, An Interior View

The preceding shows how to get at M_k by way of a manifold which it bounds. In case K is a homology sphere (for example, K a simple knot), it is convenient to work directly with M_k. This was Seifert's original approach.

Let $M = S^{2n+1} - \mathrm{Int}(\tilde{F})$. Then M is a manifold with boundary: $\partial M = F_+ \cup_K F_-$. $S^{2n+1} = M \cup \tilde{F}$ with $\partial M = \partial\tilde{F}$. $F = F_0 \subset \tilde{F}$ is a deformation retract. Now $M_k = \bigcup_{i=0}^{k=1} x^i M$ where $x^i F_+ \xleftrightarrow{T} x^{i+1} F_-$. Denote $x^i F_+ \equiv x^{i+1} F_-$ by $x^i \mathcal{F}$. Then $\partial M = \mathcal{F} \cup x\mathcal{F}$ and $\partial x^i M = x^i\mathcal{F} \cup x^{i+1}\mathcal{F}$. The idea is to first examine the homology of M and then the pasting which creates M_k.

A. Homology of M

First look at the Mayer-Vietoris sequence for $\partial M = \mathcal{F} \cup x\mathcal{F}$, $\mathcal{F} \cap x\mathcal{F} = K$.

Lemma 1.7. $0 \to H_n(\mathcal{F}) \oplus H_n(x\mathcal{F}) \to H_n(\partial M) \to 0$.

Proof. $\to H_{n+1}(\partial M) \to H_n(K) \to H_n(\mathcal{F}) \oplus H_n(x\mathcal{F}) \to H_n(\partial M) \to H_{n-1}(K) \to H_{n-1}(\mathcal{F}) \oplus H_{n-1}(x\mathcal{F}) \to$. Assuming K^{2n-1} is a homology sphere, the lemma follows immediately for $n > 1$ and just as easily for $n = 1$.

Lemma 1.8. $0 \to H_n(F) \oplus H_n(F) \xrightarrow{\psi} H_n(M) \oplus H_n(F) \to 0$ is exact where $\psi(a,b) = (i_*a + i^*b, a + b)$ (regard $i_*\colon H_n(F) \to H_n(S^{2n+1} - F)$).

Proof. $H_{n+1}(S^{2n+1}) \to H_n(\mathcal{F} \cup x\mathcal{F}) \to H_n(M) \oplus H_n(\tilde{F}) \to H_n(S^{2n+1})$. Ends of this sequence vanish and can use previous lemma on $H_n(\mathcal{F} \cup x\mathcal{F})$.

Lemma 1.9. Given an exact sequence of abelian groups $0 \to A \xrightarrow{f} B \oplus C \to 0$, $f(a) = (f_1(a), f_2(a))$. Then

$$f_1\colon \operatorname{Ker} f_2 \xrightarrow{\approx} B, \qquad f_2\colon \operatorname{Ker} f_1 \xrightarrow{\approx} C.$$

Proof. Obvious.

Corollary 1.10. $0 \to H_n(F) \xrightarrow{i_* - i^*} H_n(M) \to 0$.

Proof. Let $f_1(a,b) = i_*a + i^*b$, $f_2(a,b) = a + b$. Then $\operatorname{Ker} f_2 = \{(a,-a)\}$, $f_1(a,-a) = i_*a - i^*a$. Apply the previous lemma.

Hence, if $\{a_i\}$ is a basis for $H_n(F)$, then $i^*a_i = (i^* - i_*)(\Sigma_k \gamma_{ki} a_k)$, whence $i^*a = (i^* - i_*)\Gamma a$, $a \in H_n(F)$ where $\Gamma\colon H_n(F) \to H_n(F)$, $\Gamma a_i = \Sigma_k \gamma_{ki} a_k$.

Notation. View $\{i^*a_i\}$ as a basis for $H_n(F_-)$. To distinguish these from elements of $H_n(M)$ denote $i^*a_i = \alpha_i \in H_n(F_-)$. Similarly, let $i_*a_i = x\alpha_i \in H_n(F_+)$. We may write $\Gamma\colon H_n(\mathcal{F}) \to H_n(\mathcal{F})$ and $x\Gamma\colon H_n(\mathcal{F}) \to H_n(x\mathcal{F})$, $x\Gamma\alpha = \Gamma x\alpha$ and this means $\Gamma i_*a_i \equiv i_*\Gamma a_i$, $\Gamma i^*a_i \equiv i^*\Gamma a_i$. Using this notation, $i^*a = (i^* - i_*)\Gamma a$ becomes $\alpha = \Gamma(\alpha - x\alpha)$ where this is to be viewed as a relation holding in $H_n(M)$ seen as a quotient of $H_n(\mathcal{F}) \oplus H_n(x\mathcal{F})$. Similarly, $\{x^i\alpha\}$ form a basis for $H_n(x^i\mathcal{F})$ and $x^i\alpha = \Gamma(x^i\alpha - x^{i+1}\alpha)$.

Proposition 1.11. The following sequence is exact.

$$0 \to H_n(F) \xrightarrow{g} H_n(F) \oplus H_n(F) \xrightarrow{i_*+i^*} H_n(M) \to 0 \text{ where } g(z) = (\Gamma z, (I - \Gamma)z).$$

Proof. Let $A = \operatorname{Ker}(i_* + i^*)$. Then $g'\colon A \approx H_n(F)$, $g'(x,y) = x + y$

by previous lemma. Hence, $g'g = 1$. We need merely prove that $gg' = 1$.

$$\begin{aligned}(x,y) \in A &\Leftrightarrow -i_*x = i^*y \\ &\Leftrightarrow (i^* - i_*)x = i^*(x+y) = (i_* - i^*)\Gamma(x+y) \\ &\Leftrightarrow x = \Gamma(x+y) \\ &\Leftrightarrow (I - \Gamma)x = \Gamma y\end{aligned}$$

whence

$$(I - \Gamma)(x + y) = \Gamma y + y - \Gamma y = y$$

$$\Gamma(x + y) = x$$

Thus,

$$g'g(x,y) = (x,y).$$

Corollary 1.12. The set of relations $\alpha = \Gamma(\alpha - x\alpha)$ is a complete set of relations for $H_n(M)$.

Proof. This is just a restatement of the proposition.

One can now continue in this fashion, using Mayer-Vietoris sequences to show that the collection of relations $x^i\alpha = \Gamma(x^i\alpha - x^{i+1}\alpha)$ is a complete set of relations for $H_n(M_k)$. It then follows a là Seifert that $\Gamma^k - (\Gamma - I)^k$ is a relation matrix for $H_n(M_k)$. We will derive this in a slightly different fashion by invoking the intersection form for N_k.

B. Relations Between Γ, the Seifert Pairing, and the Homology of M_k

Since M is homotopy equivalent to $S^{2n+1} - F$, the pairing ℓ: $H_n(F) \times H_n(M) \to \mathbb{Z}$ is non-singular (Alexander duality). Let $\{\hat{a}_i\}$ be a dual basis for $H_n(M)$ such that $\ell(a_i,\hat{a}_j) = \delta_{ij}$. Express i^*: $H_n(F) \to H_n(M)$ in these bases: $i^*(a_j) = \Sigma_k c_{jk}\hat{a}_k$.

Lemma 1.13. $c_{ji} = \theta(a_i,a_j)$. Hence, the matrix of i^*, $[i^*] = V'$, the transpose of the Seifert matrix.

Proof. $\theta(a_i,a_j) = \ell(a_i,i^*a_j) = \Sigma_k c_{jk}\delta_{ik} = c_{ji}$.

Lemma 1.14. $\theta(a,b) = \langle a,\Gamma b\rangle$ where Γ: $H_n(F) \to H_n(F)$ as before,

$a, b \in H_n(F)$ and $\langle\ ,\ \rangle$ denotes the intersection pairing on F.

Proof. Note $i^* = (i^* - i_*)\circ\Gamma$.

$$\theta(a,b) = \ell(a, i^*b) = \ell(a, i^*\Gamma b) - \ell(a, i_*\Gamma b)$$
$$= \ell(a, i^*\Gamma b) + (-1)^n \ell(i_*\Gamma b, a)$$
$$= \theta(a, \Gamma b) + (-1)^n \theta(\Gamma b, a)$$
$$\theta(a,b) = \langle a, \Gamma b\rangle.$$

Corollary 1.15. Let Δ be the intersection matrix for F, $\Delta_{ij} = \langle a_i, a_j\rangle$. Then $V = \Delta\Gamma$.

Proof.
$$V_{ij} = \theta(a_i, a_j) = \langle a_i, \Gamma a_j\rangle$$
$$= \Sigma_k \langle a_i, \gamma_{kj} a_k\rangle$$
$$= \Sigma_k \Delta_{ik}\gamma_{kj}$$
$$= (\Delta\gamma)_{ij}.$$

Note. If K is a homology sphere, then Δ is invertible, hence, $\Gamma = \Delta^{-1}V$.

Proposition 1.16. $\Gamma^k - (\Gamma - I)^k$ is a relation matrix for $H_n(M_k)$.

Proof. Recall that
$$N = \begin{bmatrix} V + (-1)^{n+1}V' & -V & \\ (-1)^n V' & V + (-1)^{n+1}V' & \\ & & \ddots \end{bmatrix}$$
is the intersection matrix for N_k and, hence, a relation matrix for $H_n(\partial N_k) = H_n(M_k)$. Since $V = \Delta\Gamma$ and $V + (-1)^n V' = \Delta$, $(-1)^n V' = \Delta - V = \Delta(I - \Gamma)$ and $V + (-1)^{n+1}V' = \Delta\Gamma + \Delta(\Gamma - I) = \Delta(2\Gamma - I)$. Hence,
$$N = \begin{bmatrix} \Delta(2\Gamma - I) & \Delta(-\Gamma) & \\ \Delta(I - \Gamma) & \Delta(2\Gamma - I) & \\ & & \ddots \end{bmatrix}.$$
Since Δ is invertible, this is equivalent, by row and column operations, to
$$\begin{bmatrix} I - 2\Gamma & \Gamma & \\ \Gamma - I & I - 2\Gamma & \\ & & \ddots \end{bmatrix}.$$

It is then a simple exercise to see that this matrix is equivalent to $\Gamma^k - (\Gamma - I)^k$.

5. The Fibered Case and Periodicity

Suppose $K^{2n-1} \subset S^{2n+1}$ is simple with $\overline{F}^{2n} \subset S^{2n+1}$ such that $K = \partial\overline{F}$, $\overline{F}$ (n-1)-connected. Suppose further that the complement of K fibers over the circle with typical fiber $F = \mathrm{Int}(\overline{F})$. Thus, there is a smooth, locally trivial fiber bundle $\phi: S - K \to S^1$, fiber $F = \phi^{-1}(1)$.

Any such bundle is obtained by a diffeomorphism $h: F \to F$ so that $S - K \approx F \times I/\{(x,0) \sim (hx,1)\}$. We may also regard $h = h_1$ where $h_t: \phi^{-1}(1) \to \phi^{-1}(\exp(2\pi it))$ is an orientation preserving diffeomorphism obtained by translating the fiber. We may assume that $h_t \circ h_{t'} = h_{t+t'}$ and that h_t translates F in the direction of its positive normal for $0 < t < 1$. Hence, we may regard $i_* = h_{1/2}$ and $i^* = h_{-1/2}$.

Lemma 1.17. $hi^* = i_*$.

Proof. $hi^* = h_1 \circ h_{-1/2} = h_{1/2} = i_*$.

As before, let $\{a_i\}$ be a basis for $H_n(F)$ and let $H = h_*: H_n(F) \to H_n(F)$. H will also denote the matrix of h_* with respect to this basis.

Lemma 1.18. $\theta(a,b) = (-1)^{n+1}\theta(b,Ha)$.

Proof.
$$\begin{aligned}\theta(a,b) &= \ell(a,i^*b)\\ &= \ell(Ha,Hi^*b)\\ &= \ell(Ha,i_*b)\\ &= (-1)^{n+1}\ell(i_*b,Ha)\\ &= (-1)^{n+1}\theta(b,Ha).\end{aligned}$$

Corollary 1.19. Let V be the matrix $V_{ij} = \theta(a_i,a_j)$. Then $V' = (-1)^{n+1}VH$.

Proof. Let $Ha_j = \Sigma_k H_{kj}a_k$. Then

$$\begin{aligned} V_{ij} &= \theta(a_i,a_j) = (-1)^{n+1}\theta(a_j,Ha_i) \\ &= (-1)^{n+1}\Sigma_k\theta(a_j,a_k)H_{ki} \\ &= (-1)^{n+1}(VH)_{ji}. \end{aligned}$$

Hence,
$$V' = (-1)^{n+1}VH.$$

Corollary 1.20. $\Delta = V(I - H)$ where Δ is the intersection matrix for F.

Proof. $\Delta = V + (-1)^n V' = V - VH = V(I - H)$.

Thus, if $(I - H)$ is invertible, then $V = \Delta(I - H)^{-1}$ and the Seifert matrix may be found in terms of the monodromy matrix H and the intersection matrix of F. This is the case for K, a homology sphere, since the Wang sequence implies that $(I - H)$ is an isomorphism. In this case V is certainly invertible. However, more is true:

Proposition 1.21. If K simple with S - K fibered over S^1, fiber F, then V is an invertible matrix. Equivalently, the pairing $\theta\colon H_n(F) \times H_n(F) \to \mathbb{Z}$ is non-singular.

Proof.
$$S - F \xleftarrow{\approx} F \times (0,1) \xleftarrow{i} F \times (\varepsilon)$$
$$h_t(x) \longleftarrow\!\!\dashv (x,t)$$

Thus, we have a commutative diagram

$$\begin{array}{ccc} H_n(F \times (0,1)) & \xrightarrow{\ \approx\ } & H_n(S - F) \\ \uparrow{\scriptstyle \approx\, i} & \nearrow{\scriptstyle i_*} & \\ H_n(F) & & \end{array}$$

Hence,
$$i_*\colon H_n(F) \xrightarrow{\approx} H_n(S - F).$$

But the matrix of i_* with respect to the basis $\{a_i\}$ for $H_n(F)$ and a linking dual basis for $H_n(S - F)$ is V'. Hence, V' is invertible and, therefore, V is invertible.

Note. K simple, S - K fibered $\Longrightarrow$ The intersection matrix N of N_k

has the form

$$N = \begin{bmatrix} V(I+H) & V(-I) & & \\ V(-H) & V(I+H) & V(-I) & \\ & V(-H) & V(I+H) & \ddots \\ & & \ddots & \ddots \end{bmatrix}$$

(Substitute for V' using previous lemma.) Hence,

$$N = \begin{bmatrix} V & & & \\ & V & & \\ & & V & \\ & & & \ddots \end{bmatrix}\begin{bmatrix} I+H & -I & & \\ -H & I+H & -I & \\ & -H & I+H & \ddots \\ & & \ddots & \ddots \end{bmatrix}$$

and since V is invertible over $\mathbb{Z}$, it follows that N is equivalent to the matrix on the right via row and column operations.

Lemma 1.22. $N \sim I + H + H^2 + \ldots + H^{k-1}$.

Proof. It suffices to show this for the matrix on the right in the above product. Let $Q_\ell = 1 + y + y^2 + \ldots + y^\ell$. Then

$$(1 - Q_\ell) + Q_\ell(1 + y) = 1 + Q_\ell y = Q_{\ell+1} - Q_\ell y = 1 - Q_{\ell+1}.$$

Hence,

$$\begin{bmatrix} Q_\ell & 1 - Q_\ell & 0 \\ -1 & 1+y & -y \end{bmatrix} \sim \begin{bmatrix} 0 & (1-Q_\ell) + Q_\ell(1+y) & -Q_\ell y \\ -1 & 1+y & -y \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & Q_{\ell+1} & 1 - Q_{\ell+1} \\ -1 & 1+y & -y \end{bmatrix}$$

Thus,

$$\begin{bmatrix} 1+y & -1 & & \\ -y & 1+y & -1 & \\ & -y & 1+y & \ddots \\ & & \ddots & \ddots \end{bmatrix} \sim \begin{bmatrix} 1+y & -y & & \\ -1 & 1+y & -y & \\ & -1 & 1+y & \ddots \\ & & \ddots & \ddots \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & \cdots & & & \\ 0 & (1+y+y^2) & -(y+y^2) & & & \\ 0 & -1 & 1+y & -y & & \\ \vdots & & -1 & 1+y & -y & \\ \vdots & & & \ddots & \ddots & \ddots \end{bmatrix} \sim \cdots$$

$$\sim \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ & & & & & (1+y+y^2+\dots+y^{k-1}) \end{bmatrix} \sim 1 + y + \dots + y^{k-1}$$

Proposition 1.23. Assume K as above (simple, complement fibered). Let A and B be free abelian groups of rank = rank $H_n(F)$. Then there is an exact sequence

$$0 \to H_{n+1}(M_k) \to A \xrightarrow{I+H+H^2+\dots+H^{k-1}} B \to H_n(M_k) \to 0.$$

A and B may be identified as appropriate submodules of $H_{n+1}(N_k)$ and $H_{n+1}(N_k,\partial N_k)$ respectively.

Proof. Apply the preceding remarks to the exact sequence for the pair $(N_k,\partial N_k)$.

Corollary 1.24. Suppose K is simple, a homology sphere and that S - K is fibered with H of finite order d so that $H^d = I$. Then $H_*(M_k) \approx H_*(M_{k+d})$.

Proof. K homology sphere $\Longrightarrow (I - H)$ invertible.

$$(I + H + \dots + H^{d-1})(I - H) = I - H^d = 0.$$

Hence,
$$I + H + \dots + H^{d-1} = 0.$$

Thus,

$$\begin{aligned} I + H + \dots + H^{k+d-1} &= I + H + \dots + H^{k-1} + H^k(I + H + \dots + H^{d-1}) \\ &= I + H + \dots + H^{k-1}. \end{aligned}$$

Hence,
$$H_*(M_{k+d}) \approx H_*(M_k).$$

Remark. Suppose K satisfies all the hypotheses of the corollary except that K is not necessarily a homology sphere. Then $I + H + \dots + H^{d-1}$ may be non-zero. Suppose

$H_n(M_d) \cong Z_{\ell_1} \oplus \dots \oplus Z_{\ell_s} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{\ell}$. Then $(I + H + \dots + H^{d-1}) \sim \text{diag}(\ell_1,\dots,\ell_s) \oplus [0]$ where the zero block is $\ell \times \ell$.

<u>Claim.</u> $H_n(M_{rd}) \cong Z_{r\ell_1} \oplus Z_{r\ell_2} \oplus \dots \oplus Z_{r\ell_s} \oplus \mathbb{Z}^{\ell}$ when $r \geq 1$.

<u>Proof.</u> $(I + H + \dots + H^{rd-1}) = r\cdot(I + H + \dots + H^{d-1})$.

This appearance of extra torsion spoils the periodicity.

<u>Examples from Knot Theory</u>

a) Let $L_{a,b}$ denote a torus link of type (a,b). If a and b are relatively prime, then $L_{a,b}$ is a knot and it is well known [see 42] that the complement of a torus link in S^3 fibers over the circle with monodromy h such that order(h) = least common multiple (a,b). Hence, if M_k = k-fold cyclic cover of S^3 branching along $L_{a,b}$, then $H_1(M_{k+d}) \cong H_1(M_k)$ where $d = \text{lcm}[a,b] = ab$ for $(a,b) = 1$. For example, the trefoil knot has monodromy of order 6.

<u>Definition.</u> A knot has <u>topologically periodic</u> monodromy if $h^d = 1$ for some d, <u>homologically periodic</u> monodromy if $H^d = 1$ for some d.

<u>Proposition 1.25.</u> If a fibered knot $K \subset S^3$ has topologically periodic monodromy, then $G = \pi_1(S^3 - K)$ has non-trivial center.

<u>Proof.</u> $S^3 - K = F \times I/\{(x,0) \sim (hx,1)\}$. $G' = [G,G] \cong \pi_1(F,Q)$ for some $Q \in F$. Let t be the element of G corresponding to $Q \times [0,1]$. Clearly, every element $x \in G$ is of the form $x = t^n y$ where $y \in G'$. h induces an automorphism $\alpha\colon G' \to G'$. It is also clear that $\alpha(y) = tyt^{-1}$. Now $h^d = 1 \Longrightarrow \alpha^d = 1 \Longrightarrow t^d y t^{-d} = y \ \forall\, y \in G'$. Hence, $t^d \in \text{Center}(G)$. It is non-zero since the subgroup

$$\{(t^{nd}y) \mid y \in G'\} \cong \pi_1(M_d - K) \cong \pi_1(S \times F) \cong Z \oplus G'.$$

<u>Corollary 1.26.</u> The torus knots are the only fibered knots with

topologically periodic monodromy.

Proof. Zieschang and Burde, Math. Ann. 167 (1966), prove that the torus knots are the only knots with non-trivial centers.

Problem. Is there a knot whose complement fibers, with homologically periodic monodromy, which is not topologically periodic? One can construct knots which are not torus knots such that $H_*(M_k)$ is periodic; the examples we know do not have complements which fiber over S^1.

b) Periodicity of the Linking Invariant. Let $G = \tau H_n(M_k)$ = torsion part of $H_n(M_k)$. Recall that there is a linking pairing $\Lambda: G \times G \to Q/\mathbb{Z}$.

Lemma 1.27. Let $\hat{G} = \tau H_n(M_k - K)$. $\hat{\Lambda}: \hat{G} \times \hat{G} \to Q/\mathbb{Z}$. If K is a simple homology sphere with fibered complement, then $G \cong \hat{G}$ and $\Lambda \equiv \hat{\Lambda}$ under this isomorphism.

Proof. We have $\phi: S - K \to S^1$ with fiber F and monodromy $h: F \to F$. Note that $M_k - K$ is the pull-back

$$\begin{array}{ccc} M_k - K & \longrightarrow & S - K \\ \downarrow{\scriptstyle \hat{\phi}} & & \downarrow{\scriptstyle \phi} \\ S^1 & \xrightarrow{\lambda_k} & S^1 \end{array} \qquad \lambda_k(x) = x^k.$$

Hence, $M_k - K$ fibers over S^1 with monodromy h^k and fiber F. Applying the Wang sequence [see 25] we find for $n > 1$

$$0 \to H_{n+1}(M_k - K) \to H_n(F) \xrightarrow{I-H^k} H_n(F) \to H_n(M_k - K) \to 0.$$

However, we have already shown that $I - H^k$ is a relation matrix for $H_n(M_k)$. In case $n = 1$, it is easy to check directly that $H_1(M_k - K) \cong H_1(M_k) \oplus Z$. Thus, $G \cong \hat{G}$. Since we know that $H_n(M_k)$ is generated by the classes in $H_n(F)$, it follows that generators and relations are carried on $M_k - K$. By this we mean that given $a \in C_n(F)$, the $(n+1)$-chain A corresponding to $\bigcup_{t\in[0,1]} h_t{}^k(a)$ is supported in $M_k - K$.

$\partial A = a - H^k a$. Let $\{a_i\}$ be a basis for $H_n(F)$, then we have A_i such that $\partial A_i = a_i - H^k a_i$. Λ is determined by intersection numbers between the $\{A_i\}$ and the $\{a_j\}$. Hence, we may identify Λ and $\hat{\Lambda}$.

Corollary 1.28. Suppose K has periodic monodromy, h of order d so that $h^d = 1$. Then $H_n(M_k) = H_n(M_{k+d})$ and their linking invariants are also identical.

Proof. $M_k - K$ is diffeomorphic to $M_{k+d} - K$ since $h^{k+d} = h^k$.

Remarks. 1) A slightly more algebraic argument shows that the corollary is in fact true for homologically periodic monodromy.

2) Note that while $M_k - K$ and $M_{k+d} - K$ are diffeomorphic, M_k and M_{k+d} may not even be homeomorphic. For example, the fundamental group of M_k gets progressively more complicated as k increases when K is a torus knot. In the high-dimensional simply-connected case, one can say more *if* N_k is parallelizable.

Problem. Suppose that the complement of K is fibered with a parallelizable fiber F. Is this enough to ensure that the manifolds N_k will be parallelizable for all k?

We can answer yes for the special case where K is the neighborhood boundary of a singularity of a complex hypersurface by reinterpreting N_k as in Chapter III. I am not sure of the answer to the general case.

c) Differentiable Periodicity for M_k Simply Connected. Here we prove a generalization of a periodicity theorem due to Allan Durfee [6] for the case of Brieskorn varieties.

Theorem 1.29. Suppose $K^{2n-1} \subset S^{2n+1}$ is a simple, differentiable knot, whose complement fibers with periodic monodromy. Assume that, M_k simply-connected, N_k is parallelizable for each k, $n \geq 4$ is even, d =

period of the monodromy, $H_{n-1}(M_k)$ has no summands of order 2 or 4. Then $M_{k+d} \approx \tau_k \cdot \Sigma \# M_k$ where Σ is the Milnor sphere, and τ_k depends only on k and the Seifert pairing for F^{2n}, the fiber of the fibering $S - K \to S^1$.

Proof. This follows immediately from the Corollary 1.28 and the classification Theorem 0.5.

Remark. 1) If F^{2n} is parallelizable, then an argument due to Su [see 37] shows that N_k is parallelizable for k odd. Conjecture: F^{2n} parallelizable $\Rightarrow N_k$ parallelizable for all k.

2) One can apply the theorem to weighted homogeneous polynomials once one verifies that $V(x^k + f(z)) \cap S_\varepsilon^{2n+1}$ is a k-fold cyclic branched cover. See the next section for a discussion of this point.

3) We wish to note that our contribution to this problem is the introduction of the geometric construction of the manifold N_k. In any case, the theorem will be in an unsatisfactory state until the parallelizability question is settled.

6. Algebraic Varieties as Branched Covers

Let f(z) be a polynomial in n complex variables $z = (z_1, z_2, \ldots, z_n)$. Let $W = \{(x,z) \in C \times C^n | 0 = x^k + f(z)\}$ for $k \geq 1$ an integer, and $W' = \{(x,z) \in C \times C^n | 0 = x + f(z)\}$, $\pi: W \to W'$, $\pi(x,z) = (x^k, z)$. Thus, π is a cyclic branched cover of W' branching along $\{(0,z) | 0 = f(z)\}$. We wish to localize this situation in the neighborhood of a singularity of f.

First recall some facts about hypersurface singularities (see [25]). A point $z \in C^n$ is said to be singular (a singularity of f) if all partials $\partial f/\partial z_i$, $i = 1,2,\ldots,n$ vanish at z. The point z is said to be an isolated singularity if it has a neighborhood in which all other points are non-singular. Let $V = V(f) = \{z \in C^n | f(z) = 0\}$. Milnor studied the topology of V in the neighborhood of a point $x \in C^n$. Let $S_\varepsilon = S_\varepsilon^{2n-1}$ be a small sphere centered at x. Consider $K = V \cap S_\varepsilon$. Let $\phi: S_\varepsilon - K \to S^1$, $\phi(z) = f(z)/|f(z)|$. Then ϕ is the projection map of a smooth fiber bundle.

Given f with isolated singularity at $0 = (0,\ldots,0)$, we wish to study $F = x^k + f(z)$ regarded as a polynomial in n+1 complex variables. Given W as above, let $K = W \cap S_\varepsilon^{2n+1}$. We wish to show that K is a branched covering space of S_ε^{2n-1} with branch set K. This will be divided into two parts. First we treat only weighted homogeneous polynomials, then general polynomials will be considered.

A. Weighted Homogeneous Polynomials

The polynomial f(z) is said to be weighted homogeneous of type $(w_1,\ldots,w_n)$ if it can be expressed as a linear combination of monomials $z_1^{i_1} \ldots z_n^{i_n}$ for which $i_1/w_1 + \ldots + i_n/w_n = 1$, where $w_1,\ldots,w_n$ are positive rational numbers. Thus, the Brieskorn polynomials $z_1^{a_1} + z_2^{a_2} + \ldots + z_n^{a_n}$ are weighted homogeneous of type $(a_1,\ldots,a_n)$.

Given f weighted homogeneous of type $(w_1,\ldots,w_n)$, define $\rho * z = (\rho^{1/w_1} z_1,\ldots,\rho^{1/w_n} z_n)$ for ρ real and positive. Clearly, $f(\rho * z) = \rho f(z)$.

Proposition 1.30. Let $f(z) = f(z_1,\ldots,z_n)$ be weighted homogeneous and suppose f has an isolated singularity at $o \in C^n$. Let $F(x,z) = x^k + f(z)$, $K = V(f) \cap S_\varepsilon^{2n-1}$, $\mathbb{K} = V(F) \cap S_\varepsilon^{2n+1}$. Define $P: \mathbb{K} \to S_\varepsilon^{2n-1}$ via $P(x,z) = \rho * z$ such that $|\rho * z| = \varepsilon$. Then P is a k-fold branched covering with branch set K.

Proof. First note that given $(x,z) \in K$, $|x|^2 + |z|^2 = \varepsilon^2$ whence $|z| \leq \varepsilon$. Since $|\rho * z|$ is strictly increasing as a function of ρ, there is a unique $\rho \geq 1$ such that $|\rho * z| = \varepsilon$. Hence, P is well-defined. (Also, $z \neq 0$ since $z = 0 \Rightarrow f(z) = 0 \Rightarrow x = 0$.) Suppose $P(x,z) = P(x',z')$. Then $\rho * z = \rho' * z'$ for some $\rho, \rho' \geq 1$. This implies that $z' = \rho'' * z$ and may assume $\rho'' \geq 1$ (otherwise choose $z = \rho'' * z'$). Now $x'^k = -f(z') = -f(\rho'' * z) = -\rho'' f(z)$ and $x^k = -f(z)$. Thus, $x'^k = \rho'' x^k$. $\varepsilon^2 = |x'|^2 + |z'|^2 = |\rho''^{1/k} x|^2 + |\rho'' * z|^2 \geq |x|^2 + |z|^2 = \varepsilon^2$. Hence, $\rho'' = 1$ and, therefore, $z' = z$, $x^k = x'^k$. Thus, $P(x,z) = P(x',z')$ $z' = z$, $x'^k = x^k$.

Given $z \in S_\varepsilon^{2n-1}$ there is certainly a unique ρ, $1 \geq \rho > 0$ such that $|\rho f(z)|^{2/k} + |\rho * z|^2 = \varepsilon^2$. Then $P^{-1}(z) = \{(x, \rho * z) | x^k + f(\rho * z) = 0\}$. Hence, $P^{-1}(z)$ consists of k distinct points except when $z \in K$ and then $P^{-1}(z) = \{(0,z)\}$.

B. The General Case

For an arbitrary f(z) we need the analogue of $\rho * z$. This will be supplied by constructing an appropriate vector field. First recall some facts from Milnor's book [25].

1. Given an analytic function $f(z) = f(z_1, \ldots, z_n)$, define $\nabla f = (\overline{\partial f/\partial z_1}, \ldots, \overline{\partial f/\partial z_n})$ where the bar denotes complex conjugation. Let $\langle\, ,\, \rangle$ denote the Hermitian inner product on C^n. Thus, given a path $z = \alpha(t)$, $df(\alpha(t))/dt = \langle d\alpha/dt, \nabla f\rangle$. If $K = V(f) \cap S_\varepsilon$, $\phi: S_\varepsilon - K \to S^1$, $\phi(z) = f(z)/|f(z)|$, then: The set of critical pointsof ϕ, $\mathrm{Crit}(\phi) = \{z \in S_\varepsilon - K | i\nabla \log f(z) = \text{real multiple of } z\}$.

2. Now view $C^n = R^{2n}$ so that the real inner product is the same as $R\langle\, ,\, \rangle$.

Lemma. Given any polynomial f which vanishes at the origin,

$\exists \varepsilon_o > 0$ such that $\forall\ z \in C^n - V(f)$ with $|z| \leq \varepsilon_o$, the vectors z and $\nabla \log f(z)$ are either linearly independent or $\nabla \log f(z) = \lambda z$ where λ is a non-zero complex number with $|\arg\lambda| < \pi/4$.

Given these facts, we can construct the needed vector field. The lemma needed is an exercise in Milnor's book.

Lemma 1.31. Given f and $\varepsilon \leq \varepsilon_o$, as in previous lemma, then $\exists$ a smooth vector field v on $D_\varepsilon^{2n} - V(f)$ such that $\langle v(z), \nabla \log f(z)\rangle$ is real positive for all $z \in D_\varepsilon - V$ and $\langle v(z),z\rangle$ has positive real part.

Proof. It will suffice to work locally in a neighborhood of $z_o \in D_\varepsilon - V$ since we can then construct the vector field by using a partition of unity.

Case 1. z_o and $\nabla \log f(z_o)$ linearly independent over C. We want $\langle v, \nabla \log f(z_o)\rangle > 0$ and $R\langle v,z_o\rangle > 0$. The linear independence insures that such v can be chosen.

Case 2. $\nabla \log f(z_o) = \lambda z_o$, $|\arg\lambda| < \pi/4$. Then

$$R\langle \lambda z_o, z_o\rangle = R(\lambda)\cdot|z_o|^2 > 0$$
$$\langle \lambda z_o, \lambda z_o\rangle = \lambda\bar{\lambda}|z_o|^2 > 0.$$

Hence, in this case we can choose $v = \lambda z_o$.

Consider solutions of $dP/dt = v(P(t))$ on $D_\varepsilon - V$. The condition $\langle dP/dt, \nabla \log f(P(t))\rangle$ real positive implies that $\arg f(P(t)) =$ constant and that $|f(P(t))|$ is strictly increasing as a function of t. $\frac{d}{dt}(|P(t)|^2) = 2R\langle dP/dt, P(t)\rangle > 0$ implies that $|P(t)|$ is also strictly increasing.

Thus, given any point $z_o \in D_\varepsilon - V$ we may travel along a trajectory with initial condition $P(0) = z_o$. The path will move away from the origin in the direction of increasing $|f|$.

Let $\rho(t) = |f(P(t))|/|f(z_o)|$. Thus, ρ is an increasing function of

t with $\rho(0) = 1$. Denote $\rho * z_o = P(t)$ for the unique t such that $\rho = |f(P(t))|/|f(z_o)|$. Since $\arg f(P(t)) = \arg f(z)$, we have

$$\underline{f(\rho * z_o) = \rho f(z_o)}.$$

Assume from now on that the vector field v was normalized so that v is replaced by $\tilde{v}(z) = v(z)/\langle v(z), \nabla \log f(z)\rangle$. Thus, $\langle \tilde{v}, \nabla \log f\rangle \equiv 1$.

Claim. $f(P(t))$ does not approach zero for any finite limit $t \to t_o$, $t_o < 0$.

Proof.
$$\begin{aligned} \left|\frac{d}{dt} \log f\right| &= |\langle dP/dt, \nabla \log f\rangle| \\ &= |\langle v, \nabla \log f\rangle| \\ &\equiv 1. \end{aligned}$$

Since the above derivative is bounded, the thing can't blow up.

Reformulation. $\rho * z \neq 0$ for any ρ such that $0 < \rho$ and $\lim_{\rho \to 0} (\rho * z) = 0$.

Lemma 1.32. $0 < \rho, \rho'$, $\rho * z_1 = \rho' * z_2$ for $z_1, z_2 \in D_\varepsilon - V \Longrightarrow z_1 = \rho'' * z_2$ for some $\rho'' > 0$.

Proof. $\rho * z_1 = \rho' * z_2 \Longrightarrow z_1$ and z_2 lie on the same trajectory, by local uniqueness of solutions to differential equations. Hence, $z_1 = \rho'' * z_2$ for some $\rho'' > 0$.

Proposition 1.33. Let $f(z)$ be any polynomial with an isolated singularity at the origin. $\mathbb{K} = V(x^k + f(z)) \cap S_\varepsilon^{2n+1}$. $K = \{(0,z) \in \mathbb{K}\}$. $P: \mathbb{K} - K \to S_\varepsilon^{2n-1} - K$, $P(x,z) = \rho * z$ where $\rho \geq 1$ is the unique real such that $|\rho * z| = \varepsilon$. Then P is a k-fold cyclic cover.

Proof. The proof is the same as the proof for weighted homogeneous polynomials. However, we only define P on $\mathbb{K} - K$ since $\rho * z$ is not defined for $z \in V(f)$.

Corollary 1.34. The following diagram is commutative.

$$\begin{array}{ccc} \mathbb{K} - K & \xrightarrow{P} & S_\varepsilon^{2n-1} - K \\ \downarrow \hat{\phi} & & \downarrow \phi \\ S^1 & \xrightarrow{\lambda_k} & S^1 \end{array} \qquad \begin{array}{l} \hat{\phi}(x,z) = x/|x| \\ \lambda_k(\mu) = -\mu^k \end{array}$$

<u>Proof</u>.
$$\begin{aligned} \lambda_k\phi(x,z) &= -x^k/|x^k| = f(z)/|f(z)| \\ &= f(\rho*z)/|f(\rho*z)| \\ &= \phi(\rho*z) = \phi P(x,z). \end{aligned}$$

So far we have not succeeded in showing that K is a <u>branched</u> covering space of the sphere. In order to do this we will remove a tubular neighborhood of K and define the branched covering maps on the tubular neighborhood and its complement so that they patch up.

First recall that there is a fibration $\phi\colon E \to S^1$, $E = \{z \in D_\varepsilon^{2n} \mid |f(z)| = \delta\}$, $0 < \delta << \varepsilon$. $\phi(z) = f(z)/|f(z)|$. In S_ε^{2n-1} let $N(K) = \{z \in S_\varepsilon^{2n-1} \mid |f(z)| < \delta\}$. Since $N(K) \xrightarrow{\pi} \mathring{D}^2\delta$, $z \to f(z)$, is regular, π is a fibration with fiber K and, hence, $N(K) \approx \mathring{D} \times K$. Thus, N(K) is a tubular neighborhood of K. Define $\eta\colon E \to S_\varepsilon^{2n-1} - N(K)$, $\eta(z) = \rho*z$ such that $|\rho*z| = 1$. Then $\eta\colon E \xrightarrow{\approx} S_\varepsilon^{2n-1} - N(K)$ and the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\eta} & S_\varepsilon^{2n-1} - N(K) \\ & \phi\searrow \quad \swarrow\phi & \\ & S^1 & \end{array}$$

Note that $\partial E \equiv \partial\overline{N}(K) = \{z \in S_\varepsilon^{2n-1} \mid f(z) = \delta e^{i\theta}\}$ and, hence, $\partial E \xrightarrow{\phi} S^1$ is the trivial bundle.

Now let E_k = k-fold cyclic cover of E formed as follows: $E_k = \{(x,z) \in D_r^{2n+2} \mid x^k = f(z), |f(z)| = \delta\}$ where $r = (\varepsilon^2 + \delta^{2/k})^{1/2}$, $q\colon E_k \to E$, $q(x,z) = z$.

Let $W_k = \{(x,z) \in S_r^{2n+1} \mid x^k = f(z), |f(z)| \geq \delta\}$ and define $\eta_k\colon E_k \xrightarrow{\approx} W_k$ by $\eta_k(x,z) = (\rho^{1/k}x, \rho*z) \equiv \rho*(x,z)$ such that $|\rho*(x,z)| = r$.

Lemma 1.35. Letting $P_k: W_k \to S_\varepsilon^{2n-1} - N(K)$ by $P_k(x,z) = \rho * z$ such that $|\rho * z| = \varepsilon$, then the following diagram commutes:

$$\begin{array}{ccc} W_k & \xrightarrow{P_k} & S_\varepsilon^{2n-1} - N(K) \\ \uparrow{\scriptstyle \eta_k} & & \uparrow{\scriptstyle \eta} \\ E_k & \xrightarrow{q_k} & E \end{array}$$

Proof. Easy.

Thus, P_k is a k-fold cyclic cover.

Lemma 1.36. Let $N'(K) = \{(x,z) \in S_r^{2n+1} | x^k = f(z), |f(z)| < \delta\}$, then for ε sufficiently small and $0 < \delta << \varepsilon$, $N'(K) \approx K \times \mathring{D}_\delta^2$.

Proof. $N'(K) \xrightarrow{\pi'} \mathring{D}_\delta^2$, $(x,z) \to f(z)$. For ε and δ very small, $V(x^k - f(z))$ and $V(f)$ will both be transversal to S_r^{2n+1} and $V(f)$ will be transverse to $V(x^k - f(z)) \cap S_r^{2n+1}$. Hence, π' is regular, therefore, a fibration and, hence, the conclusion.

Thus, we may define $P_k: \overline{N}'(K) \to \overline{N}(K)$ corresponding to the map $K \times D^2 \to K \times D^2$. $(y,x) \to (y,x^k)$ viewing D^2 as complex disk, such that on the boundaries this agrees with the original P_k. Hence, P_k extends to $P_k: W_k \cup N'(K) \to S_\varepsilon^{2n-1}$. Since $W_k \cup N'(K) = V(x^k - f(z)) \cap S_r^{2n+1}$, we have proved the

Proposition 1.37. $P_k: V(x^k - f(z)) \cap S_r^{2n+1} \to S_\varepsilon^{2n-1}$ is a k-fold cyclic branched cover with branch set $K = V(f) \cap S_\varepsilon^{2n-1}$.

7. Orbit Space for a Hypersurface with O(n)-Action

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in C^k$, $z = (z_1, \ldots, z_n) \in C^n$. Let $f(\alpha)$ be a polynomial with an isolated singularity at the origin; $F(\alpha,z) = f(\alpha) + z_1^2 + z_2^2 + \ldots + z_n^2$. Let $M = V(F) \cap S_\varepsilon^{2(n+k)-1}$ with ε chosen small enough so that the intersection is transverse. Letting $z = x + iy$, $x,y \in R^n$, we see that

$M = \{(\alpha,x,y) \in C^k \times R^n \times R^n \mid f(\alpha) + |x|^2 - |y|^2 + 2i\langle x,y\rangle = 0,$
$|\alpha|^2 + |x|^2 + |y|^2 = \varepsilon^2\}.$

Thus, M admits an O(n)-action via $g\cdot(\alpha,x,y) = (\alpha,g\cdot x,g\cdot y)$ where $g\cdot x$ denotes the usual action of O(n) on R^n. Note that α completely determines an orbit. Hence, if $\pi\colon M \to C^k$, $\pi(\alpha,z) = (\alpha)$, then $\pi(M) \approx M/O(n)$.

To examine $\pi(M)$ we first consider the case where f is weighted homogeneous.

<u>Proposition 1.38.</u> If f is weighted homogeneous, $n > 1$, then $\pi(M) \approx D^{2k}$.

<u>Proof.</u> From the definition of M, it follows that
$|f(\alpha)|^2 = (|x|^2 - |y|^2)^2 + 4\langle x,y\rangle^2$ and $|x|^2 + |y|^2 = \varepsilon^2 - |\alpha|^2$.

Hence,

$$|f(\alpha)|^2 = (\varepsilon^2 - |\alpha|^2)^2 + 4(\langle x,y\rangle^2 - |x|^2|y|^2).$$

Thus,

$$|f(\alpha)|^2 \le (\varepsilon^2 - |\alpha|^2)^2 \text{ or } |f(\alpha)| + |\alpha|^2 \le \varepsilon^2.$$

Define $\mathcal{D} = \{\alpha \in C^k \mid |f(\alpha)| + |\alpha|^2 \le \varepsilon^2\}$.

<u>Lemma.</u> $\pi\colon M \to \mathcal{D}$ is surjective.

<u>Proof.</u> Given $\alpha \in \mathcal{D}$, choose $x,y \in R^n$ such that $|x|^2 + |y|^2 = c$, $c = \varepsilon^2 - |\alpha|^2$; in fact, let $|x| = \frac{\sqrt{2}}{2}(c - f_1)^{1/2}$, $|y| = \frac{\sqrt{2}}{2}(c + f_1)^{1/2}$ where $f(\alpha) = f_1 + if_2$ and, hence, $(f_1^2 + f_2^2)^{1/2} \le c$ so that $|f_1| \le c$. For $f_2 \ne 0$, adjust $2\langle x,y\rangle = 2|x||y|\cos\theta = -f_2$; that is, set $\cos\theta = -f_2/(c^2 - f_1^2)^{1/2}$. Note that $f_1^2 + f_2^2 \le c^2$ whence $f_2^2 \le c^2 - f_1^2$ and $\therefore\ |-f_2/(c^2 - f_1^2)^{1/2}| \le 1$. With these choices $(\alpha,x,y) \in M$.

Thus, $\quad \pi(M) = \mathcal{D} = \{\alpha \in C^k \mid |f(\alpha)| + |\alpha|^2 \le \varepsilon^2\}$.

Define $\psi\colon \mathcal{D} \to D^{2k}_\varepsilon$ by $\psi(\alpha) = \rho*\alpha$ such that $|\rho*\alpha|^2 = |f(\alpha)| + |\alpha|^2$. Since f is weighted homogeneous, $\rho*\alpha$ is defined on all of C^k such that $f(\rho*\alpha) = \rho f(\alpha)$. Clearly, $\psi\colon \mathcal{D} \xrightarrow{\approx} D^{2k}_\varepsilon$ and this completes the proof of the proposition.

Corollary 1.39. Given f as above but not necessarily weighted homogeneous,

$$\pi(M) \approx \mathcal{D} = \{\alpha \in C^k \mid |f(\alpha)| + |\alpha|^2 \leq \varepsilon^2\}.$$

Proof. Except for the last paragraph the above was quite general.

Remark. We now wish to analyze $\mathcal{D}$ more closely in the general case by using the vector field constructed in the section on algebraic branched covers.

Proposition 1.40. Suppose f has an isolated singularity at the origin of C^k, ε chosen as above. Then $\mathcal{D}$ is homeomorphic to D_ε^{2k}.

Proof. Consider π: $V(x^2 - f(\alpha)) \cap D_\varepsilon^{2k+2} \to C^k$, $\pi(x,\alpha) = \alpha$. Then $\text{Image}(\pi) = \{\alpha \mid x^2 = f(\alpha),\ |x|^2 + |\alpha|^2 \leq \varepsilon^2\} = \{\alpha \mid |f(\alpha)| + |\alpha|^2 \leq \varepsilon^2\} = \mathcal{D}$. We know that $V(x^2 - f(\alpha)) \cap D_\varepsilon^{2k+2}$ is homeomorphic to the cone C(K) where $K = V(x^2 - f(\alpha)) \cap S_\varepsilon^{2k+1}$. (See [25].) This homeomorphism is accomplished by means of a vector field on D_ε^{2k+2} which points away from the origin and is tangential to $V(x^2 - f(a)) - \{0\}$. Let $W = V(x^2 - f(\alpha)) \cap D_\varepsilon^{2k+2}$. Then one can see that the O(1)-actions on K and W $((x,\alpha) \to (-x,\alpha))$ are compatible with this cone construction. Hence, $W/O(1) = C(K/O(1))$. Whence $\mathcal{D} = C(K/O(1))$. However, the section on branched covers shows that $K/O(1) \approx S_\varepsilon^{2k-1}$. Hence, $\mathcal{D} = C(S^{2k-1}) = D_\varepsilon^{2k}$.

Remark. Since $\mathcal{D}$ contains a tiny ball about the origin and the map from $\mathcal{D}$ to D_ε^{2k} is a diffeomorphism everywhere except possibly at the origin, we see that we may assign a differentiable structure to $\mathcal{D}$ so that $\partial\mathcal{D}$ is the usual structure on S_ε^{2k-1} and the orbit map π: $M \to \mathcal{D}$ is differentiable. Note that $\mathcal{D}$ could possibly be some non-standard smoothing for $k = 2$.

C. The Quadratic Form of a Link

Let $L \subset S^3$ be a link, F a Seifert surface for L, $W = H_1(F)$. We have discussed the Seifert pairing $\theta: W \times W \to \mathbb{Z}$ and the pairing $f: W \times W \to \mathbb{Z}$, $f(x,y) = \theta(x,y) + \theta(y,x)$. Using the construction of Chapter 0 there is an associated bilinear pairing $b(f): G \times G \to Q/\mathbb{Z}$ where $G = \tau(\mathrm{cok}\ f)$, and a quadratic form $q(f): G \to Q/\mathbb{Z}$ for which $b(f)$ is the associated bilinear form.

Note that since $b(f)$ and $q(f)$ may be interpreted as the linking forms for the double branched cover for L, they are invariants of the link type of L.

Definition. $q(f)$ is the quadratic form of the link L. It will be denoted by $q(L)$. Note: This may be at variance with the usual terminology which takes the integral quadratic form associated with f as the quadratic form of L.

Example. Let $K \subset S^3$ be a knot, M_2 and N_2 as before. Thus, N_2 has intersection form $N = V + V'$. Since $N = V - V' + 2V' = \Delta + 2V'$ and $\det(\Delta) = 1$ for K a knot, it follows that $r = \det(N) \neq 0$. Choosing a basis $\{a_i\}$ for $H_2(N_2,M_2)$ and Poincare dual basis $\{A_i\}$ for $H_2(N_2)$ such that $\langle A_i, a_j \rangle = \delta_{ij}$, there is an exact sequence
$H_2(N_2) \xrightarrow{N} H_2(N_2,M_2) \xrightarrow{P} H_1(M_2) \to 0$. Hence, $H_1(M_2)$ is a torsion group. Let $\bar{a}_i = P(a_i)$.

Proposition 2.4. Let $\mathbb{\Lambda} = (\Lambda(\bar{a}_i,\bar{a}_j))$ where $\Lambda: H_1(M_2) \times H_1(M_2) \to Q/\mathbb{Z}$ is the linking invariant, then $\mathbb{\Lambda} = N^{-1}$ over $Q/\mathbb{Z}$.

Remark. This proposition is due to Seifert [32] by quite different arguments.

Proof. The linking invariant is calculated as follows: Given

$\bar{a},\bar{b} \in H_1(M_2) \Longrightarrow ra = N\alpha$, $rb = N\beta$, $\Lambda(\bar{a},\bar{b}) = \frac{1}{r}\langle\alpha,b\rangle$. Using the above bases, $ra_i = N(\Sigma_j c_{ji} A_j) = (NC)A_i$, $C = (c_{ij})$, $\Lambda(\bar{a}_i,\bar{a}_j) = \frac{1}{r}\langle CA_i,a_j\rangle = \frac{1}{r}\Sigma_k c_{ki}\langle A_k,a_j\rangle$, $\Lambda(\bar{a}_i,\bar{a}_j) = \frac{1}{r}c_{ji}$. But $NCA_i = ra_i$ whence $NC = r\cdot I$. Hence, $N^{-1} = \frac{1}{r}C$. Since $N' = N$, then $C' = C$. Therefore, $\Lambda = N^{-1}$ over $Q/\mathbb{Z}$.

D. The Z_2-Quadratic Form for a Link

Let $L \subset S^3$ be a link with Seifert surface F, $W = H_1(F)$, $\theta: W \times W \to \mathbb{Z}$, $f: W \times W \to \mathbb{Z}$ as before. Note that there is also the intersection pairing $\langle\ ,\ \rangle: W \times W \to \mathbb{Z}$. This is a skew-form and $\langle x,y\rangle = \theta(x,y) - \theta(y,x)$.

Let $\{a_1,\ldots,a_r\}$ be a basis for W, $V = (v_{ij})$ the matrix of θ with respect to this basis, $v_{ij} = \theta(a_i,a_j)$. Let $\overline{W} = W \otimes Z_2$ with Z_2-basis $\{\bar{a}_1,\ldots,\bar{a}_r\}$. Define a quadratic form $\psi: \overline{W} \to Z_2$ as follows: $x = x_1\bar{a}_1 + \ldots + x_r\bar{a}_r \in \overline{W}$, $\psi(x) = \sum_{i,j=1}^{r} x_i \bar{v}_{ij} x_j$.

Remark. This definition is due to Robertello [see 31] for knots. He proves that $c(\psi)$ is an invariant of knot-cobordism.

Lemma 2.5. $\psi(x + y) = \psi(x) + \psi(y) + \langle x,y\rangle \pmod 2$.

Proof. Easy.

Thus, ψ is a Z_2-quadratic form associated to the skew form $\langle\ ,\ \rangle$.

Definition. Let A be a module over $\mathbb{Z}$, $g: A \times A \to \mathbb{Z}$ an even symmetric bilinear form. Define a Z_2-quadratic form on $A/2A$ by $\phi: A/2A \to Z_2$, $\phi(\bar{a}) = \overline{\frac{1}{2}f(a,a)}$ (where —— denotes mod 2 class). ϕ is called the mod 2-reduction of g.

Remark. The above definition works just as well for A an $R(2)$-module ($R(2)$ = 2-adic integers). Thus, beginning with a Z-module A, one may view g as a form on $R(2)$ to simplify the mod-2 reduction.

Lemma 2.6. ψ is the mod-2 reduction of f.

Proof. Let $x = x_1a_1 + \ldots + x_ra_r \in W$, $\bar{x} = \bar{x}_1\bar{a}_1 + \ldots + \bar{x}_r\bar{a}_r \in \overline{W}$ so that

$$\begin{aligned}\psi(\bar{x}) &= \Sigma_{ij}\bar{x}_i\bar{v}_{ij}\bar{x}_j \\ &= \overline{\tfrac{1}{2}(2\Sigma_{ij}x_iv_{ij}x_j)} \\ &= \overline{\tfrac{1}{2}(\Sigma_{ij}x_i(v_{ij}+v_{ji})x_j)} \\ \psi(\bar{x}) &= \overline{\tfrac{1}{2}f(x,x)}.\end{aligned}$$

Note: Since cok(< , >) = $\tilde{H}_0(L)$ is torsion free, it follows from Section 3 of Chapter 0 that to classify the pair (< , >,ψ), it is sufficient to classify < , > and ψ separately.

Certainly < , > and ψ depend on the choice of Seifert surface. However, the same remarks about behavior under elementary transformations of the link projection apply here to show that < , > is determined up to direct sums with $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} = S(1)$ and ψ is determined up to direct sums with the mod-2 reduction of $U = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$; call this ϕ_0. Since the Arf invariant $c(\phi_0) = 0$, we conclude:

Lemma 2.7. The Arf invariant $c(\psi)$, when defined, is an invariant of link type.

Conjecture. $c(\psi)$ should be an invariant of link cobordism for some suitable notion of link cobordism.

Example. Let L_k be a (2,k) torus link, k = 1,2,3,... . Then, with respect to the orientation and projection indicated in Figure 20, L_k has Seifert matrix

$$V_k = \begin{bmatrix} -1 & 1 & & \bigcirc \\ & -1 & \ddots & \\ & & \ddots & 1 \\ \bigcirc & & & -1 \end{bmatrix} \quad (k-1) \times (k-1).$$

Hence, the matrix of the bilinear form f is $N_k = V_k + V_k'$ and < , > has matrix $\Delta_k = V_k - V_k'$. One finds (Durfee observed this in the context of Brieskorn varieties) that [see 6, pages 92-101]

1) $\Delta_k \simeq S(1) \oplus \ldots \oplus S(1)$ ((k-1)/2 copies) k odd

$\Delta_k \simeq [0] \oplus S(1) \oplus \ldots \oplus S(1)$ (k/2 - 1 copies) k even

2) <u>k odd</u>, $k \equiv \pm 1 \bmod 8$ $\quad N_k \simeq U \oplus \ldots \oplus U$ over R(2)

$\psi \simeq \phi_o \oplus \ldots \oplus \phi_o$

<u>k odd</u>, $k \equiv \pm 3 \bmod 8$ $\quad N_k \simeq T \oplus U \oplus \ldots \oplus U$ over R(2)

$\psi \simeq \phi_1 \oplus \phi_o \oplus \ldots \oplus \phi_o$

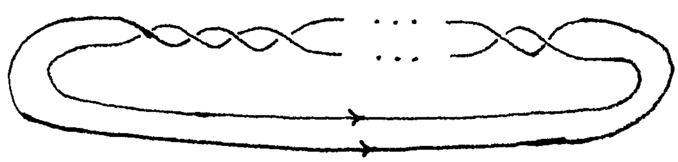
$\left(U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$

<u>k even</u> $\quad N_k \simeq N_{k-1} \oplus [(k-1)k]$ over R(2).

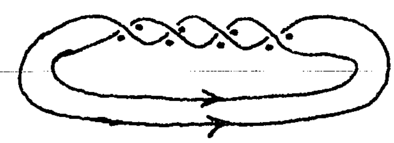
Thus, disregarding the forms U, S(1), ϕ_o, there is a periodicity of 8 in k for these invariants.

This periodicity manifests itself as a periodicity in the list of Brieskorn manifolds $K_k = V(z_o^k + z_1^2 + \ldots + z_n^2) \cap S^{2n+1}$ for n even. This example will be mentioned again in Section 3. Our point of view is that the periodicity in the invariants of these links "explains" the periodicity in the list K_k, k = 1,2,... .

<u>Figure 20</u>



L_k



$$V_4 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

2. The Symmetry Group of a Link

Let $L \subset S^3$ be a (differentiable) link with μ components $K_1, K_2, \ldots, K_\mu$. Write $L = K_1 \cup K_2 \cup \ldots \cup K_\mu$. Assume that each component has been oriented, with $+K_i$ standing for this orientation, $-K_i$ for the opposite orientation. S^3 will also be given an orientation.

If ψ is an auto-diffeomorphism of the pair (S^3, L), then ψ may permute the components of L and change some orientations. Wishing to catalogue such possibilities, we define the symmetry group of L, $\Sigma(L)$ following Whitten [41]:

Let S_μ = the symmetric group on μ letters, $Z_2 = \{\pm 1\}$, $Z_2^{\mu+1} = Z_2 \oplus Z_2 \oplus \ldots \oplus Z_2$ $\mu+1$ times. Define a split extension $1 \to Z_2^{\mu+1} \to \Gamma_\mu \to S_\mu \to 1$ via $W\colon S_\mu \to \mathrm{Aut}(Z_2^{\mu+1})$. $W(p) = W_p$, $W_p(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_\mu) = (\varepsilon_0, \varepsilon_{p_1}, \ldots, \varepsilon_{p_\mu})$. Elements $\gamma \in \Gamma_\mu$ are written $\gamma = (\varepsilon_0, \ldots, \varepsilon_\mu, p)$ for $\varepsilon_i = \pm 1$, $p \in S_\mu$. We say that L admits γ if $L = +K_1 \cup \ldots \cup +K_\mu$ is taken to $L^\gamma = \varepsilon_1 K_{p_1} \cup \ldots \cup \varepsilon_\mu K_{p_\mu}$ under an auto-diffeomorphism $\psi\colon S^3 \to S^3$ such that $\psi(+S^3) = \varepsilon_0 S^3$, $\psi(+K_i) = \varepsilon_i K_{p_i}$. The symmetry group $\Sigma(L) \equiv \{\gamma \in \Gamma_\mu \mid L \text{ admits } \gamma\}$.

Example. A torus link is a link which can be inscribed on a torus, standardly embedded in S^3. One sees that all the components of a torus link have the same knot type. Any torus knot may be described as a closed curve winding a times in the meridian direction and b times in the longitude direction on a torus (here a and b are relatively prime). Let $L_{a,b}$ denote a torus link of type (a,b), where, given $d = \gcd(a,b)$, $a = d\alpha$, $b = d\beta$. Thus, $L_{a,b}$ consists of d torus knots each of type (α, β). (See Figure 1.)

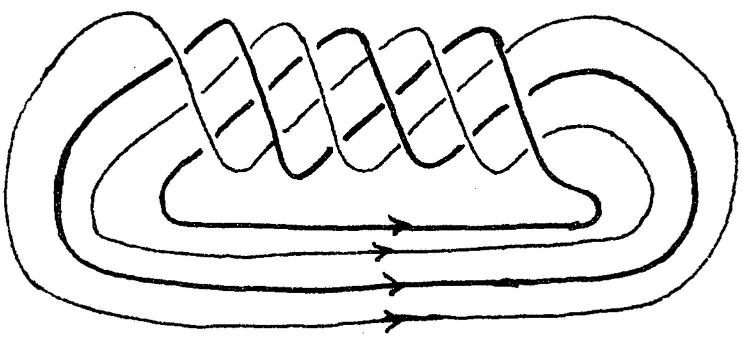


Figure 1

Give each component of $L_{a,b}$ an orientation as indicated in Figure 1.

Proposition 2.8. Let $L = L_{a,b}$, $d = \gcd(a,b)$, $a = d\alpha$, $b = d\beta$, $\alpha,\beta > 1$. Then $\Sigma(L) \cong Z_2 \times S_d$.

Proof. Let the components of L be $K_1,\ldots,K_d$. These are non-trivial torus knots of type (α,β). Letting ℓ denote linking number in S^3, one can easily see that $\ell(K_i,K_j) = \alpha\beta$ for $i \neq j$.

Note that a non-trivial torus knot is not amphicheiral [see 30]. This means that K_i can never be carried to K_j by any diffeomorphism which reverses the orientation of S^3. Hence, we may restrict attention to those diffeomorphisms which preserve the orientation of S^3. However, such diffeomorphisms preserve linking numbers. Hence, given $g: (S^3,L) \to (S^3,L)$, $\ell(g(K_i),g(K_j)) = \ell(K_i,K_j)$. Suppose $g(K_i) = \varepsilon_1 K_{i'}$, $g(K_j) = \varepsilon_2 K_{j'}$. Then $\varepsilon_1\varepsilon_2\alpha\beta = \ell(gK_i,gK_j) = \ell(K_i,K_j) = \alpha\beta$. Whence $\varepsilon_1\varepsilon_2 = +1$. Thus, $\varepsilon_1 = \varepsilon_2$. The upshot is that, at best, a symmetry can only reverse all of the link orientations. In fact, each torus link has such a symmetry. It is obtained by turning the link around and then rotating it about its central axis by 180 degrees.

On the other hand, for each permutation $p \in S_d$ there is a diffeomorphism $g(p)$ such that $g(p)(K_i) = K_{p_i}$. This is constructed by noting that the components of the link may be viewed as lying on concentric tori.

Hence, $$\Sigma(L) = Z_2 \times S_d.$$

3. Equivariant Classification of O(n)-Manifolds

Recall the outline of Jänich's classification theory for group actions [see 14, 15, 16]. Roughly, one has a manifold X with O(n) acting on it such that all isotropy groups are conjugate to either O(n-1) or O(n-2). Thus, in the case where the orbit space is D^2, the interior points correspond to orbits of type O(n)/O(n-2) and the boundary points to orbits of type O(n)/O(n-1). Inasmuch as $X \xrightarrow{\pi} D^2$ can be viewed as a pasting together of the bundle of orbits over the interior of D^2 and the bundle of orbits over $S^1 = \partial D^2$, the "pasting data" is given by a certain reduction of structural group. This turns out to correspond to a map $\sigma\colon \partial D^2 \to S^1$. Under appropriate conditions, O(n)-manifolds with two orbit types and orbit space D^2 are classified up to equivariant diffeomorphism by $|\text{degree}\ (\sigma)|$.

If, along with orbits homeomorphic to O(n)/O(n-2), O(n)/O(n-1) one allows fixed points, then under appropriate conditions the classification can be reduced to the case of two orbit types by removing a tubular neighborhood of the fixed point set. For example, if the orbit space is D^4 with fixed points corresponding to a link $L \subset S^3 = \partial D^4$, then it turns out that sufficient pasting data is given by a map $\sigma\colon S^3 - N(L) \to S^1$ where $N(L) = N(K_1) \cup \ldots \cup N(K_r)$ for $L = K_1 \cup \ldots \cup K_r$, K_i = component of L, $N(K_i)$ = tubular neighborhood of K_i chosen so that $N(K_i) \cap N(K_j) = \phi$ for $i \neq j$. σ is of degree ± 1 when restricted to a meridian circle on the boundary of $N(K_i)$.

The technical conditions under which the above remarks hold are restrictions on the slice representations, that is, the representations of the isotropy group normal to an orbit. Letting ρ_n denote the standard representation of $O(n)$ on R^n, and ε^k the k-dimensional trivial representation, one must require that the slice representations are as follows:

1) $O(n)$: $\varepsilon^1 \oplus \rho_n \oplus \rho_n$

2) $O(n-1)$: $\varepsilon^3 \oplus \rho_{n-1}$

3) $O(n-2)$: ε^4

Manifolds satisfying these conditions for either two or three orbit types will be referred to as O(n)-manifolds.

Example. Let $X = R^n \times R^n - \{0\}$ and let $O(n)$ act via $g\cdot(x,y) = (gx,gy)$, $x,y \in R^n$. One easily checks that this is an O(n)-manifold with two orbit types.

Proposition 2.9. If $\alpha \in C^k$, $z \in C^n$, $z = (z_1,\ldots,z_n)$, $F(\alpha,z) = f(\alpha) + z_1^2 + z_2^2 + \ldots + z_n^2$ where $f(\alpha)$ is a polynomial with an isolated singularity at the origin, let $M = V(F) \cap S_\varepsilon^{2(n+k)-1}$. Then M is an O(n)-manifold with orbit space homeomorphic to D^{2k} (diffeomorphic for $k \neq 2$).

Proof. We have already defined an action of $O(n)$ on M with the given orbit space. It remains to check the slice representations. Recall that $M = \{(\alpha,x,y) | f(\alpha) + |x|^2 - |y|^2 + 2i\langle x,y\rangle = 0,\ |\alpha|^2 + |x|^2 + |y|^2 = \varepsilon^2,\ x,y \in R^n\}$ and, for $g \in O(n)$, $g\cdot(\alpha,x,y) = (\alpha,gx,gy)$. Thus, the fixed point set is $K = \{(\alpha,0,0) | f(\alpha) = 0,\ |\alpha|^2 = \varepsilon^2\}$. The action of $O(n)$ on M is the restriction of a standard action on $C^k \times R^n \times R^n$. It follows that M is an O(n)-manifold by the same arguments as in [13, page 33].

Note that for $k = 2$ in the above $M/O(n) \approx D^4$ and the fixed point set corresponds to a link in S^3.

Definition. A link-manifold is an $O(n)$-manifold M with orbit space D^4 such that the fixed point set corresponds to a link in $S^3 = \partial D^4$.

Jänich proves a classification theorem [16] for link-manifolds: Given $L \subset S^3$ a differentiable link, let $N(L) = \bigcup N(K_i)$ be as above. Let S_i^1 be a meridian circle on the boundary of $N(K_i)$,

$$\mathcal{S} = [\sigma\colon S^3 - L \to S^1 \mid \deg\sigma | S_i^1 = \pm 1,\ i = 1,\ldots,r]$$

where [ ] denotes homotopy classes of maps. Suppose $g \in \mathrm{Diff}(D^4,L)$. Then $\sigma \in \mathcal{S}$ $g \circ \sigma \in \mathcal{S}$. Hence, $\mathrm{Diff}(D^4,L)$ acts on $\mathcal{S}$. Also, Z_2 acts on $\mathcal{S}$ via composition with $S^1 \to S^1$ of degree -1.

Theorem 2.10. (Jänich) Let $S_n(D^4,L)$ denote the set of equivariant diffeomorphism classes of $O(n)$-link manifolds with fixed point set corresponding to the given link $L \subset S^3$. Then $S_n(D^4,L)$ is a bijective correspondence with $\mathcal{S}/Z_2 \times \mathrm{Diff}(D^4,L)$.

It is interesting to observe that this result has a reformulation in terms of the symmetry group of the link L. This follows from a series of observations.

1) By Cerf [5] any element of $\mathrm{Diff}(S^3,L)$ extends to an element of $\mathrm{Diff}(D^4,L)$.

2) Since elements of $\mathcal{S}$ are determined by their restrictions to the meridian circles S_i^1, they really correspond to the 2^r possible orientations for these circles. Also, orienting a meridian circle is equivalent to choosing an orientation for the corresponding link component. Since any $g \in \mathrm{Diff}(S^3,L)$ may, for our purpose, be assumed to carry a tubular neighborhood of L to itself, an orientation for L will go to a new one under g, and the orientations on meridian circles will be correspondingly altered.

3) Letting $L = K_1 \cup K_2 \cup \ldots \cup K_r$, choose orientations for each component. Let $\mathcal{O}(L) = \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r) | \varepsilon_i = \pm 1\}$ denote the set of possible orientations for L ((1,1,...,1) denoting the chosen orientation). Z_2 operates on $\mathcal{O}(L)$ via $(\varepsilon_1, \ldots, \varepsilon_r) \to (-\varepsilon_1, \ldots, -\varepsilon_r)$. Clearly, the action of Diff(S^3, L) on $\mathcal{O}(L)$ corresponds precisely to the action of $\Sigma(L)$ on $\mathcal{O}(L)$ given by

$$\gamma \cdot (\varepsilon_1, \ldots, \varepsilon_r) = (\bar{\varepsilon}_{p^{-1}(1)} \varepsilon_{p^{-1}(1)}, \bar{\varepsilon}_{p^{-1}(2)} \varepsilon_{p^{-1}(2)}, \ldots, \bar{\varepsilon}_{p^{-1}(r)} \varepsilon_{p^{-1}(r)})$$

where $\gamma = (\bar{\varepsilon}_0, \bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r, p) \in \Sigma(L)$. Hence,

Theorem 2.11. $S_n(D^4, L) \longleftrightarrow \mathcal{O}(L)/Z_2 \times \Sigma(L)$.

Corollary 2.12. There are at most 2^{r-1} elements in $S_n(D^4, L)$.

Corollary 2.13. Let $L = L_{2,b}$ be a torus link of type a,b with $d = \gcd(a,b)$, $a = d\alpha$, $b = d\beta$, $\alpha, \beta > 1$. Then $S_n(D^4, L) \longleftrightarrow \mathcal{O}(L)/Z_2 \times S_d$ where S_d = symmetric group on d letters acting on $\mathcal{O}(L)$ by permutation. Hence, $S_n(D^4, L)$ has $\frac{1}{2}(d + 2)$ elements for d even and $\frac{1}{2}(d + 1)$ elements for d odd.

Proof. $\Sigma(L_{a,b}) = Z_2 \times S_d$ by the section on the symmetry group of a link. The rest follows by a simple counting argument.

For example: $L_{4,6}$, $d = 2$, $\frac{1}{2}(d + 2) = 2$. Hence, there are two equivariant classes corresponding to this link.

Examples. a) L = two circles with linking number ±1. There are two possible orientations (see Fig. 1) L_1 and L_2.

Claim. If $M_1, M_2 \in S_n(D^4, L)$ correspond to L_1 and L_2 respectively, then $M_1 \not\approx M_2$.

Proof. Let $g: (S^3, L) \to (S^3, L)$ be the composition $g = g_1 g_2$ where g_2 reverses orientation of S^3, and g_1 turns the right-hand circle over. Then $g(L_1) = L_2$. (See Fig. 2.)

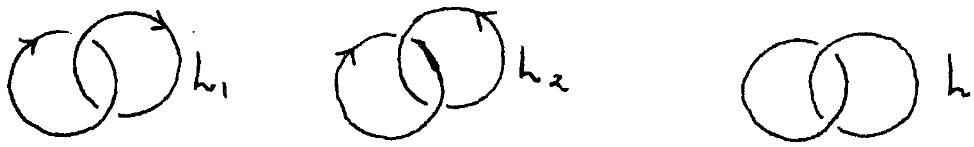


Figure 1

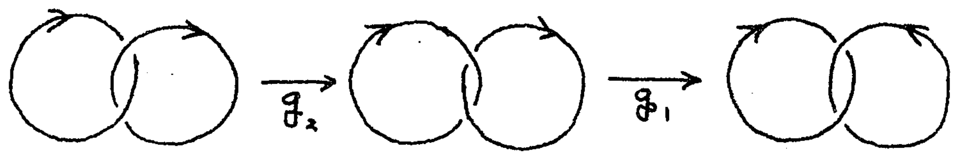


Figure 2

Thus, there is only one element in $S_n(D^4,L)$. This may be described variously as the tangent sphere bundle to S^{n+1}, the Brieskorn variety $\Sigma(2,2,\ldots,2)$, etc. The above provides an amusing way to prove that these seemingly different examples are really the same.

b) If L is a torus link of type (2,2k), k > 1, then the two choices of orientation yield different elements of $S_n(D^4,L)$. This follows from a signature check of the links in question. For example, let L be as pictured in Figure 3. Then, we wish to show that there is no symmetry taking L_1 to L_2. Since L_1 and L_2 have opposite self-linking numbers, such a symmetry would have to be orientation reversing. Hence, the question boils down to the existence of an orientation preserving symmetry between L_1 and L_2'. However, $\sigma(L_1) = 3$, $\sigma(L_2') = 1$. Hence, there is no such symmetry.

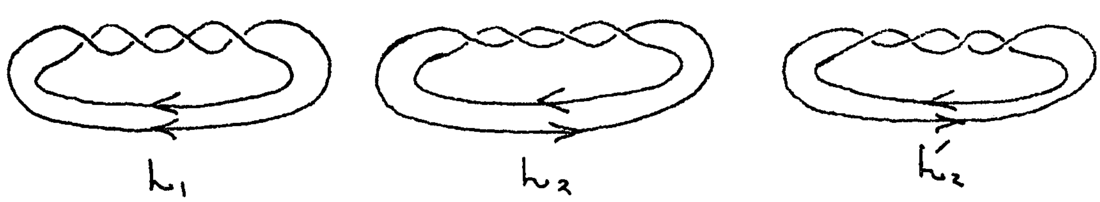


Figure 3

c) Let L be the <u>Borommean rings</u> (Fig. 4).

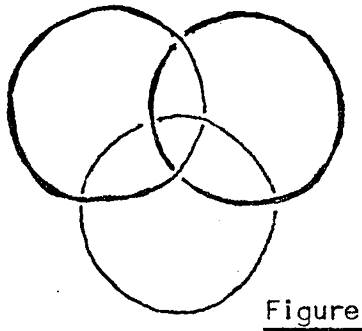


Figure 4

These have a great deal of symmetry. It is an easy exercise to check that $S_n(D^4,L)$ has only one element.

See the next section for more details on the manifolds in these last two examples.

d) Regard $S^{2n+1} = \{(z,x,y) \in C^2 \times R^n \times R^n \mid |z|^2 + |x|^2 + |y|^2 = 1\}$. Then $O(n)$ acts in standard fashion via $g\cdot(z,x,y) = (z,gx,gy)$, making S^{2n+1} an $O(n)$-manifold with fixed point set corresponding to an unknotted circle in D^4. Regard $D^4 = \{(z_1,z_2) \mid |z_1|^2 + |z_2|^2 = 1,\ z_1,z_2 \in C\}$. Letting a and b be integers ≥ 1, define $f: D^4 \to D^4$ by $f(z_1,z_2) = \rho(z_1{}^a,z_2{}^b)$ where ρ is chosen so that $|(\rho z_1{}^a,\rho z_2{}^b)| = |(z_1,z_2)|$. Let $X_{a,b}$ be the pull-back

$$\begin{array}{ccc} X_{a,b} & \longrightarrow & S^{2n+1} \\ \downarrow & & \downarrow \pi \\ D^4 & \xrightarrow{f_{a,b}} & D^4 \end{array}$$

Then the fixed point set of the induced action on $X_{a,b}$ will correspond in D^4 to the inverse image of the fixed point circle under f. We can assume that the circle is embedded in S^3 as $S^1 = \{(\lambda\sqrt{2}/2,-\lambda\sqrt{2}/2) \mid |\lambda| = 1,\ \lambda \in C\}$. Then $f^{-1}_{a,b}(S^1) = \{\frac{\sqrt{2}}{2}(\lambda,\mu) \mid |\lambda| = |\mu| = 1,\ \lambda^a + \mu^b = 0\}$. This is an (a,b) torus link. Hence, $X_{a,b} \in S_n(D^4,L_{a,b})$. One can check from the pasting data or directly, using branched coverings, that $X_{a,b}$ is equivariantly diffeomorphic to the Brieskorn variety $\Sigma(a,b,2,2,\ldots,2)$.

4. Diffeomorphism Classification of Link Manifolds

Definition. $B_{2n} \equiv$ the set of diffeomorphism classes of $O(n-1)$ link manifolds. Given a link $L \subset S^3$, let $B_{2n}(L) \equiv$ the set of diffeomorphism classes of link manifolds corresponding to this given link. Thus, $B_{2n}(L) \subset B_{2n}$.

Note that $M \in B_{2n} \Rightarrow \dim M = 2n-1$.

Recall that BP_{2n} denotes the set of diffeomorphism classes of (n-2)-connected 2n-1 manifolds that bound parallelizable manifolds. We will observe that $B_{2n} \subset BP_{2n}$ and use the known results about BP_{2n} to classify elements of B_{2n}. The technique is an equivariant surgery procedure used by Hirzebruch and Erle [7] for the case of knot-manifolds. Since their methods go over almost verbatim for the link case, we will explain the geometry for n = 2 and indicate the relevant general results.

A. Three Dimensional Geometry

If n = 2, then we are dealing with $O(1) = Z_2$ actions on a 3-manifold M with orbit space S^3 and fixed point set corresponding to an oriented link $L \subset S^3$. M is the double branched cover of S^3 with branch set L. The following sequence of remarks explains how to obtain M from S^3 by a sequence of O(1)-equivariant surgeries.

1) Regard $S^3 = R^3 \cup \{\infty\}$, $T: S^3 \to S^3$. $T(x,y,z) = (1/|(x,y,z)|)\cdot(x,y,-z)$. Thus, T defines an O(1)-action on S^3 with fixed points $\mathrm{Fix}(T) = \{(x,y,0)\,|\,x^2 + y^2 = 1\} = S^1$.

2) One can choose a spanning surface for the link in the form of a disk with (twisted) bands (see Fig. 1). Assume that the disk is $D^2 = \{(x,y,0)\,|\,x^2 + y^2 \le 1\}$. Each surgery on S^3 will correspond to one of

the bands. Each band will be specified by a map $f: [0,1] \to \{(x,y,z)|z \geq 0\}$ such that $f(0), f(1) \in S^1$; $\ell = f[0,1]$ meets S^1 orthogonally and corresponds to the core of the hand. The twisting of the band is described by a vector field v defined on ℓ and normal to it; v will be tangent to S^1 as it approaches $f(0)$ and $f(1)$. (Convention: v is tangent to S^1 in the clockwise direction at $f(0)$.)

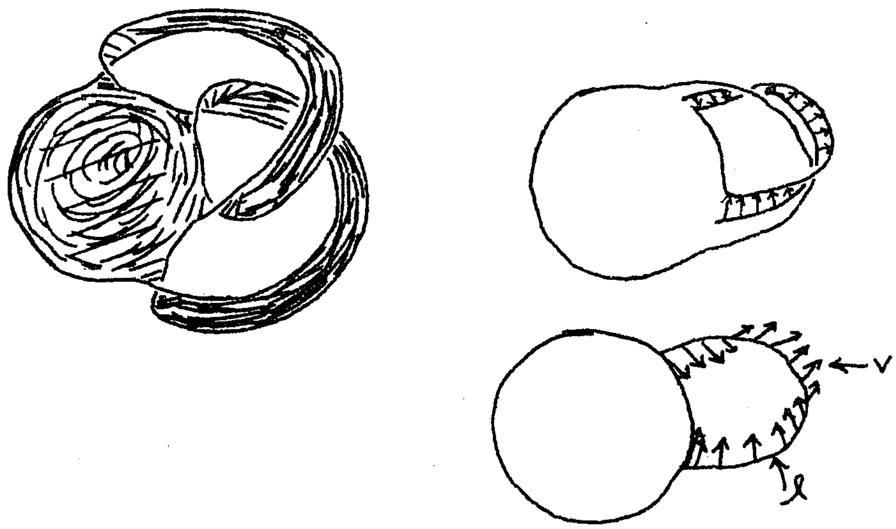


Figure 1

3) A surgery is associated to each band as follows: $\ell \cup T\ell$ gives an extension $f: S^1 \to S^3$. Letting $w = v \times f'/|v \times f'|$ be a unit vector field perpendicular to ℓ and v, define $f: S^1 \times D^2 \to S^3$ by

$$f(\alpha,\beta) = f(\alpha) + \beta_0 v(\alpha) + \beta_1 w(\alpha)$$

where $\quad S^1 \times D^2 = \{(\alpha,\beta)|\alpha_0{}^2 + \alpha_1{}^2 = 1,\ \beta_0{}^2 + \beta_1{}^2 \leq 1\}.$

Here we have extended the fields v and w to ℓ $T\ell$ via $v(Tp) = Tv(p)$, $w(Tp) = Tw(p)$ and, by definition, $f'(Tp) = Tf'(p)$ (f' = vector $\frac{d}{dt}f$).

Define $T: S^1 \times D^2 \to S^1 \times D^2$ by

$$T(\alpha_0,\alpha_1;\beta_0,\beta_1) = (\alpha_0,-\alpha_1;\beta_0,-\beta_1).$$

Lemma 2.14. $f: S^1 \times D^2 \to S^3$ is $O(1)$-equivariant.

Proof. Note that if $a,b \in S^3 - \{\infty\}$, then $Ta \times Tb = -T(a \times b)$ where "x" denotes the standard vector cross product. We want to show that $fT = Tf$.

$$f(T(\alpha,\beta)) = f(T\alpha) + \beta_0 v(T\alpha) - \beta_1 w(T\alpha).$$

But

$$\begin{aligned} -w(T\alpha) &= -(v(T\alpha) \times f'(T\alpha))/|\sim| \\ &= -(Tv(\alpha) \times Tf'(\alpha))/|\sim| \\ &= Tw(\alpha). \end{aligned}$$

Hence,

$$\begin{aligned} f(T(\alpha,\beta)) &= Tf(\alpha) + \beta_0 Tv(\alpha) + \beta_1 Tw(\alpha) \\ &= Tf(\alpha,\beta). \end{aligned}$$

Thus,

$$fT = Tf.$$

Lemma 2.15. Suppose L is the boundary of $D^2 \cup B_1 \cup \ldots \cup B_r$ where B_i denotes the i-th band. Let M be obtained by doing a sequence of equivariant surgeries, as above, one surgery for each band. Then $M/O(1) \approx S^3$ and the fixed point set of the action of O(1) on M projects to a subset of S^3 isotopic to L.

Proof. It suffices to check this for a single surgery. Suppose L is the boundary of $D^2 \cup B$. It is easy to see that

$$\begin{aligned} M/O(1) &\approx (S^3 - \text{Int}(3\text{-cell})) \cup_f (3\text{-cell}) \\ &\approx S^3. \end{aligned}$$

To prove the second part, let

$$F_1 = \text{Fix}(T\colon M \to M),\ F_2 = \text{Fix}(T\colon D^2 \times S^1 \to D^2 \times S^1).$$

Then

$$F_2 = \{(\alpha_0, 0; \beta_0, 0) \mid |\alpha_0| \le 1,\ \beta_0 = \pm 1\} = D^1 \times S^0.$$

Note that each component of F_2 deforms across embedded disks $D^2 \times S^0$ to the boundary $S^1 \times S^1$. Now

$F_1 = (S^1 - \mathrm{Int}(f(F_3))) \cup_f F_2$ where $F_3 = \mathrm{Fix}(T: S^1 \times D^2 \to S^1 \times D^2)$.

But $f(F_2)$ deforms onto two paths on $\partial f(S^1 \times D^2)$ in the upper hemisphere given by $f(t) \pm v(t)$. Hence,

$$F_1 = (S^1 - \mathrm{Int}\, f(F_3)) \cup \{f(t) \pm v(t)\}$$
$$= \partial(D^2 \cup B) \qquad \text{(see Fig. 2).}$$

This completes the proof of the lemma.

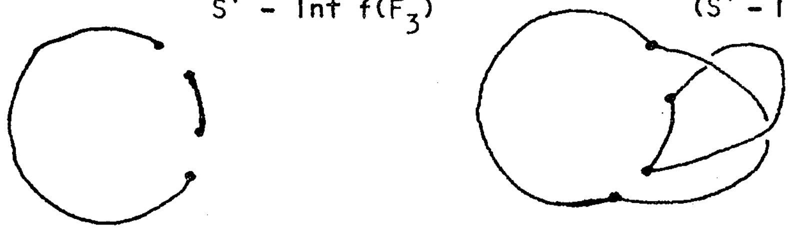


Figure 2

Note that we have also constructed a 4-manifold $N = D^4 \cup \{\bigcup_{i=1}^{r} (D^2 \times D^1)_i\}$ such that $\partial N = M$. Let $a_1,\ldots,a_r \in H_2(N)$ denote the homology classes represented by these handles.

If ℓ_i = core of the i-th band B_i, let $\bar{\alpha}_i = \ell_i - T\ell_i$ as a chain on S^3. Then $\bar{\alpha}_i$ is the chain corresponding to the i-th handle on N. Thus, if $\langle\ ,\ \rangle: H_2(N) \times H_2(N) \to \mathbb{Z}$ denotes the intersection pairing, then $\langle a_i, a_j \rangle = \ell(\bar{\alpha}_i, \bar{\alpha}_j)$ where ℓ denotes linking number in S^3.

Letting F denote the spanning surface for L, there is also the Seifert pairing $\theta: H_1(F) \times H_1(F) \to \mathbb{Z}$. Here $\theta(\alpha,\beta) = \ell(\alpha, i^*\beta) = \ell(i_*\alpha, \beta)$ where i_* = normal push in +z direction on D^2 and in +w direction on the bands;

i^* = normal push in -z direction on D^2 and in -w direction on the bands.

Note that if $\alpha_i = \ell_i - d_i$ where d_i denotes any path from $f_i(0)$ to $f_i(1)$ in D^2, then $\{\alpha_i | i = 1,\ldots,r\}$ is a basis for $H_1(F)$.

<u>Proposition 2.16</u>. $\langle a_i, a_j \rangle = \theta(\alpha_i, \alpha_j) + \theta(\alpha_j, \alpha_i)$. Thus, if $V = (v_{ij}) = (\theta(\alpha_i, \alpha_j))$, then the intersection matrix for N is $V + V'$.

<u>Proof</u>. Note that $F' = T(F)$ is another orientable surface in S^3. The boundary of F' is a link which is a mirror image of the link L. Regard $T: H_1(F) \to H_1(F')$. <u>Define</u> i^* on F' by $i^*(Tp) = T(i_* p)$. In this sense $Ti_* = i^*T$. Using $\theta: H_1(F') \times H_1(F') \to \mathbb{Z}$, we can speak of $\theta(T\alpha, T\beta) = \ell(T\alpha, i^*T\beta) = \ell(T\alpha, Ti_*\beta) = \ell(\alpha, i_*\beta) = \theta(\beta, \alpha)$. Let $\bar{\alpha}$ and $\bar{\beta}$ be cores of handles with $\bar{\alpha} = \ell - T\ell$, $\bar{\beta} = \ell' - T\ell'$; $\alpha = \ell - d$, $\beta = \ell' - d'$. We may assume that d and d' lie on the boundary S^1 of D^2. Then, since points of S^1 are fixed by T, $\bar{\alpha} = \alpha - T\alpha$, $\bar{\beta} = \beta - T\beta$.

$$\begin{aligned} \ell(\bar{\alpha}, \bar{\beta}) &= \ell(\alpha - T\alpha, \beta - T\beta) = \ell(\alpha - T\alpha, i^*(\beta - T\beta)) \\ &= \ell(\alpha, i^*\beta) + \ell(T\alpha, i^*T\beta) \\ &\quad - \ell(\alpha, i^*T\beta) - \ell(T\alpha, i^*\beta). \end{aligned}$$

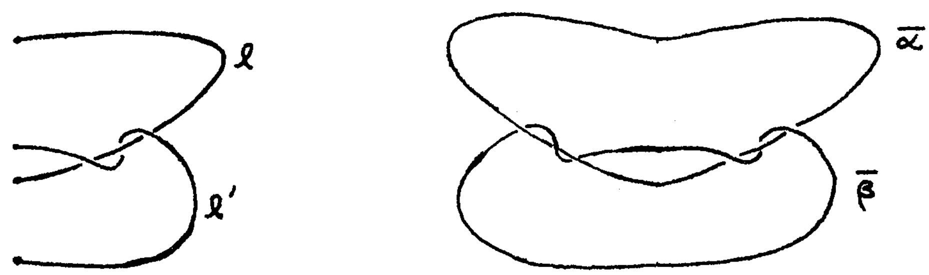
Since α and β lie in the exterior of S^2, $T\alpha$ and $T\beta$ lie within it. Thus,

$$\ell(\alpha, i^*T\beta) = 0 = \ell(T\alpha, i^*\beta).$$

Hence,

$$\begin{aligned} \ell(\bar{\alpha}, \bar{\beta}) &= \ell(\alpha, i^*\beta) + \ell(T\alpha, i^*T\beta) \\ &= \theta(\alpha, \beta) + \theta(T\alpha, T\beta) \\ &= \theta(\alpha, \beta) + \theta(\beta, \alpha). \end{aligned}$$

This proves the proposition.



<u>Figure 3</u>

Thus, although this construction of M and N is seemingly quite different from that of Chapter I, the algebraic results about it are the same. In fact, this is really another version of the construction in the earlier chapter. Let M' denote the usual construction of the double branched cover, obtained by splitting S^3 along the spanning surface and pasting.

Lemma 2.17. M is M' in disguise.

Proof. The surface F consists of D^2 together with bands attached to S^1 and extending into the upper half-space exterior to S^2. Let $f_i: S^1 \times D^2 \to S^3$ be the maps corresponding to the bands. Let $\overline{M} = S^3 - \text{Int}\{\bigcup f_i(S^1 \times D^2)\}$. Then $\overline{M} = \overline{M}_- \cup \overline{M}_+$ where $\overline{M}_- = \overline{M} \cap D^3$, $\overline{M}_+ = \overline{M} \cap \text{Exterior}(D^3)$. It is easy to see that each of $\overline{M}_-$ and $\overline{M}_+$ are homeomorphic to S^3 split along F, and that the process of doing the surgeries, i.e., forming $\overline{M} \cup \{\bigcup (D^2 \times S^1)_i\}$ corresponds to pasting $\overline{M}_-$ to $\overline{M}_+$. Hence, M is really the usual construction M'.

This completes the sketch of the three dimensional case. The next section outlines the generalization of this construction which produces O(n)-manifolds by equivariant surgery on S^{2n+1}.

B. View $S^3 \to S^{2n+1}$ via $(x_o,x_1;y_o,y_1) \to (x_o,x_1,0,\ldots,0;y_o,y_1,0,\ldots,0)$. Here $S^3 = \{(x,y) \in R^2 \times R^2 \mid |x|^2 + |y|^2 = 1\}$ with O(1)-action $T\cdot(x_o,x_1;y_o,y_1) = (x_o,-x_1;y_o,-y_1)$. $S^{2n+1} = \{(x,y) \in R^{n+1} \times R^{n+1} \mid |x|^2 + |y|^2 = 1\}$ with O(n)-action

$g\cdot(x_o,x_1,\ldots,x_n;y_o,y_1,\ldots,y_n) = (x_o,g(x_1,\ldots,x_n);y_o,g(y_1,\ldots,y_n))$.
$S^1 = \{(x_o,0;y_o,0) \in S^3\}$.

The path $f: [0,1] \to S^3$ such that $f(0),f(1) \in S^1$ may be regarded as $f: [0,1] \longrightarrow S^{2n+1}$. Let $\lambda = f[0,1] \subset S^{2n+1}$. Let $G(\lambda) = O(n)\cdot\lambda$. Then since $O(n)\cdot p \approx S^{n-1}$ for $p \in \lambda$, $p \neq f(0),f(1)$ and $O(n)\cdot f(0) = f(0)$,

$O(n) \cdot f(1) = f(1)$, one sees that $G(\lambda) \approx S^n$. In this way one obtains an embedding $\hat{f}: S^n \to S^{2n+1}$. Similarly, by using the vector fields v and w, one extends this to an O(n)-equivariant embedding $\hat{f}: S^n \times D^{n+1} \to S^{2n+1}$. If M^{2n+1} is the manifold obtained from S^{2n+1} by doing a sequence of surgeries corresponding to the bands on a spanning surface for a link $L \subset S^3$, then $M^{2n+1}/O(n) \approx D^4$ and M^{2n+1} is an O(n) link manifold in $B_{2n+2}(L)$.

As in the three dimensional case one also obtains a manifold N^{2n+2} such that $\partial N^{2n+2} = M^{2n+1}$. N^{2n+2} is parallelizable for $n \geq 1$. (This follows from the same argument as in [7, page 211].) Hence, $M^{2n+1} \in BP_{2n+2}$.

Letting λ_i be the core of the i-th band, α_i, as in Section A, θ the Seifert pairing for F in S^3, Erle [7, page 197] shows:

Proposition 2.18. $\ell(G(\lambda_i), G(\lambda_j)) = \theta(\alpha_i, \alpha_j) + (-1)^{n+1}\theta(\alpha_j, \alpha_i)$ (ℓ = linking in S^{2n+1}). Hence, N^{2n+1} has intersection form $V + (-1)^{n+1}V'$ where V is the Seifert matrix for F in S^3.

Remark. The proof of this proposition depends on some specific case checking in three dimensions. Thus, it does not generalize directly for higher dimensional orbit spaces. It is very likely, however, that most everything done for orbit space D^4 will carry over for orbit space D^{2k} with fixed point set embedded in codimension 2 in ∂D^{2k}. See Chapter III for a special case of this conjecture.

C. Recall that manifolds in B_{2n+2} corresponding to a given link $L \subset S^3$ are also determined by a map $\sigma: S^3 - L \to S^1$ (see Section 3). Specifying such a map is equivalent to specifying an orientation for each link component. Let M_1^{2n+1} = manifold in B_{2n+2} constructed by pasting bundles with pasting data σ, M_2^{2n+1} = manifold in B_{2n+2} obtained by equivariant surgery on S^{2n+1} using an orientable spanning surface whose oriented boundary is the link L with orientation specified by σ.

<u>Proposition 2.19</u>. Under the above hypotheses M_1 and M_2 are $O(n)$-equivariantly diffeomorphic.

<u>Proof</u>. To check that M_1^{2n+1} is equivariantly diffeomorphic to M_2^{2n+1}, it suffices to check this for M_1^3 and M_2^3 under the corresponding Z_2 action. This is the case since pasting data for the 2-fold branched cover is the same as pasting data for the higher dimensional manifold and in both cases one has inclusions:

$$\begin{array}{ccc} M_1^3 & \hookrightarrow & M_1^{2n+1} \\ \downarrow & & \downarrow \\ S^3 & \hookrightarrow & D^4 \end{array} \qquad\qquad \begin{array}{ccc} M_2^3 & \hookrightarrow & M_2^{2n+1} \\ \downarrow & & \downarrow \\ S^3 & \hookrightarrow & D^4 \end{array}$$

However, since we have identified the usual construction of the double cover with that obtained by surgery, it will suffice to examine the ordinary double cover.

Given $S^3 - L \xrightarrow{\sigma} S^1$, form the pull-back

$$\begin{array}{ccc} M & \longrightarrow & S^1 \\ \downarrow & & \downarrow \phi \\ S^3 - L & \xrightarrow{\sigma} & S^1 \end{array} \qquad \phi(x) = x^2$$

$\hat{M}$ is the double cover of $S^3 - L$ corresponding to σ. Its completion to a branched cover is M_1^3. M_2^3 is identified with the manifold obtained by splitting S^3 along $\sigma^{-1}(x)$ for x a regular value of σ and pasting two such split S^3's together. Hence, M_2^3 is also the completion of M. Thus, M_1^{2n+1} and M_2^{2n+1} are diffeomorphic.

<u>Corollary 2.20</u>. $B_{2n+2} \subset BP_{2n+2}$.

Proof. Any element of B_{2n+2} is a manifold M_1^{2n+1}. Since $M_1^{2n+1} \approx M_2^{2n+1} \subset BP_{2n+2}$, this proves the corollary.

D. Knowing that $B_{2n} \subset BP_{2n}$, it is natural to ask how large a subset this forms. We will show that for $k > 1$, $B_{4k} \equiv BP_{4k}$ and discuss the case B_n for n odd. Thus, by indirect means, one has the result that every element of BP_{4k} admits a particularly nice orthogonal group action.

Let $\mathcal{H}_{2n}$ ≡ the set of diffeomorphism classes of parallelizable (n-1)-connected 2n-manifolds with (n-2)-connected boundary.

For n even we know [38] that $\mathcal{H}_{2n}$ is in 1-1 correspondence with equivalence classes of even symmetric bilinear forms over $\mathbb{Z}$ (via middle dimensional intersection form).

Definition. Let A be a symmetric square matrix with even entries on the diagonal. We say that A is of link type if no row contains more than one odd entry.

Lemma 2.21. If A is of link type, then $A = V + V'$ where V is a Seifert matrix for some link $L \subset S^3$.

Proof. Given the matrix A, the lemma will be proven by constructing a surface F as a disk with attached bands, such that if V is the Seifert matrix for F, then $A = V + V'$. Thus, we want a surface F with n bands, for A $n \times n$. Let $\alpha_i \in H_1(F)$, $i = 1,\ldots,n$, be the homology classes corresponding to the hands. $A = (a_{ij})$. Then we require $a_{ij} = \theta(\alpha_i,\alpha_j) + \theta(\alpha_j,\alpha_i)$. First, some observations:

1) $a_{ii} = 2\theta(\alpha_i,\alpha_i)$ specifies the twisting of the band corresponding to α_i.

2) $\theta(\alpha_i,\alpha_j)$ for $i \neq j$ is independent of the twisting of the i-th and j=th bands. It is specified by the embeddings of their cores.

3) Consider the two points of intersection of a band core with D^2. Call these the feet of the band.

Choose an orientation for the disk and therefore for its boundary. Given two points $p,q \in S^1 = \partial D^2$ dividing S^1 into unequal segments, let $[p,q]$ be the smaller segment. We say $p < q$ if, when this segment is oriented from p to q, the orientation agrees with the orientation of S^1.

Assume that the feet of each band divide S^1 into unequal segments. If p and q are the feet of a band with $p < q$, we say that a point x is between p and q if $x \in [p,q]$.

4) Letting $\psi(\alpha,\alpha') = \theta(\alpha,\alpha') + \theta(\alpha',\alpha)$, note that we can obtain $\psi(\alpha,\alpha')$ = odd by planting one foot of α' between the feet of α and adjusting the linking accordingly. (See Fig. 4.)

5) One can obtain $\psi(\alpha,\alpha')$ = even by keeping both feet of α' out from in between the feet of α. (See Fig. 5.)

We wish to show that one can make all of these indicated adjustments (foot placement, linking, band twists) without any one of them interfering with any other.

Induction Hypothesis: The lemma is true for all matrices A of link type and size $r \times r$ for $r \leq n$. $A = V + V'$ where V is the Seifert matrix for an orientable surface F realized as a disk with r attached bands. The homology classes of the bands are represented by $\alpha_i \in H_1(F)$. By the feet of α_i we mean the feet of the i-th band. Assume that the feet of a band are never both placed between the feet of another band. Assume also that if $\psi(\alpha,\alpha')$ is even, then α has no feet between the feet of α' and vice versa. If $\psi(\alpha,\alpha')$ is odd, then each band has one foot between the feet of the other.

Case I. If A is 1×1, $A = (a)$, take a disk with a single band having a half-twists. This obviously satisfies the lemma and the induction hypothesis.

Case II. Suppose $A = (a_{ij})$ is $(n+1) \times (n+1)$ and of link type. Let $\overline{A} = (a_{ij})$, $i \leq n$, $j \leq n$. Certainly $\overline{A}$ is of link type and, therefore, we may apply the induction hypothesis. Since $a_{n+1,j}$ is odd for at most one j, $1 \leq j \leq n$, choose the feet of α_{n+1} as follows: Let p be a point in between the feet of α_j, if $a_{n+1,j}$ is odd, and choose $q > p$ such that q does not lie between the feet of any band. Then p and q will be the feet of α_{n+1}. Note that such p and q can be chosen since $a_{n+1,j}$ = odd and, hence, $a_{j,k}$ = even for $1 \leq k \leq n$ (since A is of link type). Therefore, no other feet stand between the feet of α_j nor do the feet of α_j stand between any feet (this follows from the induction hypothesis).

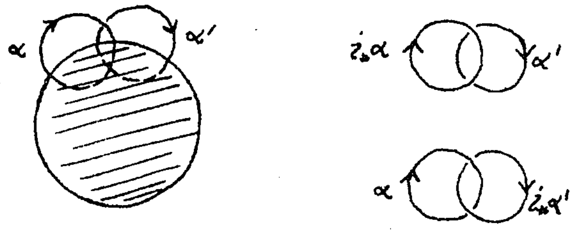
If $a_{n+1,j}$ is even for $1 \leq j \leq n$, then simply choose $p < q$ such that p and q stand outside the feet of all the bands. Again, p and q will be the feet of α_{n+1}.

Now construct the core of the (n+1)-st band by first doing the odd linking with α_j (if there is such) without creating any linking with the other bands. This can be done since the linking can be done locally in the neighborhood of a band. If there is no odd linking, just embed an arc from p to q such that there is no linking with any band.

Finally, change the arc by cutting small segments from it and replacing these by segments linking the α_i so that $a_{n+1,i} = \psi(\alpha_{n+1},\alpha_i)$. All of this can be accomplished by our remarks. The result is the construction of the core of the (n+1)-st band. Now thicken this core into a band and introduce $a_{n+1,n+1}$ half-twists. The result is a new surface satisfying the induction hypothesis and such that $a_{ij} = \psi(\alpha_i,\alpha_j)$ for $i \leq n+1$, $j \leq n+1$.

Hence, the lemma is proved by induction.

Remark. The process outlined in the lemma is illustrated in Figure 6.

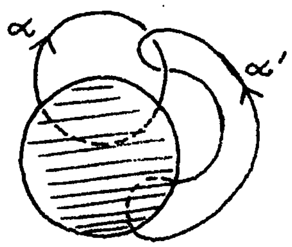


$\theta(\alpha,\alpha') = 0$

$\psi(\alpha,\alpha') = 1$

$\theta(\alpha',\alpha) = 1$

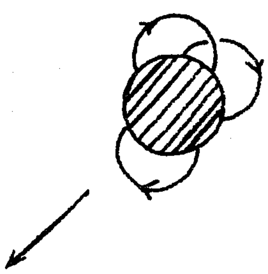
Figure 4



$$\psi(\alpha,\alpha') = \theta(\alpha,\alpha') + \theta(\alpha',\alpha)$$
$$= 1 + 1$$
$$\psi(\alpha,\alpha') = 2$$

Figure 5

$$A = \begin{bmatrix} 6 & 1 & 4 \\ 1 & 2 & 2 \\ 4 & 2 & -2 \end{bmatrix}$$



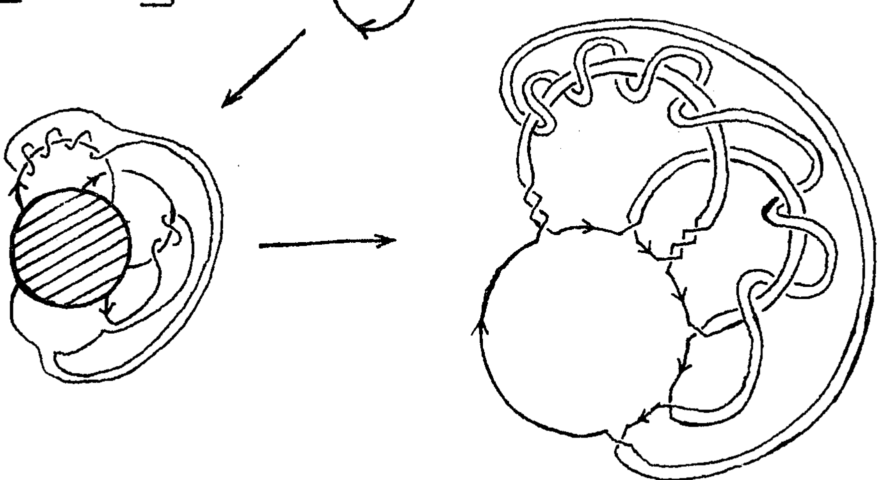


Figure 6

Lemma 2.22. Let A be any symmetric square matrix with even entries on the diagonal. Then there exists a unimodular matrix P such that PAP' is of link type.

Proof. A is a matrix over $\mathbb{Z}$. Let $\bar{A} = (\bar{a}_{ij})$ be the matrix of mod-2 residue classes over Z_2. A matrix over Z_2 will be said to be of link type if no row contains more than one non-zero entry. Thus, A is of link type $\Rightarrow$ $\bar{A}$ is of link type. However, over Z_2, the symmetric matrix $\bar{A}$ is congruent to a matrix $\bar{B}$ of the form

$$\bar{B} = \begin{bmatrix} 0 & & \cdots & & 0 & 1 \\ 0 & & \cdots & 0 & 1 & 0 \\ & & \cdots & & & \\ 0 & 1 & \cdots & & & 0 \\ 1 & 0 & \cdots & & & 0 \end{bmatrix} \oplus [0] \qquad \bar{B} = \bar{P}\bar{A}\bar{P}'$$

where $\bar{P}$ is invertible over Z_2.

Let P be the matrix of zeroes and ones over $\mathbb{Z}$ such that $\bar{P} = P$. Since $\bar{P}$ is a product of elementary matrices each of which corresponds to changing a row by adding another row to it, it is clear that P is invertible over $\mathbb{Z}$, hence, unimodular. Thus, $\overline{PAP'} = \bar{P}\bar{A}\bar{P}' = \bar{B}$. Hence, PAP' is of link type, proving the lemma.

Definition. Let $N^{2n}(L)$ denote the manifold whose boundary $M^{2n-1}(L)$ is obtained by equivariant surgery on S^{2n-1} to produce an $O(n-1)$-link manifold corresponding to the link $L \subset S^3$.

Proposition 2.23. For n even, >2 any element of $\mathcal{H}_{2n}$ is represented by a manifold $N^{2n}(L)$ for some link $L \subset S^3$.

Proof. $N \in \mathcal{H}_{2n} \Longrightarrow N$ is represented by the equivalence class of its intersection form. If N_1 and N_2 have intersection matrices A_1, A_2 such that $A_2 = PA_1P'$, P unimodular, then $N_1 \approx N_2$. Any symmetric matrix with even diagonal entries is realized by a manifold in $\mathcal{H}_{2n}$. Thus, given $N \in \mathcal{H}_{2n}$, let A = intersection matrix. Choose P unimodular such that PAP' is of link type. Then the previous lemma implies that there is a link $L \subset S^3$ such that $N^{2n}(L)$ has intersection matrix PAP'. Hence, $N \approx N^{2n}(L)$.

Corollary 2.24. $BP_{4k} \equiv B_{4k}$ for $k > 1$.

Proof. $BP_{4k} = \partial \mathcal{H}_{4k}$.

Remark. If n is odd, then $B_{2n} \neq BP_{2n}$. In fact, since $N^{2n}(L)$ has intersection form $V - V' = \Delta$ where Δ = intersection form on the spanning surface for the link, it follows that $M^{2n-1}(L)$ has no torsion in its homology groups.

E. The Diffeomorphism Classification

Recall the classification result of Chapter 0. If n is even and $M^{2n-1} \in BP_{2n}$ then we have $M = \partial N^{2n}$, $f: H_n(N) \times H_n(N) \to \mathbb{Z}$, the intersection pairing, $b(f): \tau H_{n-1}(M) \times \tau H_{n-1}(M) \to Q/\mathbb{Z}$, and $q(f): \tau H_{n-1}(M) \to Q/\mathbb{Z}$ the linking form and linking quadratic form.

Lemma 2.25. Let $M^{2n-1} = M^{2n-1}(L)$ for $L \subset S^3$. Suppose $M^{2n-1}(L)$ is constructed from a spanning surface F for L and let $f: H_1(F) \times H_1(F) \to \mathbb{Z}$ be the bilinear form of L, $f(x,y) = \theta(x,y) + \theta(y,x)$. Let $b(L)$ and $q(L)$ be the corresponding linking pairing and quadratic form. Then, for n even, we may identify f with the pairing $H_n(N) \times H_n(N) \to \mathbb{Z}$, $N = N^{2n}(L)$, and $b(L) = b(f)$, $q(L) = q(f)$.

Proof. This follows directly from the fact that $N^{2n}(L)$ has intersection pairing $V + V'$.

Theorem 2.26. Let $n > 2$ be even. Then

1) $BP_{2n} \equiv B_{2n}$, the set of $O(n-1)$-link manifolds.

2) Given $M^{2n-1} \in B_{2n}$ corresponding to a link $L \subset S^3$, then the diffeomorphism type of M^{2n-1} is determined by invariants of the link $L \subset S^3$. In particular, suppose given $M_1, M_2 \in B_{2n}$ with $M_1 \leftrightarrow L_1 \subset S^3$, $M_2 \leftrightarrow L_2 \subset S^3$.

Suppose that $q(L_1) \cong q(L_2)$ and $cok(f(L_1)) \cong cok(f(L_2))$, then, if $\sigma(L_1) \geq \sigma(L_2)$, one has

$$M_1 \approx M_2 \# \frac{1}{8}(\sigma(L_1) - \sigma(L_2))\cdot\Sigma$$

where Σ is the Milnor sphere. (If $cok(f(L_i))$ have no summands of order 2 or 4, we may replace $q(L_1) \cong q(L_2)$ with $b(L_1) \cong b(L_2)$.)

Proof. This follows immediately from the lemma and Theorem 0.5.

Remark. Note that if $f = f(L)$, then $cok(f(L))$ is an invariant of the link L. The form $f(L)$ depends on the choice of spanning surface. However, we may assume that the spanning surface F is derived by Seifert's algorithm for a planar projection of L. Then, f_1 and f_2 differ at most by direct sums with $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ if f_1 and f_2 are both bilinear forms for L. Thus, $f_1 \oplus U \oplus \ldots \oplus U \cong f_2 \oplus U \oplus \ldots \oplus U$. Hence, $cok(f_1) \cong cok(f_2)$. Also, if $N_1(L)$ and $N_2(L)$ denote the corresponding handle-bodies in $\mathcal{H}_{2n}$, then, since direct sum with U corresponds to connected sum with $S^n \times S^n$-(small n-disk), we see that $\partial N_1(L) \approx \partial N_2(L)$. This is another way of proving that the diffeomorphism type of $M^{2n-1}(L)$ is independent of the choice of spanning surface.

Theorem 2.27. Let $M_1, M_2 \in B_{2n}$, for n-odd, $n > 2$. Suppose $M_1 = M(L_1)$, $M_2 = M(L_2)$, $L_1, L_2 \subset S^3$. Let $\psi_i = \psi(L_i)$ denote the Z_2-quadratic forms for these links. Suppose that L_1 and L_2 have the same number of components. Then

1) If $n = 3$ or 7, $M_1 \approx M_2$.

2) If $\psi_i | rad(\psi_i) \equiv 0$, $i = 1,2$, then

$$M_1 \approx M_2 \# (c(\psi_1) + c(\psi_2))\cdot\Sigma_1.$$

3) If $\psi_i | rad(\psi_i) \not\equiv 0$, $i = 1,2$, then

$$M_1 \approx M_2 \approx M_2 \# \Sigma_1$$

where Σ_1 = Kervaire sphere.

Proof. Again, this is an application of the general classification Theorem 0.6 for BP_{2n}. The only point to check is that the Z_2-quadratic form of the link L is the quadratic form associated with the skew form on $H_n(N(L))$. However, Erle [7, page 215] shows that if x is the class on the Seifert surface corresponding to a band and, hence, corresponding to a class $x \in H_n(N(L))$, then $\frac{1}{2} f(x,x) = 0$ or 1 according as x has trivial or non-trivial normal bundle. We showed that $\psi(L)(x) = \frac{1}{2} f(x,x)$ in Section 1.

These two theorems generalize Hirzebruch's theorem on the relation between the signature of a knot and its corresponding knot manifold.

Notation. In the examples to follow we refer to 2.26 as Theorem A and 2.27 as Theorem B.

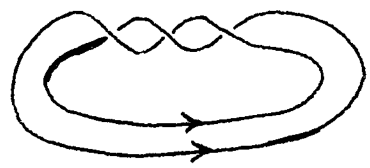
F. Applications and Examples

1) Let L_k be the (2,k) torus link as in the example on page 55. Let $M_k = M^{2n-1}(L_k)$ for $n \geq 3$, odd. Then, applying Theorem B to the link invariants listed on that page, one finds:

$M_1 \approx S^{2n-1}$	$M_5 \approx \Sigma^{2n-1}$	
$M_2 \approx T$	$M_6 \approx T$	
$M_3 \approx \Sigma^{2n-1}$	$M_7 \approx S^{2n-1}$	$M_{k+8} \approx M_k$
$M_4 \approx (S^{n-1} \times S^n) \# \Sigma^{2n-1}$	$M_8 \approx S^{n-1} \times S^n$	

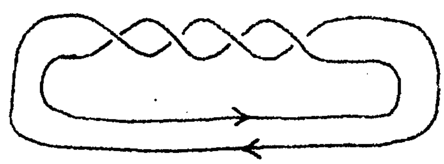
where T = tangent sphere bundle to S^n and Σ is the Kervaire sphere.

This periodicity example is due to Durfee. He observed it in the context of Brieskorn varieties ($M_k \approx \Sigma(k,2,2,\ldots,2)$). We include it here because it fits very well into our link manifold context. In a sense, the periodicity in the invariants of the links L_k explains the periodicity in the list of manifolds M_k.

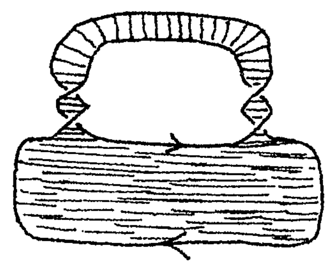


L_3

2) L_{2k} is a 2-component torus link with a particular orientation. Let $\hat{L}_{2k}$ denote the same link with the orientation of one of the components reversed.



L_4



It is easy to see that $\hat{L}_{2k}$ has a Seifert surface F which is simply a disk with one band having 2k half-twists. Thus,

$$\theta: H_1(F) \times H_1(F) \to \mathbb{Z}, \quad \theta(a,a) = k$$

$$f: H_1(K) \times H_1(K) \to \mathbb{Z}, \quad f(a,a) = 2k.$$

$\langle a,a\rangle = 0$, where a denotes the generator of $H_1(F)$. The Z_2 quadratic form of $\hat{L}_k$ is $\mathcal{V}_o$ (the zero form on the 1-dimensional Z_2-vector space). Hence, by Theorem 1, $M^{2n-1}(\hat{L}_{2k}) \approx S^{n-1} \times S^n$ for $n \geq 3$ and n <u>odd</u>. Thus, when L_{2k} is replaced by $\hat{L}_{2k}$, the periodicity collapses from 4 to 1.

For n <u>even</u>, the manifolds $M^{2n-1}(L_{2k})$ are not at all periodic. In fact, since $f \approx [2k]$, we see that $H_{n-1}(M^{2n-1}(\hat{L}_{2k})) = Z_{2k}$ when n is even.

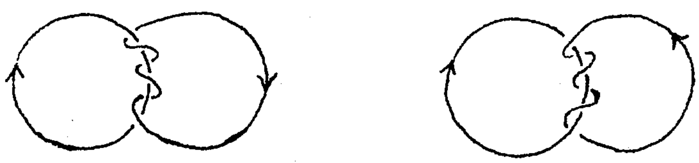
Let $\hat{M}^{4n-1}_{2k} \equiv$ the S^{2n-1} sphere bundle over S^{2n} with characteristic element $k\tau$ where $\tau \in \pi_{2n-1}(SO(2k))$ is the characteristic map for the tangent bundle of S^{2n-1}. Then $\hat{M}_{2k}$ is the boundary of the corresponding disk bundle, which has intersection form [2k]. Thus,

$$\underline{M^{4n-1}(\hat{L}_{2k}) \approx \hat{M}^{4n-1}_{2k} \text{ for } n > 1}$$

by <u>Theorem A</u>.

For $n = 1$, $M^3(L_{2k})$ is a standard lens space as can be seen directly from the description of it as one surgery on S^3.

It is beguiling to note that, in the last analysis, all of these differences between $M(L_k)$ and $M(\hat{L}_k)$ arise out of the fact that two links of the type pictured below are of different link symmetry type.

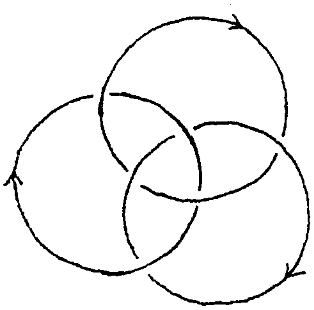


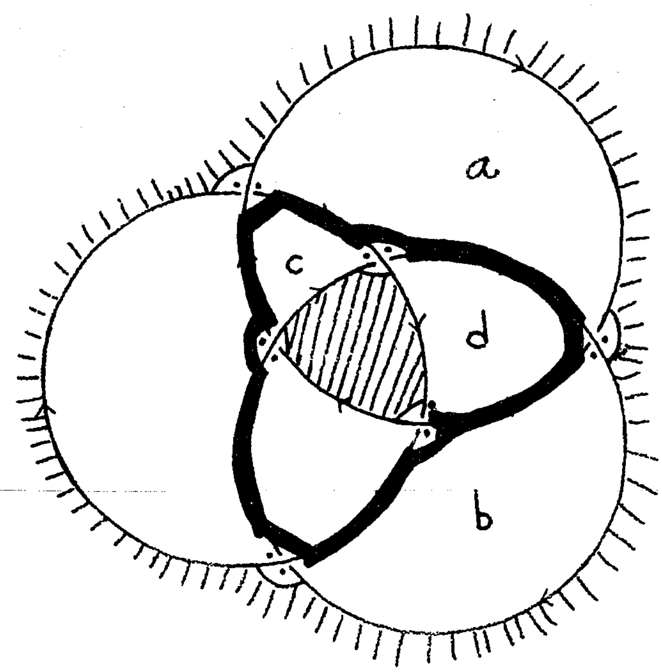
3) Note that if $M^{2n-1}(L) \varepsilon B_{2n}$ for n = 3 or 7 and L has r+1 components, then Theorem 1 implies that

$M \approx (S^{n-1} \times S^n) \# (S^{n-1} \times S^n) \# \ldots \# (S^{n-1} \times S^n)$ where there are r terms in the connected sum. Thus, for example, letting

$K(r) = (S^2 \times S^3) \# \ldots \# (S^2 \times S^3)$, r terms, then K(r) admits a vast collection of inequivalent O(2)-actions.

4) Borommean Rings. Recall that the Borommean rings have only one symmetry class (see Section 2). Hence, there is only one Borommean ring manifold in each dimension. To analyze it, we can use any convenient projection and orientation for the link.





$\theta(\ ,\ )$	a	b	c	d
a	1	0	1	-1
b	-1	1	0	1
c	0	0	-1	1
d	0	0	0	-1

Thus, V is the matrix above. The pairing $f(x,y) = \theta(x,y) + \theta(y,x)$ has matrix $M = V + V'$.

$$M = \begin{bmatrix} 2 & -1 & 1 & -1 \\ -1 & 2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ -1 & 1 & 1 & -2 \end{bmatrix}$$

A simple reduction of M reveals that $\mathrm{cok}(f) \cong Z_4 \oplus Z_4$. To find the rational congruence class of M, note that given $M = \begin{bmatrix} A & C' \\ C & B \end{bmatrix}$, A invertible and A and B symmetric, then, letting $P \begin{bmatrix} I & -A^{-1}C' \\ 0 & I \end{bmatrix}$, one has

$$P'MP = \begin{bmatrix} A & \bigcirc \\ 0 & [-CA^{-1}C'+B] \end{bmatrix}.$$

Here $$P'MP = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \oplus \left(\frac{-4}{3}\right)\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Thus, $\underline{\sigma(L) = 0.}$

Note that $$P = \begin{bmatrix} 1 & 0 & \frac{-2}{3} & \frac{1}{3} \\ 0 & 1 & \frac{-1}{3} & \frac{-1}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

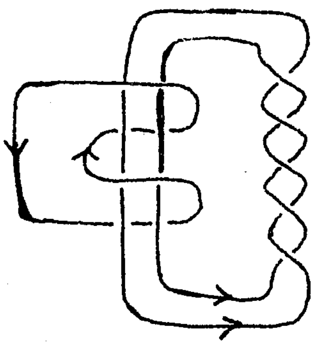
is, in fact, unimodular over R(2). Hence, the Z_2 quadratic form $\psi(L)$, being the mod-2 reduction of f, is given by $\psi = \phi_1 \oplus n_o \oplus n_o$.

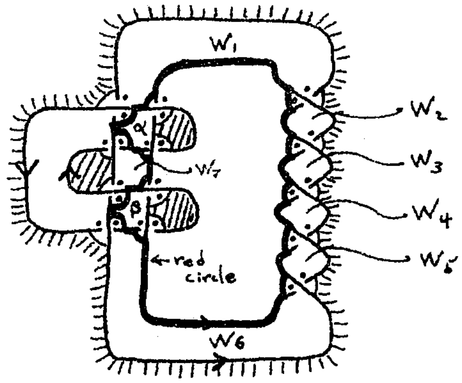
$\Delta = V - V'$ is easily seen to be congruent to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus [0] \oplus [0]$ over $\mathbb{Z}$. Hence, for n <u>odd</u>, the corresponding manifold in B_{2n} is $\Sigma_1 \# (S^{n-1} \times S^n) \# (S^{n-1} \times S^n)$. Formally,

<u>Proposition</u>. Assume $n \geq 5$, n odd, Σ_1 the Kervaire sphere, then

$\Sigma_1 \# (S^{n-1} \times S^n) \# (S^{n-1} \times S^n)$ admits the structure of an O(n)-manifold with orbit space D^4 and fixed point set corresponding to the Borommean rings in $\partial D^4 = S^3$. For n = 3 or 7, the same statement is true for $(S^{n-1} \times S^n) \# (S^{n-1} \times S^n)$.

5) Let L be the link pictured below:



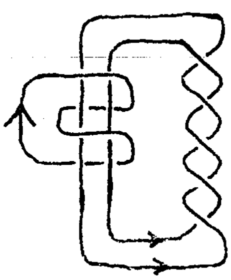


V =

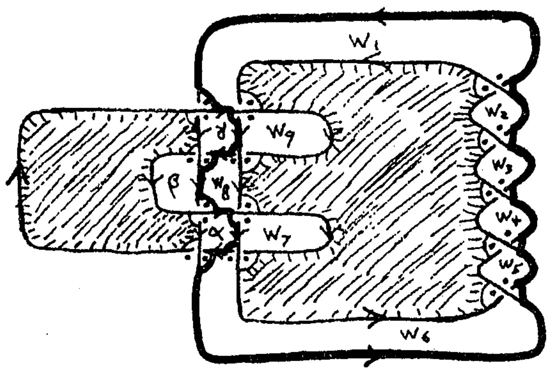
	w_1	w_2	w_3	w_4	w_5	w_6	w_7	α	β
w_1	0	-1	0	0	0	0	0	0	0
w_2	0	1	-1	0	0	0	0	0	0
w_3	0	0	1	-1	0	0	0	0	0
w_4	0	0	0	1	-1	0	0	0	0
w_5	0	0	0	0	1	-1	0	0	0
w_6	0	0	0	0	0	0	0	0	0
w_7	0	0	0	0	0	0	-1	0	0
α	1	0	0	0	0	0	-1	-1	0
β	0	0	0	0	0	-1	1	0	-1

M = V + V' is the matrix of f. A lengthy but straightforward computation reveals that M is congruent to $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus [0]$ over $\mathbb{Z}$. Hence, $\text{cok}(f) = \mathbb{Z}$, $\sigma(L) = 0$. Thus, for n even, n > 2, Theorem A implies that $M^{2n-1}(L) \approx S^{n-1} \times S^n$.

6) Let L be the link below.



This is, of course, the previous example with the orientation of one component reversed.



	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	α	β	γ
w_1	1	-1	0	0	0	0	0	0	0	0	-1
w_2	0	1	-1	0	0	0	0	0	0	0	0
w_3	0	0	1	-1	0	0	0	0	0	0	0
w_4	0	0	0	1	-1	0	0	0	0	0	0
w_5	0	0	0	0	1	-1	0	0	0	0	0
w_6	0	0	0	0	0	1	-1	0	1	0	0
w_7	0	0	0	0	0	0	1	-1	0	0	0
w_8	0	0	0	0	0	0	0	1	-1	0	1
α	0	0	0	0	0	0	0	0	1	-1	0
β	0	0	0	0	0	0	0	0	0	1	-1
γ	0	0	0	0	0	0	0	0	0	0	1

Replacing w_1 by w_9 for convenience, we see that $M = V + V'$ has the following form:

$$
M = \begin{bmatrix}
2 & -1 & & & & & & & 0 & 0 & 0 \\
-1 & 2 & -1 & & & & & & 0 & 0 & 0 \\
 & -1 & 2 & -1 & & & & & 0 & 0 & 0 \\
 & & -1 & 2 & -1 & & & & 0 & 0 & 0 \\
 & & & -1 & 2 & -1 & & & +1 & 0 & 0 \\
 & & & & -1 & 2 & -1 & & 0 & 0 & 0 \\
 & & & & & -1 & 2 & -1 & -1 & 0 & 1 \\
 & & & & & & -1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 2
\end{bmatrix}
$$

An easy check reveals that $\operatorname{cok}(f) \approx \mathbb{Z}$. A somewhat long but simple computation shows that, rationally, M is congruent to

$$\operatorname{diag}(2,3/2,4/3,5/4,6/5,7/6,8/7,9/8,4/9,-1/4,0).$$

Hence, $\sigma(L) = 8$. Thus, Theorem A applies to show that $M^{2n-1}(L) \approx \Sigma \# (S^{n-1} \times S^n)$ where Σ is the Milnor sphere, and n even ≥ 4.

Remarks. a) It is interesting to note that the change in orientation between the last two examples is reflected in the addition of an exotic sphere.

b) Let $K^{2n-1} = V[(z_o + z_1^2)(z_o^2 + z_1^5) + z_2^2 + \dots + z_n^2] \cap S_\varepsilon^{2n+1}$. Then K^{2n-1} is an $O(n-1)$-manifold with orbit space homeomorphic to D^4. However, since $(z_o + z_1^2)(z_o^2 + z_1^5)$ is not weighted homogeneous, we cannot assert that the orbit space has the usual smoothing. If this were the case, then the corresponding link would be that of example 6 and one could conclude that $K \approx \pm\Sigma \# (S^{n-1} \times S^n)$.

CHAPTER III

ALGEBRAIC BRANCHED COVERS AND DEGENERATION

In a sense, this chapter may be regarded as a continuation of the last sections of Chapter I. We take a second look at algebraic branched covers and, specializing to weighted homogeneous polynomials, prove the homological periodicity result (Corollary 1.24) of Chapter I by deriving it from a lemma about the monodromy of $x^k + f(z)$. Then, by deforming cycles in $\mathbb{F}_k = \{(x,z) \in C^{n+1} | x^k + f(z) = 1\}$, we calculate the intersection and Seifert pairings in terms of these pairings for $F = \{z | f(z) = 1\}$. This leads to an inductive procedure which allows one to compute the intersection form for $\sum_{i=1}^{n} x_i^{a_i} + f(z)$ in terms of the Seifert pairing for $f(z)$. In particular, one may recover the intersection form for a Brieskorn variety. These results about intersection forms parallel the results obtained via branched coverings and O(n)-actions. They are suggestive of more general theorems about group actions.

1. More on Algebraic Branched Covers

Let $z \in C^n$ and $f(z)$ be a polynomial with an isolated singularity at the origin. Choosing $\varepsilon > 0$ sufficiently small, let $M_1 = V(x - f(z)) \cap S_\varepsilon^{2n+1}$ and $M_k = \{(x,z) \in C^{n+1} | x^k = f(z), |x|^{2k} + |z|^2 = \varepsilon^2\}$ for $k \geq 1$ an integer. Define $p: M_k \to M_1$ by $p(x,z) = (x^k,z)$. Thus, p is a k-fold cover of M_1 branching along $K = \{(x,z) | x = 0 = f(z)\}$.

Lemma 3.1. $M_k \approx V(x^k - f(z)) \cap S_\varepsilon^{2n+1}$.

Proof. Letting $\mathbb{K} = V(x^k - f(z)) \cap S_\varepsilon^{2n+1}$, note that the following diagram commutes:

$$\begin{array}{ccccc} M_k - K & \xrightarrow{p} & M_1 - K & \xrightarrow{\approx} & S_\varepsilon^{2n-1} - K \\ \downarrow \psi'' & & \downarrow \psi' & \swarrow \psi & \\ S^1 & \xrightarrow{\lambda_k} & S^1 & & \end{array}$$

where $\psi''(x,z) = x/|x|$, $\psi'(x,z) = x/|x|$, $\psi(z) = f(z)/|f(z)|$ and $\lambda_k(x) = x^k$. Hence, by the previous section on branched covers, we may identify $\mathbb{K} - K \approx M_k - K$. Thus, it suffices to check the behavior of $p\colon M_k \to M_1$ on a tubular neighborhood of K.

Let $N = N(K) = \{(x,z) \in M_1 \mid |x| \le \delta\}$ for $0 < \delta \ll \varepsilon$.
$\hat{N} = p^{-1}(N) = \{(x,z) \in M_k \mid |x|^k \le \delta\}$.

$$\begin{array}{ccc} \hat{N} & \xrightarrow{\hat{q}} & D_\delta^2 \\ \downarrow p & & \downarrow \bar{p} \\ N & \xrightarrow{q} & D_\delta^2 \end{array} \qquad \begin{aligned} \hat{q}(x,z) &= x \\ q(x,z) &= x \\ \bar{p}(x,z) &= x^k \end{aligned}$$

Choosing δ small enough so that q is regular, we see that $p\colon \hat{N} \to N$ may be identified with $K \times D^2 \to K \times D^2$, $(a,b) \to (a,b^k)$. Hence, we may identify M_k and K.

Letting $\mathbb{F}_k = \{(x,z) \mid x^k - f(z) = \delta, |x|^{2k} + |z|^2 \le \varepsilon^2\}$ where $0 < \delta \ll \varepsilon$ we may identify $\mathbb{F}_k$ as the k-fold cyclic branched cover of $\mathbb{F} = \mathbb{F}_1$. Since $\partial\mathbb{F}_1 \approx M_1$, it follows that $\partial\mathbb{F}_k \approx M_k \approx \mathbb{K}$.

Lemma 3.2. $\mathbb{F}_1 \approx D_\varepsilon^{2n}$.

Proof. $\mathbb{F}_1 \simeq C(V(x - f(z) - \delta) \cap S_\varepsilon^{2n+1}) \simeq C(S_\varepsilon^{2n-1}) \simeq D_\varepsilon^{2n}$.

Remark. Thus, K may be regarded as the boundary of the manifold obtained by taking the k-fold cyclic covering of D^{2n} branching along a submanifold $F \subset D^{2n}$, $\partial F \approx K \subset S^{2n-1}$. Note that this description agrees with the general construction in the first chapter. Note also that $\mathbb{K}$ is the k-fold cyclic branched cover of S_ε^{2n-1} branching along K and $S_\varepsilon^{2n-1} - K \xrightarrow{\psi} S^1$ is a fibration of the type considered in Chapter 1; hence, Proposition 1.23 applies to $H_*(\mathbb{K})$. However, in this case there is even more structure since the complement of K in S_ε^{2n+1} also fibers over the circle. This leads to an independent proof of Proposition 1.23 at least in the case of weighted homogeneous polynomials.

From now on we assume that f(z) is a weighted homogeneous polynomial.

Recall some facts about weighted homogeneous polynomials:

1) If $K = V(f) \cap S^{2n-1}$ and F is the fiber of the fibration $S^{2n-1} - K \xrightarrow{\psi} S^1$, then F is homotopy equivalent to $\{z \in C^n | f(z) = 1\}$. The fibration is equivalent to the fibration $E \xrightarrow{\bar{\psi}} S^1$ where $E = \{z \in C^n | |f(z)| = 1\}$ and $\bar{\psi}(z) = \arg f(z)$. Thus, from now on we may regard $F \equiv \{z \in C^n | f(z) = 1\}$.

2) Given that f is of type $(w_1, w_2, \ldots, w_n)$, the monodromy for E, $h: F \to F$, is given by $h(z_1, z_2, \ldots, z_n) = (\bar{w}_1 z_1, \bar{w}_2 z_2, \ldots, \bar{w}_n z_n)$ where $\bar{w}_j = \exp(2\pi i/w_j)$, $j = 1, 2, \ldots, n$. Hence, if $F(x,z) = x^k + f(z)$ and $\mathbb{E}_k \to S^1$ is the associated fibration with fiber $\mathbb{F}_k = \{(x,z) \in C^{n+1} | F(x,z) = 1\}$, then $H: \mathbb{F}_k \to \mathbb{F}_k$ is given by $H(x,z) = (\bar{w}x, h(z))$ where $\bar{w} = \exp(2\pi i/k)$ and h as above.

3) The full fiber $\mathbb{F}_k$ is a branched cover of $\mathbb{F}_1$ with branch set $F = \{(0,z) | f(z) = 1\}$. $\pi: \mathbb{F}_k \to \mathbb{F}_1$, $\pi(x,z) = (x^k, z)$.

4) Recall that $\rho * z = (\rho^{1/w_1} z_1, \ldots, \rho^{1/w_n} z_n)$ and $f(\rho * z) = \rho f(z)$.

<u>Lemma 3.3.</u> $F \hookrightarrow \mathbb{F}_1$ via $z \to (0,z)$ for $f(z) = 1$ extends to an inclusion $cF \to \mathbb{F}_1$.

<u>Proof.</u> Regard $cF = I \times F/\{(0,z) \sim (0,z')\}$. Map $(\rho, z) \to (1-\rho, \rho * z)$, $0 \leq \rho \leq 1$. Note that $(1-\rho) + f(\rho * z) = 1-\rho + \rho f(z) = 1-\rho+\rho = 1$.

Lemma 3.4. The above inclusion $cF \to F_1$ is a homotopy equivalence.

Proof. This is obvious since $\mathbb{F}_1$ has the homotopy type of D^{2n}.

Now $\pi\colon \mathbb{F}_k \to \mathbb{F}_1$ by $\pi(x,z) = (x^k,z)$. Hence,

$$\pi^{-1}(cF) = \{(w^i(1-\rho)^{1/k}, \rho * z) \mid 0 \le i \le k-1,\ 0 \le \rho \le 1,\ z \in F\}.$$

Let $x^i cF \equiv \{(w^i(1-\rho)^{1/k}, \rho * z) \mid 0 \le \rho \le 1,\ z \in F\}$ $(w = \exp(2\pi i/k))$. Thus, $\pi^{-1}(cF) = \bigcup_{i=0}^{k-1} x^i cF$. By the same arguments as in Chapter 1, the inclusion $\pi^{-1}(cF) \to \mathbb{F}_k$ is a homotopy equivalence. Hence, if $\{a_i\}_{i=1}^{r}$ is a set of $(n-1)$ spheres imbedded in F which form a basis for $H_{n-1}(F)$ (recall that F has the homotopy type of a bouquet of spheres), then we may regard $ca_i \hookrightarrow cF \hookrightarrow \mathbb{F}_1$ and $x^j ca_i \hookrightarrow x^j cF \subset \mathbb{F}_k$. Let $\alpha_i = ca_i \cup xca_i$ and $x^j\alpha_i = x^j ca_i \cup x^{j+1} ca_i$. Then $\mathcal{B} = \{x^j\alpha_i \mid j = 0,\dots,k-2,\ i = 1,2,\dots,r\}$ is a set of spheres imbedded in F_k forming a basis for $H_n(\mathbb{F}_k)$. Note that as an element of the chain group $C_n(\mathbb{F}_k)$, we should write $x^j\alpha_i = x^j ca_i - x^{j+1} c\, a_i$. Regard $c\colon H_{n-1}(F) \to C_n(\mathbb{F}_k)$, $c(a_i) = ca_i$ as a homomorphism of groups. $h\colon F \to F$ induces $h\colon H_{n-1}(F) \to H_{n-1}(F)$. Let h denote this map and also the matrix of the map with respect to the basis $\{a_i\}$ of $H_{n-1}(F)$. Similarly, $H\colon H_n(\mathbb{F}_k) \to H_n(\mathbb{F}_k)$.

Define $h(\alpha_i) = cha_i - xcha_i$.

Lemma 3.5. $H(x^j\alpha_i) = x^{j+1}h(\alpha_i)$.

Proof.
$$\begin{aligned} H(ca_i) &= H\{(y,z) \mid (y,z) \in ca_i\} \\ &= \{(wy,hz) \mid (y,z) \in ca_i\} \\ &= x\{(y,hz) \mid (y,z) \in ca_i\} \\ &= x\{((1-\rho)^{1/k}, h(\rho * z)) \mid 0 \le \rho \le 1,\ z \in a_i,\ f(z) = 1\}. \end{aligned}$$

But
$$h(\rho * z) = \rho * h(z).$$

Hence,
$$\begin{aligned} H(ca_i) &= x\{((1-\rho)^{1/k}, \rho * h(z)) \mid 0 \le \rho \le 1,\ z \in a_i\} \\ &= xc(ha_i). \end{aligned}$$

Thus,
$$\begin{aligned} H(x^j\alpha_i) &= H(x^j ca_i - x^{j+1} ca_i) \\ &= x^j H(ca_i) - x^{j+1} H(ca_i) \\ &= x^{j+1}(cha_i - xcha_i) \\ H(x^j\alpha_i) &= x^{j+1} h(\alpha_i). \end{aligned}$$

Corollary 3.6. With respect to the basis $\mathcal{B}$ for $H_n(\mathbb{F}_k)$, H is given by the matrix

$$[H]_{\mathcal{B}} = \begin{bmatrix} 0 & h & & & \bigcirc \\ & 0 & h & & \\ & \bigcirc & \ddots & \ddots & \\ & & & 0 & h \\ -h & -h & -h & \cdots & -h \end{bmatrix}$$

Thus, the monodromy $H\colon H_n(\mathbb{F}_k) \to H_n(\mathbb{F}_k)$ is completely determined by h and k.

Conjecture. This result about the monodromy of $x^k + f(z)$ is true for any polynomial f(z) with isolated singularity at the origin.

We now use this result to examine $H_*(\mathbb{K})$, $\mathbb{K} = V(x^k + f(z)) \cap S^{2n+1}$.

Proposition 3.7. Letting $A \equiv$ the subgroup of $H_n(F_k)$ generated by $\{\alpha_i\}$, there is an exact sequence

$$0 \to H_n(K) \to A \xrightarrow{\mathcal{H}} A \to H_{n-1}(K) \to 0$$

where $\mathcal{H} = I + h + h^2 + \ldots + h^{k-1}$.

Proof. Consider the Wang sequence for the fibration $S^{2n+1} - \mathbb{K} \to S^1$.

$$\begin{array}{ccccccccc} 0 \longrightarrow & H_{n+1}(S - \mathbb{K}) & \longrightarrow & H_n(\mathbb{F}_k) & \xrightarrow{I-H} & H_n(\mathbb{F}_k) & \longrightarrow & H_n(S - \mathbb{K}) & \longrightarrow 0 \\ & \wr\| & & & \text{Alexander Duality} & & & \wr\| & \\ & H^{n-1}(\mathbb{K}) & & & & & & H^n(\mathbb{K}) & \\ & \wr\| & & & \text{Poincare Duality} & & & \wr\| & \\ & H_n(\mathbb{K}) & & & & & & H_{n-1}(\mathbb{K}) & \end{array}$$

Hence, $$0 \to H_n(\mathbb{K}) \to H_n(\mathbb{F}_k) \xrightarrow{I-H} H_n(\mathbb{F}_k) \to H_{n-1}(\mathbb{K}) \to 0.$$

By the previous corollary

$$[I - H]_{\mathcal{B}} = \begin{bmatrix} I & -h & & & \\ & I & -h & & \\ & & \ddots & \ddots & \\ & & & & -h \\ h & h & \cdots & & h+I \end{bmatrix}$$

<u>Claim.</u> The above matrix is equivalent to $I + h + \ldots + h^{k-1}$ via row and column operations.

<u>Proof of claim.</u>

$$\begin{bmatrix} 1 & -x & & & \\ & 1 & -x & & \\ & & \ddots & \ddots & \\ & & & & -x \\ x & x & \cdots & & 1+x \end{bmatrix} \sim \begin{bmatrix} 1 & -x & & & \\ & 1 & & -x & \\ & & \ddots & \ddots & \\ & & & & -x \\ 0 & x+x^2 & x & \cdots & 1+x \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \\ & \begin{bmatrix} 1 & -x & & \\ & 1 & -x & \\ x+x^2 & x & \cdots & 1+x \end{bmatrix} \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & (1+x+x^2+\ldots+x^{k-1}) \end{bmatrix}$$

Since row and column operations amount to basis changes in $H_n(\mathbb{F}_k)$, the proposition follows from this claim.

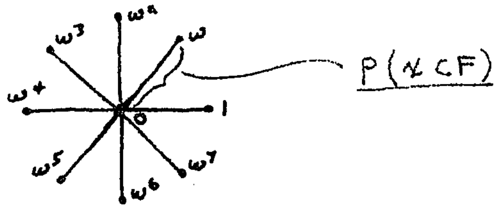
<u>Remark.</u> As noted before, this proposition follows from another argument unrelated to the monodromy. Once again we get the corollary that $H_*(\mathbb{K}_g) \cong H_*(\mathbb{K}_{g+d})$ when K is a homology sphere, $\mathbb{K}_g = V(x^g + f(z)) \cap S^{2n+1}$ and $d = \text{order } (h)$.

2. <u>Degeneration Technique</u>

Consider $\mathbb{F}_k = \{(x,z) \in C^{n+1} | x^k + f(z) = 1\}$ for f weighted homogeneous. We showed that $\bigcup_{i=0}^{k-1} x^i cF \to \mathbb{F}_k$ was a homotopy equivalence and in this manner obtained information about the topology of $\mathbb{F}_k$ and its "boundary" $\mathbb{K} = V(x^k + f(z)) \cap S^{2n+1}$. Now consider this approach from another viewpoint. Let $p: \mathbb{F}_k \to C$, $p(x,z) = x$. Then

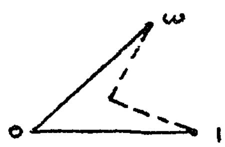
$$p^{-1}(x) \simeq \begin{cases} F & \text{for } x^k \neq 1 \\ V(f) & \text{for } x^k = 1 \end{cases} .$$

Recall that $cF = \{((1-\rho)^{1/k}, \rho * z) \mid f(z) = 1,\ 0 \leq \rho \leq 1\}$. Thus, $p(x^i cF) \equiv$ the line segment from 0 to w^i where $w = \exp(2\pi i/k)$.



The cones which piece together to generate $H_n(\mathbb{F}_k)$ arise from degeneration, as cycles on F collapse in approaching the singular fibers of the map p. This suggests a method of calculating the intersections of classes in $H_n(\mathbb{F}_k)$.

Let $a = c\alpha - xc\alpha$, $b = c\beta - xc\beta \in H_n(\mathbb{F}_k)$ with $\alpha, \beta \in H_n(F)$, $F = p^{-1}(0)$. Thus, $p(a) = p(b) =$ union of the line segments from 0 to 1 and 0 to w. Imagine deforming b until it lies over the dotted lines in the figure below. Call this d(b).



Then a and d(b) intersect only at two points. We may then calculate these intersection numbers directly or by referring them to linking numbers in spheres about the points of intersection.

<u>Lemma 3.8.</u> Given a path $\alpha: [0,1] \to C$ and f(z) a weighted homogeneous polynomial, $z \in C^n$, then there is an induced path $\hat{\alpha}: [0,1] \to C^n$ such that $f(\hat{\alpha}(t)) = \alpha(t) \cdot f(z_o)$ for any specified z_o.

<u>Proof.</u> Suppose f is of type $(w_1, w_2, \ldots, w_n)$. Then given any $y \in C$

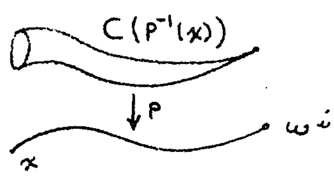
we may define $y*z = (y^{1/w_1}z_1,\ldots,y^{1/w_n}z_n)$ where y^{1/w_i} is usually understood to be a principal root. Here we can surely define $\alpha(t) = $ "$\alpha(t)*z_o$" where it is understood that we take principal roots for $\alpha(0)*z_o$ and obtain the rest of the curve by analytic continuation.

Lemma 3.9. Suppose $\alpha: [0,1] \to C$ as above and that $\alpha(t)^k \neq 1$ for $0 \leq t < 1$. Then there is a path $\gamma: [0,1] \to \mathbb{F}_k$ such that for $(x,z) \in p^{-1}(\alpha(0))$, $\gamma(0) = (x,z)$ and $p(\gamma(t)) = \alpha(t)$ for $p: \mathbb{F}_k \to C$, $p(x,z) = x$.

Proof. Note that $(x,z) \in p^{-1}(\alpha(0))$ implies that $x = \alpha(0)$ and, hence, $x^k \neq 1$. Define $\rho(t) = (1 - \alpha(t)^k)/(1 - x^k)$ and let $\rho: [0,1] \to C^n$ as above so that $f(\rho(t)) = \rho(t)f(z)$. Define $\gamma(t) = (\alpha(t),\rho(t))$. Need only check that $\gamma(t) \in \mathbb{F}_k$.

$$\alpha(t)^k + f(\hat{\rho}(t)) = \alpha(t)^k + \rho(t)\cdot f(z)$$
$$= (\alpha(t)^k - \alpha(t)^k x^k + f(z) - \alpha(t)^k f(z))/(1 - x^k)$$
$$= f(z)/(1 - x^k) = f(z)/f(z) = 1.$$

Remark. Note that under the hypotheses of the above lemma we have actually constructed a map $B: p^{-1}(x) \times [0,1] \to \mathbb{F}_k$, $B((x,z),t) = (\alpha(t),\hat{\rho}(t))$ with $pB((x,z),t) = \alpha(t)$. If $\alpha(1)^k = 1$ and $\alpha[0,1]$ is a simple arc, then $\mathrm{Image}(B) = c(p^{-1}(x))$.



We will use this construction shortly to form the deformed cycles.

Note also that if $\alpha(t) = t$, then $B: F \times [0,1] \to \mathbb{F}_k$ $(F = p^{-1}(0))$ is given by $B((0,z),t) = (t,(1 - t^k)*z)$.

$$B(F \times [0,1]) = \{(t,(1 - t^k)*z) \mid 0 \leq t \leq 1,\ f(z) = 1\}.$$

Hence, $B(F \times [0,1]) = cF$ as described before.

Observe that the tips of all cones under consideration consist of the set $\{(w^i,0) | i = 0,1,\ldots,k-1\}$. Since intersection numbers may be referred to linking numbers in a small sphere about the cone tip, it is necessary to determine $cF \cap S_\varepsilon^{2n-1}$ where S_ε^{2n-1} is a small sphere in F_k about $(1,0)$ (similarly for $(w^i,0)$).

Proposition 3.10. One may choose $\hat{S}_\varepsilon^{2n-1}$ about $(1,0)$ in $\mathbb{F}_k$ such that $cF \cap \hat{S}_\varepsilon^{2n-1} \approx \{z \in S_\varepsilon^{2n-1} | f(z) \in R^+\}$. That is, if $K = \mathbf{V}(f) \cap S_\varepsilon^{2n-1}$, then $cF \cap \hat{S}_\varepsilon^{2n-1}$ = to the fiber of the fibration $S_\varepsilon^{2n-1} - K \xrightarrow{\phi} S^1$.

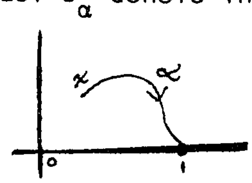
Proof. $\pi: \mathbb{F}_k \to C^n$, $\pi(x,z) = z$ is a cyclic branched cover branching along $F = \{z | f(z) = 1\} \subset C^n$. Since $z = 0$ is a zero of $f(z)$, we may choose $0 < \varepsilon << 1$ such that $D_\varepsilon^{2n} \cap F = \phi$. Then $\pi^{-1}(D_\varepsilon^{2n})$ is just k disjoint copies of D_ε^{2n} and similarly for $\pi^{-1}(S_\varepsilon^{2n-1})$. Since $\pi^{-1}(0) = \{(w^i,0)\}$, let $\hat{S}_\varepsilon^{2n-1}$ = the copy of S_ε^{2n-1} surrounding $(1,0)$.

$$cF = \{((1-\rho)^{1/k}, \rho * z) | f(z) = 1,\ 0 \leq \rho \leq 1\}$$

$$\pi: cF \cap \hat{S}_\varepsilon^{2n-1} \xrightarrow{\approx} \{\rho * z | f(z) = 1,\ 0 \leq \rho \leq 1\} \cap S_\varepsilon^{2n-1}$$
$$\cong$$
$$\{z | f(z) \in R^+\} \cap S_\varepsilon^{2n-1}.$$

Thus, $$cF \cap \hat{S}_\varepsilon^{2n-1} \approx \{z \in S_\varepsilon^{2n-1} | f(z) \in R^+\}. \qquad \text{Q.E.D.}$$

Next we deform cones. Let $x \in C$ such that $x^k \neq 1$ and let $\alpha: [0,1] \to C$ be a path with no self intersections such that $\alpha(0) = x$, $\alpha(1) = w^i$. Let C_α denote the cone $C_\alpha = B(p^{-1}(x) \times \mathbf{I})$.



Suppose that $\alpha(1) = 1$. We want to choose α so that C_α is a suitable deformation of cF. In particular, $C_\alpha \cap \hat{S}_\varepsilon^{2n-1} \equiv J_\alpha$ should be recognizable.

Now $J_\alpha \xrightarrow{\approx} \pi(J_\alpha)$ and $\pi(J_\alpha) = \{\rho(t) * z | f(z) = 1 - x^k\} \cap S_\varepsilon^{2n-1}$ where

$\rho(t) = (1 - \alpha(t)^k)/(1 - x^k)$. Suppose that near $t = 1$, $\alpha(t)$ is chosen so that $1 - \alpha(t)^k = (1 - t)(1 - x^k)$. Then $\rho(t) = (1 - t)$ and $J_\alpha \approx \{(1 - t)*z \,|\, f(z) = 1 - x^k\} \cap S_\varepsilon^{2n-1} \approx \{z \in S_\varepsilon^{2n-1} - K \,|\, \arg f(z) = \arg(1 - x^k)\} \approx \phi^{-1}(\arg(1 - x^k))$ where $\phi\colon S_\varepsilon^{2n-1} - K \to S^1$ is the Milnor fibration. Summing this up we have:

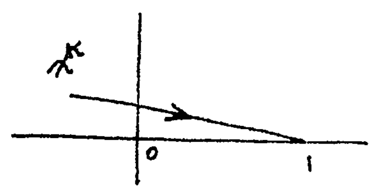
Lemma 3.11. Let $J_\alpha = C_\alpha \cap \hat{S}_\varepsilon^{2n-1}$. Suppose α is chosen so that near $t = 1$, $1 - \alpha(t)^k = (1 - t)(1 - x^k)$. Then $J_\alpha \approx \phi^{-1}(\arg(1 - x^k))$ where $\phi\colon S_\varepsilon^{2n-1} - K \to S^1$ is the Milnor fibration.

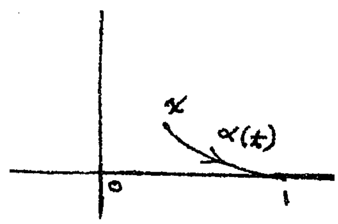
Discussion. $1 - \alpha(t)^k = (1 - t)(1 - x^k)$

$$\Rightarrow \alpha(t)^k = x^k + t(1 - x^k)$$

So $\alpha(t) = (x^k + t(1 - x^k))^{1/k}$ (principal k-th root).

Note that if we want $\alpha(1) = 1$, then we must choose x so that $0 < \arg x < \frac{1}{2}\arg w$.





If x is chosen so that $\frac{1}{2}\arg w < \arg x < \arg w$, then x^k will lie in the lower half space and if $\alpha'(t) = (x^k + t(1 - x^k))^{1/k}$, then $\alpha'(0) = x$, $\alpha'(1) = w$.

In the first case, $C_\alpha \cap cF = \{(1,0)\}$, and in the second, $C_{\alpha'} \cap xcF = \{(w,0)\}$.

Now to form an appropriate deformation of $\Sigma F = cF \cup xcF$, let

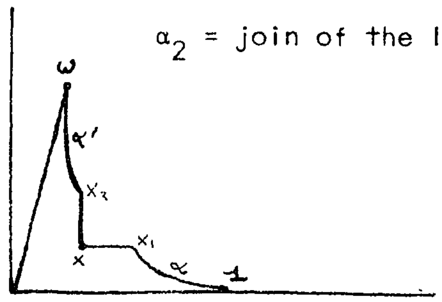
$x = \varepsilon'\tau$, $\tau^2 = w$

$x_1 = \varepsilon'\tau + \varepsilon''$ α as above such that $\alpha(0) = x_1$, $\alpha(1) = 1$.

$x_2 = \varepsilon'\tau + \varepsilon'' i$ α' such that $\alpha'(0) = x_2$, $\alpha'(1) = w$.

α_1 = join of the line from x to x_1 with α.

α_2 = join of the line from x to x_2 with α'.



Define

$$d(\Sigma F) = C_{\alpha_1} \cup C_{\alpha_2}.$$

Remark. Given x_1 and x_2 as above we have $\arg(1 - x_1^k) < 0$ and $\arg(1 - x_2^k) < 0$. Also, it follows from the last lemma that $C_{\alpha_1} \cap \hat{S}_\varepsilon^{2n-1} \approx$ fiber of ϕ over $\arg(1 - x_1^k)$ and $C_{\alpha_2} \cap x\hat{S}_\varepsilon^{2n-1} \approx$ fiber of ϕ over $\arg(1 - x_2^k)$.

Now given $a \in H_{n-1}(F)$ represented by a sphere $\underline{a}$ embedded in $F = p^{-1}(0)$, we have defined $\Sigma a = ca \cup xca \subset \Sigma F$. Transport a to a cycle $a' \subset p^{-1}(x)$ by trivializing the bundle $p^{-1}(\{tx|0 \leq t \leq 1\})$ and define $d(\Sigma a) = C_{\alpha_1}(a') \cup C_{\alpha_2}(a')$.

Certainly Σa and $d(\Sigma a)$ represent the same homology class in $H_n(\mathbb{F}_k)$ and given $b \in H_{n-1}(F)$, we see that $d(\Sigma a)$ and Σb intersect in only two points.

Recall the Seifert pairing. Let $\mathcal{F}$ denote $\phi^{-1}(1)$ where $\phi: S_\varepsilon^{2n-1} - K \to S^1$ as before. Then there is a pairing $\theta: H_{n-1}(\mathcal{F}) \times H_{n-1}(\mathcal{F}) \to \mathbb{Z}$ given by $\theta(a,b) = \ell(a,i^*b) = \ell(i_*a,b)$ where i_* = transport to ϕ^{-1} (pos argument) and i^* = transport to ϕ^{-1} (neg argument).

Note that F and $\mathcal{F}$ have the same homotopy type and given $a \subset F$ have corresponding sphere $\bar{a} = \{\rho * z | |\rho * z| = 1, z \in a\}$ in $\mathcal{F}$. Thus, we may confuse F with $\mathcal{F}$.

Proposition 3.12. Given $a,b \in H_{n-1}(F)$ as above, then

$$\langle \Sigma a, \Sigma b \rangle = \theta(a,b) + (-1)^n \theta(b,a).$$

Proof. By the remark, we may interpret $C_{\alpha_1} \cap \hat{S}_\varepsilon^{2n-1} \approx i^*\mathcal{F}$, $C_{\alpha_2} \cap x\hat{S}_\varepsilon^{2n-1} \approx i_*\mathcal{F}$. Similarly, we may interpret $C_{\alpha_1}(a') \cap \hat{S} \approx i^*a$, $C_{\alpha_2}(a') \cap x\hat{S} \approx i_*a$. Now

$$\begin{aligned}
<\Sigma a,\Sigma b> &= <d(\Sigma a),\Sigma b> \\
&= <C_{\alpha_1}(a') - C_{\alpha_2}(a'), cb - xcb> \\
&= <C_{\alpha_1}(a'), cb> + <C_{\alpha_2}(a'), xcb> \\
&= \ell(C_{\alpha_1}(a') \cap \hat{S}, cb \cap \hat{S}) + \ell(C_{\alpha_2}(a') \cap x\hat{S}, xcb \cap x\hat{S}) \\
&= \ell(i^*a,b) + \ell(i_*a,b) \\
&= (-1)^n \ell(b,i^*a) + \ell(a,i^*b) \\
&= \theta(a,b) + (-1)^n\theta(b,a).
\end{aligned}$$

Computations of the other intersection numbers go the same way. Here is a sketch: Abusing notation, write $\Sigma a = ca - xca$, $d(\Sigma a) = ci^*a - xci_*a$. Then

$$\begin{aligned}
<\Sigma a, x\Sigma b> &= <d(\Sigma a), x\Sigma b> \\
&= <ci^*a - xci_*a, xcb - x^2cb> \\
&= <-xci_*a, xcb> \\
&= <-ci_*a, cb>.
\end{aligned}$$

Whence
$$\begin{aligned}
<\Sigma a, x\Sigma b> &= -\ell(i_*a,b) \\
&= -\ell(a,i^*b) = -\theta(a,b).
\end{aligned}$$

Similarly,
$$<x\Sigma b, \Sigma a> = (-1)^{n+1}\theta(b,a).$$

Theorem 3.13. Let $\{a_1,\ldots,a_r\}$ denote a basis for $H_{n-1}(F)$ and $\{x^i\Sigma a_j \mid 0 \le i \le k-2,\ 1 \le j \le r\}$ = the corresponding basis for $H_n(\mathbb{F}_k)$. Let V be the matrix of θ: $H_{n-1}(F) \times H_{n-1}(F) \to \mathbb{Z}$, $\theta(a_i,a_j) = v_{ij}$. Then the intersection matrix for $H_n(\mathbb{F}_k)$ has the form

$$N = \begin{bmatrix} V + (-1)^n V' & -V & & & \\ (-1)^{n+1} V' & V + (-1)^n V' & -V & & \\ & (-1)^{n+1} V' & \ddots & \ddots & \\ & & \ddots & \ddots & -V \\ & & & (-1)^{n+1} V' & V + (-1)^n V' \end{bmatrix}$$

Proof. This follows from the preceding discussion.

Note that this result exactly parallels the result that would be obtained via branched covers. In the algebraic case actual computations are often possible.

3. Examples and Computations of Intersection Forms

1) $\mathbb{F}_2$: $x^2 + f(z) = 1$, hence, $N = V + (-1)^n V'$.

2) $\mathbb{F}$: $x_1^2 + x_2^2 + \dots + x_k^2 + f(z) = 1$, $z \in C^n$. $f(z)$ weighted homogeneous. F: $f(z) = 1$. θ: $H_{n-1}(F) \times H_{n-1}(F) \to \mathbb{Z}$ basis $\{a_i\}$, $i = 1,\dots,r$ for $H_{n-1}(F)$. $V = (v_{ij})$, $v_{ij} = \theta(a_i, a_j)$.

Proposition 3.14. Since each a_i is an $(n-1)$-sphere embedded in F, we may speak of $\Sigma^k a_i$ = k-th suspension of a_i. For each a_i there is an embedding of $\Sigma^k a_i$ in $\mathbb{F}$. The set $\{\Sigma^k a_i\}$ forms a basis of $H_{n+k-1}(\mathbb{F})$.

Proof. Let $\Sigma^k a = \{(x_1, x_2, \dots, x_k, t*z) | x_i \in R, \sum_{i=1}^{k} x_i^2 \leq 1, t = (\Sigma x_i^2)^{1/2}$, $z \in a$. The $\{\Sigma^k a_i\}$ are a basis for $H_{n+k-1}(\mathbb{F})$ since they are exactly the cycles obtained by iterating the degeneration procedure. That is, begin with $x_1^2 + f(z) = 1$, $\Sigma a = \{(x_1, t*z) | x_1 \in R, \dots\}$. Then take $f_1(z) = x_1^2 + f(z)$ and work with $x_2^2 + f_1(z) \dots$.

Proposition 3.15. Let $\mathbb{F}$: $x_1^2 + x_2^2 + \dots + x_k^2 + f(z) = 1$ have usual orientation as a complex submanifold of C^{n+k}. Let $\mu = (-1)^{n(k-1)+\frac{k(k-1)}{2}}$. Then

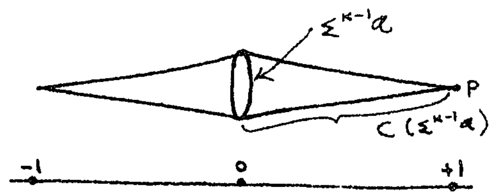
$$\langle \Sigma^k a, \Sigma^k b \rangle = \mu(\theta(a,b) + (-1)^{n+k-1}\theta(b,a)).$$

Thus, the intersection matrix for $\mathbb{F}$ is given by $N = \mu(V + (-1)^{n+k-1}V')$ where V is the Seifert matrix for the mebedding of F: $f(z) = 1$ in S^{2n-1}.

Proof. To determine $<\Sigma^k a, \Sigma^k b>$, fiber F over one of the variables, say x_k. Then let $\mathbb{F}_t$: $x_1{}^2 + \ldots + x_{k-1}^2 + f(z) = 1 - t^2$ and

$$C(\Sigma^{k-1}a) = \{(x_1,\ldots,x_k,z) \in \Sigma^k a | x_k \geq 0\}$$
$$C'(\Sigma^{k-1}a) = \{(x_1,\ldots,x_k,z) \in \Sigma^k a | x_k \leq 0\}.$$

Thus, $$\Sigma^k a = C(\Sigma^{k-1}a) \cup C'(\Sigma^{k-1}a).$$



We can choose coordinates about $p = (0,\ldots,0,1,0)$ such that

$$F_t: x_1{}^2 + \ldots + x_{k-1}^2 + f(z) = t^2$$
$$C\Sigma^{k-1}a: x_i \text{ real}, \Sigma x_i{}^2 \leq 1, x_k \geq 0 \ldots$$

Thus, we may regard the coordinates of this cone about p to be coordinates for ca plus $x_1,\ldots,x_{k-1}$. Thus, $(ca, x_1,\ldots,x_{k-1})$ gives orientation for $C\Sigma^{k-1}a$ at p. Now $d(C\Sigma^{k-1}b)$ has orientation $(i^*cb, i^*x_1,\ldots,i^*x_{k-1}) = (ci^*b, -ix_1,\ldots,-ix_{k-1})$. Together these define an orientation for $\mathbb{F}$ at p:

$$(ca, x_1,\ldots,x_{k-1}, ci^*b, -ix_1,\ldots,-ix_{k-1})$$
$$= (-1)^{n(k-1)}(ca, ci^*b, x_1,\ldots,x_{k-1}, -ix_1,\ldots,-ix_{k-1})$$
$$= (-1)^{n(k-1)}\cdot(-1)^{k(k-1)/2}(ca, ci^*b, x_1, ix_1, x_2, ix_2,\ldots,x_{k-1}, ix_{k-1}).$$

Let $\mu = (-1)^{n(k-1)}\cdot(-1)^{k(k-1)/2}$. Then the above shows that

$$<C\Sigma^{k-1}a, C\Sigma^{k-1}i^*b> = \mu\cdot<ca, ci^*b>$$
$$= \mu\cdot\ell(a, i^*b)$$
$$= \mu\cdot\theta(a,b).$$

Similarly, $\langle C'\Sigma^{k-1}a, C'\Sigma^{k-1}i_*b\rangle = \mu\cdot(-1)^{k-1}\ell(a,i_*b)$

$$= \mu\cdot(-1)^{n+k-1}\ell(b,i^*a)$$

$$= \mu\cdot(-1)^{n+k-1}\theta(b,a).$$

Hence, $\langle\Sigma^k a,\Sigma^k b\rangle = \mu(\theta(a,b) + (-1)^{n+k-1}\theta(b,a))$.

Remark. Let $\mathbb{F}$: $x_o^2 + x_1^2 + \ldots + x_n^2 = 1$ and let Σ^n denote the generator of $H_n(\mathbb{F})$. Then the same method as above shows that $\langle\Sigma^n,\Sigma^n\rangle = (-1)^{n(n-1)/2}(1 + (-1)^n\cdot 1)$.

Remarks. For $k = 1$ this proposition agrees with the previous discussion, giving $N = V + (-1)^n V'$.

If $n = 2$ and $k \geq 1$, then $N = V + (-1)^{k-1}V'$ up to sign. The intersection matrix depends only on three dimensional topology associated with $f(z_1,z_2)$. We treated this case more generally from the viewpoint of $O(k)$-actions.

The calculation suggests a generalization of the theorem on $O(n)$-actions with orbit space D^4 (Chapter II) to higher dimensional orbit spaces.

3) Generalizing 2) we pose

Problem A. Find the intersection matrix for
$\mathbb{F}$: $x_o^{m_o} + x_1^{m_1} + \ldots + x_k^{m_k} + f(z) = 1$ for f weighted homogeneous, $z \in C^n$, $m_i \geq 2$, in terms of the Seifert matrix for F: $f(z) = 1$.

Problem B. Express the Seifert pairing for $\mathbb{F}_m$: $x^m + f(z) = 1$ in terms of the Seifert pairing for F: $f(z) = 1$.

Note that $B \Rightarrow A$. Just iterate B to find Seifert pairing for F and then automatically get the intersection pairing.

Let $\{a_1,\ldots,a_r\}$ be a basis for $H_{n-1}(F)$, $\{x^i\Sigma a_j\}$ the corresponding basis for $H_n(\mathbb{F}_m)$.

$\theta\colon H_{n-1}(F) \times H_{n-1}(F) \to \mathbb{Z}$

$\theta\colon H_n(\mathbb{F}_m) \times H_n(\mathbb{F}_m) \to \mathbb{Z}$

Denote both pairings by θ.

<u>Lemma 3.16.</u> Let $\mathbb{F}_k$ have its natural orientation. Then

$$\theta(x^i\Sigma a, x^i\Sigma b) = (-1)^{n-1}\theta(a,b) \qquad 0 \le i \le k-2$$

$$\theta(x^{i-1}\Sigma a, x^i\Sigma b) = (-1)^{n}\theta(a,b) \qquad 1 \le i \le k-2$$

and otherwise $\theta(\ ,\ ) = 0$.

In matrix form, let $\mathbb{V} = [\theta_{\mathbb{F}}]$, $V = [\theta]$. Then

$$\mathbb{V} = (-1)^{n-1}\begin{bmatrix} V & -V & & & \\ & V & -V & & \\ & & V & \ddots & \\ & & & \ddots & \\ & & & & V \end{bmatrix}$$

<u>Proof.</u> Let $\mathbb{E} = \{(x,z) \mid x^k + f(z) \in S^1\}$, $\hat{\phi}\colon \mathbb{E} \to S^1$, $\hat{\phi}(x,z) = x^k + f(z)$. Recall that this is equivalent to the fibration $\phi\colon S^{2n-1} - K \to S^1$. $\mathbb{F}_k = \mathbb{F} = \phi^{-1}(1)$. $\theta\colon H_n(\mathbb{F}) \times H_n(\mathbb{F}) \to \mathbb{Z}$ may be defined by $\theta(\Sigma a, \Sigma b) = \ell(i_*\Sigma a, \Sigma b)$ where ℓ is linking number in $\mathbb{E}$, i_* = translate fiber of ϕ by small positive argument. Let $p\colon \mathbb{E} \to C$, $p(x,z) = x$. Call $p(\Sigma a)$ the "shadow" of Σa. Compare the shadows of $i_*\Sigma a$ and Σb.

To obtain $\theta(\Sigma a, \Sigma b)$, regard

$$\ell(i_*\Sigma a, \Sigma b) = \langle c i_*\Sigma a, c\Sigma b\rangle$$

$$= \langle \Sigma c i_* a, \Sigma c b\rangle.$$

Let t denote the suspension coordinate. Then the above gives an orientation

$$(ci_*a, i_*t, cb, t) = (-1)^{n+1}(ci_*b, cb, t, i_*t).$$

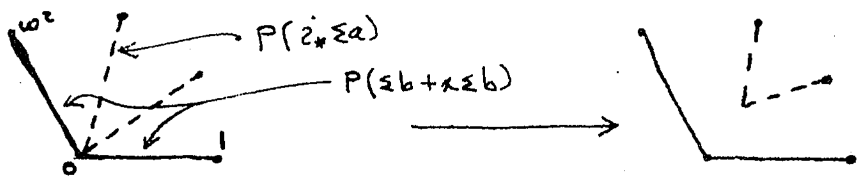
This implies that $\langle \Sigma c i_* a, \Sigma c b\rangle = (-1)^{n-1}\langle c i_* a, cb\rangle$. Hence,

$$\theta(\Sigma a, \Sigma b) = (-1)^{n-1}\theta(a,b).$$

It then follows immediately that

$$\theta(x^i\Sigma a, x^i\Sigma b) = \theta(\Sigma a, \Sigma b) = (-1)^{n-1}\theta(a,b).$$

Now consider $\Sigma b + x\Sigma b$. If $\Sigma b = c - xc$ where $p(c) = [0,1]$, then $x\Sigma b = xc - x^2c$ and $\Sigma b + x\Sigma b = c - x^2c$. Comparing the shadows of $i_*\Sigma a$ and $\Sigma b + x\Sigma b$, we see that these cycles may be deformed away from one another in $\mathbb{E}$.



Hence, $$\ell(i_*\Sigma a, \Sigma b + x\Sigma b) = 0.$$

Thus, $$\theta(\Sigma a, x\Sigma b) = -\ell(i_*\Sigma a, \Sigma b) = (-1)^n\theta(a,b).$$

Remark. The lemma will be applied by iteration. Thus, $\theta(\Sigma^2 a, \Sigma^2 b) = (-1)^{n-1}(-1)^n\theta(a,b)$, $\theta(\Sigma^3 a, \Sigma^3 b) = (-1)^{3(n-1)+1+2}$, and generally, $\theta(\Sigma^k a, \Sigma^k b) = (-1)^{k(n-1)+1+2+\ldots+k-1}\theta(a,b) = (-1)^{kn+k\frac{(k+1)}{2}}(a,b)$. For example, if $\mathbb{F}$: $x_1{}^2 + \ldots + x_k{}^2 + f(z) = 1$, then

$$\begin{aligned}\langle\Sigma^k a, \Sigma^k b\rangle &= \theta(\Sigma^{k-1}a, \Sigma^{k-1}b) + (-1)^{n+k-1}\theta(\Sigma^{k-1}b, \Sigma^{k-1}a)\\ &= (-1)^{n(k-1)+\frac{k(k-1)}{2}}(\theta(a,b) + (-1)^{n+k-1}\theta(b,a))\end{aligned}$$

recovering the proposition in 2).

Take $\mathbb{F}$: $x_0^{m_0} + x_1^{m_1} + \ldots + x_k^{m_k} + f(z) = 1$, $m_i \geq 2$, where $z \in C^n$ and $f(z)$ is weighted homogeneous. Let $\Gamma = Z_{m_0} \times Z_{m_1} \times \ldots \times Z_{m_k}$ a direct product of cyclic groups, and let x_i, $i = 0,\ldots,k$ represent generators of Z_{m_i}. Thus, elements of Γ are represented by $x = x_0^{s_0}x_1^{s_1}\ldots x_k^{s_k}$. Then $H_{n+k}(\mathbb{F})$ has a basis $\{x\Sigma^{k+1}a \mid x = \Pi x_i^{s_i}, 0 \leq s_i \leq m_i-2$, a runs over basis for $H_{n-1}(F)\}$.

Define $\tau: Z_N \to Z_2$ by $\tau(x_\alpha) = -1$, $\tau(x_\alpha{}^i) = 0$ for $i \not\equiv 1 \pmod N$ where x_α denotes the generator of Z_N. Extend τ to a function $\tau: \Gamma \to Z_2$ by $\tau(\Pi_i x_i^{\varepsilon_i}) = \Pi_i\tau(x_i^{\varepsilon_i})$. Let $\bar{x}$ = inverse of x for $x \in \Gamma$.

<u>Proposition 3.17.</u> Under the above assumptions, the intersection pairing $<\ ,\ >: H_{n+k}(\mathbb{F}) \times H_{n+k}(\mathbb{F}) \to \mathbb{Z}$ is given by

$$<x\Sigma^{k+1}a, y\Sigma^{k+1}b> = \mu(\tau(\bar{x}y)\theta(a,b) + (-1)^{n+k}\tau(\bar{y}x)\theta(b,a))$$

where $x,y \in \Gamma$; $a,b \in H_{n-1}(F)$; $\mu = (-1)^{nk+\frac{k(k+1)}{2}}$.

<u>Proof.</u> First let $\mathbb{F}': x_1^{m_1} + \ldots + x_k^{m_k} + f(z) = 1$. $\Gamma' = Z_{m_1} \times Z_{m_2} \times \ldots \times Z_{m_k}$. By iterating the lemma, we may compute $\theta: H_{n+k-1}(\mathbb{F}') \times H_{n+k-1}(\mathbb{F}') \to \mathbb{Z}$. It is easy to see that

$$\theta(x'\Sigma^k a, y'\Sigma^k b) = \mu\tau(\bar{x}'y')\theta(a,b),$$

for $x',y' \in \Gamma'$. Express $x\Sigma^{k+1}a = x_o^{s_o}x'\Sigma^{k+1}a = x_o^{s_o}\Sigma(x'\Sigma^k a)$, $x \in \Gamma$, $x' \in \Gamma'$. Recall Proposition 3.12 which says that

$$<\Sigma A, \Sigma B> = \theta(A,B) + (-1)^{n+k}\theta(B,A)$$

$$<\Sigma A, x_o\Sigma B> = -\theta(A,B)$$

$$<x_o\Sigma A, \Sigma B> = (-1)^{n+k+1}\theta(B,A)$$

where $A,B \in H_{n+k-1}(\mathbb{F}')$.

In our notation this clearly becomes

$$<x_o^{s_o}\Sigma A, x_o^{t_o}\Sigma B> = \tau(x_o^{s_o}x_o^{t_o})\theta(A,B) + (-1)^{n+k}\tau(x_o^{t_o}x_o^{s_o})\theta(B,A).$$

Letting $A = x'\Sigma^k a$, $B = y'\Sigma^k b$ and substituting, we get

$$<x\Sigma^{k+1}a, y\Sigma^{k+1}b> = \mu(\tau(\bar{x}y)\theta(a,b) + (-1)^{n+k}\tau(\bar{y}x)\theta(b,a)).$$

This proposition generalizes Pham's calculation for the intersection matrix of a Brieskorn variety [see 28]. It also, we hope, puts it in a more conceptual context and may lend some insight into the harder problem of obtaining the intersection matrix for an arbitrary nonsingular polynomial hypersurface. Various particular calculations suggest themselves.

<u>Example.</u> <u>Intersection Matrix for a Brieskorn Variety.</u> Let $\mathbb{F}: x_o^{m_o} + \ldots + x_k^{m_k} = 1$ and $\Gamma = Z_{m_o} \times \ldots \times Z_{m_k}$. Note that we may regard

$H_k(\mathbb{F})$ as generated by $\{x\Sigma^k a \mid x = \Pi x_i^{\ell_i} \in \Gamma,\ 0 \le \ell_i \le m_i - 2\}$ where $a = w-1$, $w = \exp(2\pi i/m_o)$, the generator of $\bar{H}_o(F)$ as a Z_{m_o}-module (F: $x_o^{m_o} = 1$). Hence, we may use the notation $x \leftrightarrow x\Sigma^k a$.

Proposition 3.18. Under the above notation $<\ ,\ >: H_k(\mathbb{F}) \times H_k(\mathbb{F}) \to \mathbb{Z}$ is given by

$$<x,y> = (-1)^{k(k-1)/2}(\tau(x\bar{y}) + (-1)^k \tau(\bar{x}y)).$$

Proof. First note that θ: $\bar{H}_o(F) \times \bar{H}_o(F) \to \mathbb{Z}$ is easily calculated. We find $\theta(x_o^{s_o}, x_o^{t_o}) = -\tau(\bar{x}_o^{s_o} x_o^{t_o})$. Then apply the previous proposition to $x_1^{m_1} + \ldots + x_k^{m_k} + f(z)$ where $f(z) = f(x_o) = x_o^{m_o}$. Then $\mu = (-1)^{n(k-1)+\frac{k(k-1)}{2}}$ and

$$<x,y> = <x' x_o^{s_o}, y' x_o^{t_o}> = \mu(\tau(\bar{x}'y')\theta(x_o^{s_o}, x_o^{t_o})$$
$$+ (-1)^k \tau(x'\bar{y}')\theta(x_o^{t_o}, x_o^{s_o})) = \mu(-\tau(\bar{x}y) + (-1)^k(-\tau(x\bar{y}))$$
$$= (-1)^{k+\frac{k(k-1)}{2}}(\tau(\bar{x}y) + (-1)^k \tau(\bar{y}x))$$
$$= (-1)^{k(k-1)/2}(\tau(x\bar{y}) + (-1)^k \tau(\bar{x}y)).$$

Remark. One can easily check that this agrees with the Pham-Hirzebruch calculation [see 13, page 85].

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Cyclic Branched Covers, O(n)-Actions, and
Hypersurface Singularities

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ABSTRACT

This thesis investigates relationships among branched covers, hypersurface singularities, and O(n)-manifolds. I extend a theorem of Hirzebruch classifying knot manifolds to the case of link manifolds (O(n)-manifolds with orbit space D^4 and fixed point set corresponding to a link in $S^3 = \partial D^4$). The result classifies a link manifold in terms of imbedding invariants of the link. Manifolds whose boundaries are branched covers are constructed and the structure of the covers is pursued from this point of view. Similar techniques enable us to calculate intersection forms corresponding to singularities of polynomials of type $x_1^{a_1} + \ldots + x_n^{a_n} + f(z)$. These provide settings for certain periodicity theorems about Brieskorn varieties and offer interesting links between low and high dimensional topology.