Math 215 - Assignment Number 1, Fall 2011
Hand these problems in on Wednesday, August 31.

1. Read in Eccles Chapters 1,2,3,4,5 (pp. 1-52) and be responsible for the exercises for Chapters 1 and 2. (The solutions are in the back of the book, so we are not asking you to hand in these exercises, but you should do them and compare your answers with the book. In class, please feel free to ask questions related to these exercises.)

2. Eccles page 53. Problems 1. 2. and 3.

3. Along with doing truth tables we will be looking at symbolic logic algebraically. For this purpose I will write
~a = not a,
a v b = a or b,
a&b = a and b (using this word processors type),
a > b = a implies b.
When I write X = Y, I will mean that X and Y have the same truth table. For example ~(~ a) = a is true because, if a = T then ~(~ T) = ~F = T and if a = F then ~(~ F ) = ~T = F. So whatever value a takes it is true that ~(~ a) = a.
Prove the following equalities.
(a) A v ~A = T
(b) A > B = (~ A) v B
(c) ~(A v B) = (~A) & (~B)
(d) ~(A & B) = (~A) v (~B)
(e) A & ( B v C) = (A & B) v (A & C)
(f) A v ( B & C) = (A v B) & (A v C).

____________________________________________

____________________________________________

Math 215 - Assignment Number 2, Fall 2011
Hand these problems in on Friday, September 9.

1. Read in Eccles Chapters 1,2,3,4,5 (pp. 1-52) and be responsible for the exercises for Chapters 1,2,3 and 4. (The solutions are in the back of the book, so we are not asking you to hand in these exercises, but you should do them and compare your answers with the book. In class, please feel free to ask questions related to these exercises.)

2. Show that the following is a tautology using only algebra: (A>B)v(B>A).

3. Use the distributive law to prove that
(A&B)v(B&C)v(C&A) = (AvB)&(BvC)&(CvA).
Hint: Rewrite the above in Boolean notation and do the problem using that notation.

4. Write a careful proof showing that the square root of 5 is irrational. (Note that you will have to examine when squares are divisible by 5.)

5. Prove that if $S = 1 + x + x^2 + x^3 + \ldots + x^n$, and x is not equal to 1, then $S = (1 - x^{(n+1)})/(1-x)$.
(Remark: If you know about mathematical induction, you can prove this by induction, but I would like you to prove it by using the algebra of expressions involving the "three dots". Thus you can do things like
$x + x^2 + x^3 + \ldots + x^n = x(1 + x + x^2 + \ldots + x^{(n-1)})$.)

6. Eccles, page 53. Problems 4,5, 9,11.

7. Write a careful proof showing that any game of Sprouts must end in a finite number of moves.

---

---

Math 215 - Assignment Number 3, Fall 2011
Hand these problems in on Monday, September 19.

1. Continue reading Eccles Chapters 1,2,3,4,5 (pp. 1-52) and be responsible for the exercises for Chapters 1,2,3, 4 and 5. Read also the notes on the web about Boolean Notation and Boolean algebra. Look at the article by Charles Sanders Peirce but you are not responsible for its contents. Look at the article about Sprouts to find out more about the game. Read the article on the website about Peano's Axioms. You are not responsible for the exercises in this article.
http://www.math.uic.edu/~kauffman/Peano.pdf

2. Prove by induction that
$1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$ for n = 1,2,... .

(Hint: You can use the formula 1 + 2 + ... + n = n(n+1)/2.)

3. Let Un be the n-th Fibonacci number as defined by Eccles in Definition 5.4.2. Show by induction that
$U_1^2 + U_2^2 + \ldots + U_n^2 = U_nU_{n+1}$ for n = 1,2,3,... .

4. Eccles page 53. Problems 12, 16 and 20.

----------------------------------------------------------------------------------------------------------------------------------------

Math 215 - Assignment Number 4, Fall 2011
Hand these problems in on Wednesday, September 29.

0. There are three new short articles on the website: ConwayArmy.pdf, Exist.pf and Desargues.pdf.
Conway Army figures in problem 6 below. The other two are, for the moment, for discussion. Read them.

1. Read Eccles Chapter 6. Eccles page 115.
Problems 1. 2. and 3.

2. Prove that if a square is odd then it is one more than a multiple of 8. For example 9 = 1 + 8, 25 = 1 + 8 x 3,
49 = 1 + 8 x 6. Prove the general result.

3. Eccles page 56. problem 19.
Eccles page 57. problem 25.

4. Write up your own inductive proof of the Euler Formula that says that v - e + f = 2 for a conncected plane graph with v nodes, e edges and f faces. (See the discussion about the Euler formula in these notes.)

5. Use the Euler formula to determine a formula for the number of regions in a game of sprouts that has a connected graph (this can take a few moves) and starts with S nodes, after M moves have been made. Check your formula against some sample game positions.

6. Read the notes posted on the website titled "Conway's Army". Learn to play the game indicated in the notes and read the proof about this game that is in the notes. Write a

one page summary, in your own words, of the contents of this article including a sketch of the proof that it is impossible to reach the fifth level.

7. Design a switching circuit with one light and three main switches such that each switch controls the light. Recall that with two switches we went over how to do this in class and that it corresponded to the logical condition (a&b)v(~a & ~b), which could then be translated into a switching circuit. Do the same job using 3 switches a,b,c.
##################################################

Math 215 - Assignment Number 5, Fall 2011
Hand these problems in on Monday, October 10, 2011.

1. Read Eccles Chapters 7 and 8 and 9. (Concentrate on the main ideas. We will work with these chapters over the next two weeks. The main ideas are
(a) The use of "for all" and "there exists". In more flexible fonts I will use an upside-down A for "for all" and a backwards E for "there exists". Here I will write
"Ax" for "for all x" and
"Ex" for "there exists and x".
Thus we can have propositions of the form
Ax, P(x):  For all x, P(x) is true.
and
Ex, P(x) : There exists an x such that P(x) is true.
It is very important to have these so-called quantifiers in our logic, particularly when we work with sets in mathematics.
Note that
~(Ax, P(x)) = Ex, ~P(x),
and
~(Ex, P(x)) = Ax, ~P(x).
This is the fundamental relationship between the general (for all) and the particular (there exists) via negation.
(b) The notion of a function from one set to another.
(c) The notions of injective, surjective and bijective functions and their properties.

2. Eccles page 57, problem 26. Draw examples for
n < = 6. Then find a formula for the number of regions as a function of n, by applying the Euler formla that relates the

number of nodes, edges and regions in a connected plane graph (v-e+f=2). In order to do this, you will need to find a formula for the number of nodes and another formula for the number of edges in the graph that corresponds to the circle with n points all connected to one another as described in the problem. We will definitely discuss this problem on Wednesday, October 5.

3. Eccles page 117, problems 11, 12, 13.

4. Read carefully the Notes on Logic Circuits
http://www.math.uic.edu/~kauffman/LogicCircuits.pdf
with their solution to the three-switch light problem from the last homework. Then solve problem 2 in those notes. Problem 2 is a problem to design (using the "crossover switch" from the notes) a circuit with N switches for an arbitrary N, such that each switch controls a single light.

5. Consider the function

$$F[n] = 1^2 + 2^2 + \cdots + n^2$$

.

The following chart shows the values of this function for several values of n = 1,2, ..., 20 and the factorizations of these values into prime factors. From this, find a pattern and guess a formula for F[n]
(similar to 1 + 2 + ... + n = n(n+1)/2).
Having guessed your formula, find a proof that it works for all n by using mathematical induction.

$1 : 1$
$2 : 5$
$3 : 14 = 2 \times 7$
$4 : 30 = 2 \times 3 \times 5$
$5 : 55 = 5 \times 11$
$6 : 91 = 7 \times 13$
$7 : 140 = 2 \times 5 \times 7$
$8 : 204 = 2 \times 2 \times 3 \times 17$
$9 : 285 = 3 \times 5 \times 19$
$10 : 385 = 5 \times 7 \times 11$
$11 : 506 = 2 \times 11 \times 23$
$12 : 650 = 2 \times 5 \times 5 \times 13$
$13 : 819 = 3 \times 3 \times 7 \times 13$
$14 : 1015 = 5 \times 7 \times 29$
$15 : 1240 = 2 \times 2 \times 2 \times 5 \times 31$
$16 : 1496 = 2 \times 2 \times 2 \times 11 \times 17$
$17 : 1785 = 3 \times 5 \times 7 \times 17$
$18 : 2109 = 3 \times 19 \times 37$
$19 : 2470 = 2 \times 5 \times 13 \times 19$
$20 : 2870 = 2 \times 5 \times 7 \times 41$

#################################################
Math 215 - Assignment Number 6, Fall 2011
Hand these problems in on Friday, October 27, 2011.

1. Read Chapters 10 -14. You should read for ideas. We will work with these chapters for quite some time. We have already begun to talk about some of the ideas in these chapters. In particular you will see discussion of choice functions and their relation to the binomial theorem, and

you will see generalizations of counting to the notion of the cardinality of a set. Two sets have the same cardinality if they can be put into 1-1 correspondence. For finite sets it is quite clear that cardinality is the same as having the same number of members. For infinite sets things become very interesting. For example, let N be the natural numbers and E be the even numbers. So N = {1,2,3,...} and E = {2,4,6,8,...}. Then N and E are in 1-1 correspondence via the map F:N -----> E given by the formula F(n) = 2n. The map F is injective and surjective. So we say that N and E have the same cardinality. And yet E is a proper subset of N. So in a certain sense E is smaller than N and in the sense of cardinality, E has the same size as N. We will be seeing a lot of this sort of phenomena. In fact, we will eventually prove that any infinite set is in 1-1 correspondence with some proper subset of itself, and that this never happens for finite sets.

2. page 115. #4,5,6,7,8,9.

3. page 117. #14,16,18,19.

4. Recall from our class discussion that given a sequence { $a_n$ } of real numbers with n = 1,2,3, ..., we say that

lim $_n$ ---> Infinity ($a_n$) = a

exactly when

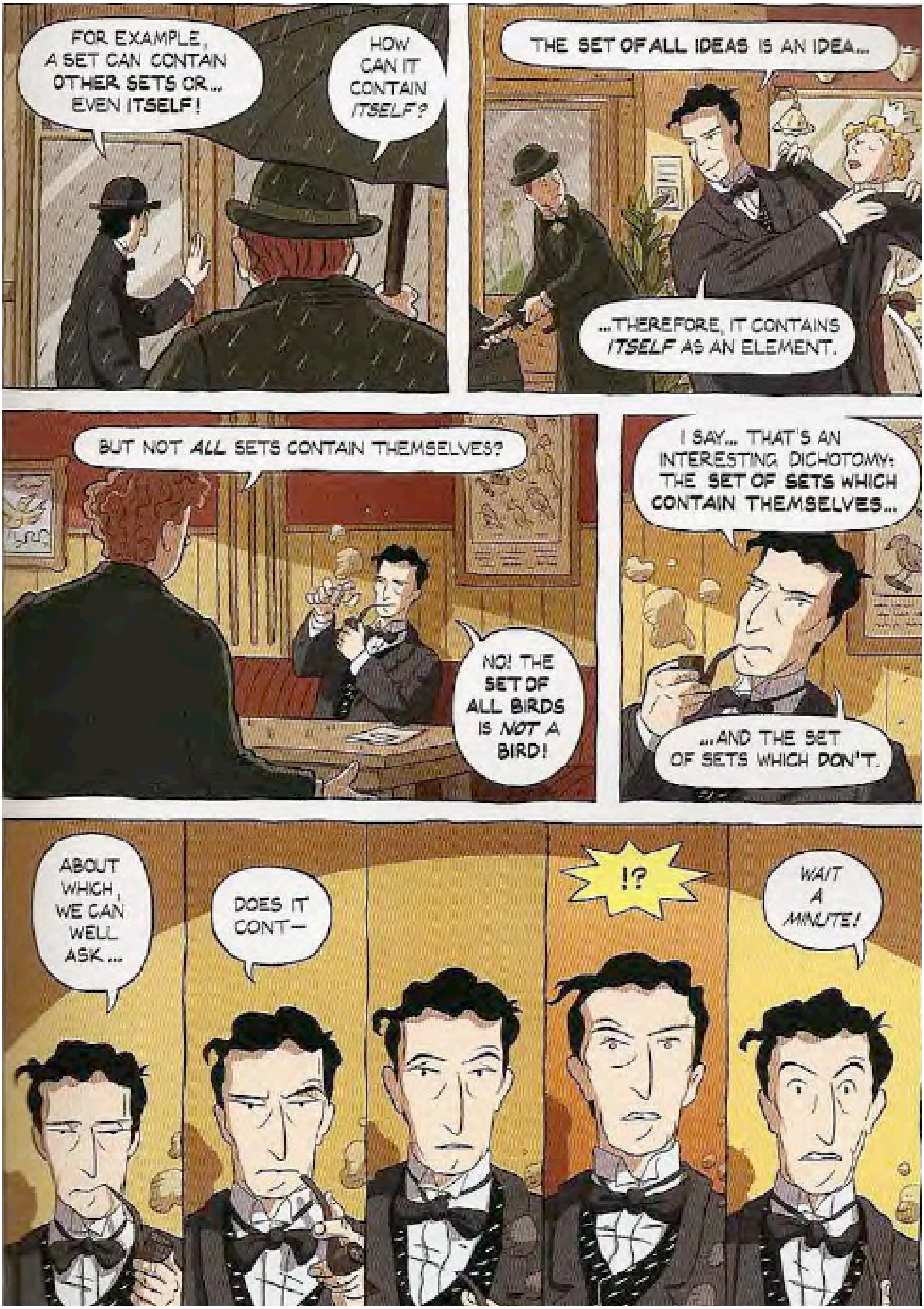
A $\varepsilon > 0$, E M in N, s.t. n > M implies $|a_n - a| < \varepsilon$.

Use this definition of limit and prove that

lim $_n$ ---> Infinity $((n^2 +1)/(n^2 -1)) = 1$.

5. Refer to the discussion in 1. above and prove that the odd numbers O = {1,3,5,7,11,...} have the same cardinality as the entire set of natural numbers N.

6. Examine the cartoon below (an excerpt from Logicomix. see our website for a reference to the book itself.)


FOR EXAMPLE, A SET CAN CONTAIN OTHER SETS OR... EVEN ITSELF!
HOW CAN IT CONTAIN ITSELF?
THE SET OF ALL IDEAS IS AN IDEA...
...THEREFORE, IT CONTAINS ITSELF AS AN ELEMENT.
BUT NOT ALL SETS CONTAIN THEMSELVES?
NO! THE SET OF ALL BIRDS IS NOT A BIRD!
I SAY... THAT'S AN INTERESTING DICHOTOMY: THE SET OF SETS WHICH CONTAIN THEMSELVES...
...AND THE SET OF SETS WHICH DON'T.
ABOUT WHICH, WE CAN WELL ASK...
DOES IT CONT—
!?
WAIT A MINUTE!

This comic strip dramatizes Bertrand Russell's discovery of the set R of all sets that are not members of themselves:

$$R = \{X \mid X \text{ is a set and } X \text{ is not a member of } X\}.$$

In the cartoon, Russell is asking himself,

"Can R be a member of R?"

Do you see the problem?

1. If R belongs to R then it must be that R is not a member of R.
2. But if R does not belong to R, then R meets the criteria for being a member of R, and so R should belong to R.

Thus it seems that we have a contradiction in the form
*R is a member of R if and only if R is not a member of R.*
This is not just a simple mistake. Russell spent ten years and the writing of the "Principia Mathematica" with Alfred North Whitehead in an attempt to solve this problem, and find a way to base mathematics on Logic.

Your task: Think about this situation, and write your own commentary on it.
##################################################
##################################################

Math 215 - Assignment Number 7, Fall 2011
Hand these problems in on Monday November 7, 2011.

1. Re-read carefully Chapter 14 in the light of our class discussions. Then read the proof of the Cantor - Schroeder-Bernstein Theorem (Theorem 14.3.4). It is stated in the book, but you will find a proof on the web at
<http://planetmath.org/?op=getobj&from=objects&id=3156>
and also linked from our website. Take notes on this proof and then rewrite the proof in your own words. Write it carefully so that it is complete for you and readable.

2. page 185. problem 19.

3. Two sets X and Y are said to have the *same cardinality* if there exists a *bijection* F:X ---> Y. Recall that a map from one set to another is a bijection if it is an injection and it is onto Y. We write $|X| = |Y|$ when X and Y have the same cardinality. This definition applies to both infinite and finite sets, but infinite sets have

different properties in regard to their cardinality. For finite cardinality we take specific finite sets as models. Thus we take
$0 = \{ \}$
$1 = \{0\}$
$2 = \{0,1\}$
$3 = \{0,1,2\}$
and inductively
$n+1 = \{0,1,2,...,n\}$.
More precisely, we have the inductive definition

$$n+1 = n \cup \{n\}$$

with

$$0 = \{ \}.$$

This inductive definition of numbers as specific sets lets us say that any finite set has cardinality n for some natural number in the set $\{0,1,2,3,4,...\} = N \cup \{0\}$ where N is the natural numbers.
For example {a,b,c} can be put in 1-1 correspondence with $3 = \{0,1,2\}$, and so we say that {a,b,c} has cardinality 3.
Using this set-theoretic inductive definition of natural numbers (and zero), prove the following facts about these sets:
*(i) n + 1 is not equal to n ( as sets).*
*(i) If n +1 = m + 1 as sets, then n = m as sets.*

**4.** Professor Squarepunkt has been lecturing his class about Cantor's diagonal argument. His lecture is transcribed below. Read the lecture and comment on the problem that it raises.
##################################################

**Cantor's Demise**

by Professor Hilbert Squarepunkt

We have seen how Cantor's diagonal argument can be used to produce new elements that are not on a listing of elements of a certain type. For example there is no complete list of all Left-Right sequences of the form A1A2A3... where An = L or R and two such sequences A and B are said to be equal when An = Bn for all n=1,2,.... We proved this by assuming that we had a list of such sequences A(1),A(2),... such that A(n)m denotes the m-th element of the sequence A(n). Then we constructed the *diagonal sequence* D defined by Dn = A(n)n. And we made the *flipped diagonal sequence* Flip(D) from this by defining Flip(D)n = L when Dn = R and Flip(D)n = R when Dn = L. Cantor argues that Flip(D) is necessarily a new sequence not equal to any Dn that is on our list. The proof is clear, since Flip(D) is constructed to differ from each sequence in the list in at least the n-th place for D(n).

Now Cantor's real intent was to prove that the real numbers are uncountable (not listable) and I have discovered a fatal flaw in his argument! Let me explain. I will use the binary notation for real numbers between 0 and 1. Thus

.0= 0

.1 = 1/2

.01 = 1/4

.001 = 1/8

.0001 = 1/16 and generally

.0000...01 = $1/2^n$ where there are n-1 zeros before the 1.

Then real numbers between zero and one are represented by binary analogues of decimals like .101001000100001...

Note that .0111111... = .1000....

since 1/4 + 1/8 + 1/16 + ... = 1/2.

We apply the Cantor argument to lists of binary numbers in the same way as for L and R. In fact L and R are analogous to 0 and 1. For example if we had the list

(1) .**1**0111...

(2) .1**0**101010...

(3) 0.11**1**0110011...

...

Then we would make the diagonal sequence

**D = .101 ...**

and flip it to form

**Flip(D) = 0.010...**

Just as we argued before Flip(D) is not on the list, and so the list is incomplete.

**Here is a counterexample to Cantor's argument!**

Consider the following list:

(1) .**1**0000000....

(2) .0**0**100...

(3) .00**0**1000...

(4) .000**0**1000...

(5) .0000**0**1000...

...

As you can see this is a very definite list. The first element of the list is .10000... and subsequent members of the list consist in n-zeros, a 1 and then zeros forever. Ok? Now the diagonal element is

D = .1000... and

Flip(D) = .01111...

But .0111... = .10000
and so Flip(D) = D = .1000... and so
Flip(D) = the first element of our list!
**Flip(D) is not a new number outside the list.**
**This is a counterexample to Cantor!**

Gentleman, Ladies: Cantor lies in ruin before us. His theory is a shambles. No longer can we speak of higher infinities. Mathematics must be revised from the ground up and built anew. I exhort you to join me in this grand project of reconstruction of the truth.
#####################################################

Please comment on Professor Squarepunkt's lecture. Do you agree with him? What has he actually shown? Can you find a proof that the real numbers are uncountable in base-2, or is Squarepunkt right and we will have to start all over again with the foundations. (The idea behind Squarepunkt's lecture is due to a real mathematician - Nathaniel Hellerstein.)

**5. Read the definition of the game of Brussels Sprouts in** <http://homepages.math.uic.edu/~kauffman/Conway.pdf>. Use the Euler formula to find a formula for the number of moves in a game of Brussels sprouts in terms of the number of initial nodes.

**6. Read the illustrated story about the Infinite Hotel at**
http://homepages.math.uic.edu/~kauffman/InfiniteHotel.pdf
(There is a direct link to it on our website.)
In that story you will read about a hotel with infinitely many rooms where the manager advertises that there is "always room for more guests". The guests get tired of being pushed around (from room to room) by the manager and decide to give him some trouble.
In the story you will find a document that reads as follows.

## The professor's plan

1. During the next few weeks there will be a series of meetings.
2. First there will be an "empty meeting" with no one present.
3. Next, every guest will go to a meeting at which only he, or she, is present.
4. Then the guests will meet in pairs: these meetings must cover all possible pairs of guests.
5. Then there will be meetings of all possible groups of three guests.
6. Then meetings of four guests, then of fives, then of sixes, and so on, until-
7. There have been meetings of all possible groups of all sizes.
8. This series of meetings must be completed by 1st. december.
9. The purpose of each meeting is to invite a guest (who may, or may not be a present guest) to stay at this hotel next Christmas.
10. When a meeting has agreed on its guest, the manager must be asked to reserve a room for that person.
11. No two meetings may invite the same person (a list will be kept to avoid this happening)
12. The professor will decide whom the "empty meeting" (rule 2) will invite, and he will ask the manager to book a room for that guest.

Although the other guests didn't really understand it, they thought it must be a very good plan. The rules were sent to all the guests.

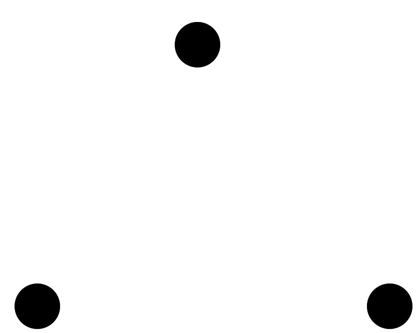
This document has the guests assembling in meetings. There are infinitely many guests and some of these meetings will have infinitely many members. That is what is meant by "all sizes" in number 7 above. Read this story very carefully and see if you can decide whether the manager made a correct decision at the end of the story.

Please write a short paragraph about your reasoning about this matter. (As far as I know, the Professor in this story never met Professor Squarepunkt. You can speculate whether these two Professors would get along with one another!)

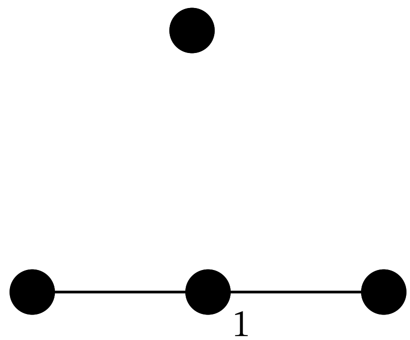
################################################
################################################
################################################
Discussion Section (Not Homework!)

This next part is for class discussion. It introduces you to some aspects of graph theory via a game played by drawing graphs on a sheet of paper. The game is called "sprouts" and it is an invention of John Horton Conway. The purpose of this problem is to give you some practice exploring a mathematical domain and seeing both results and proofs emerging naturally from questions and explorations.

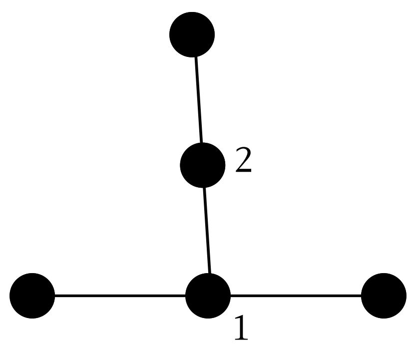
Sprouts is a game played by two players, using a sheet of paper and a pen or a pencil. The game begins with a choice by the players of a collection of "spots" or "nodes" on the paper. We illustrate here with three spots.



A move in Sprouts is accomplished by connecting two spots with an edge and placing a new spot in the middle of the edge. Below you see the result of the first player connecting two spots and placing the new spot.




We have labeled the new spot with the number 1 just so we can remember who did the move. Now the second player makes a move.

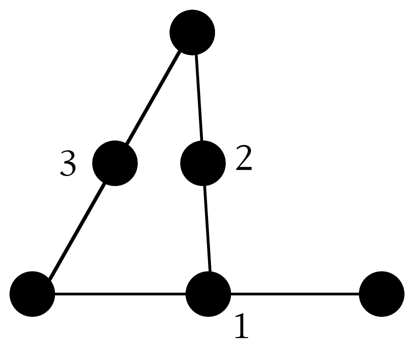


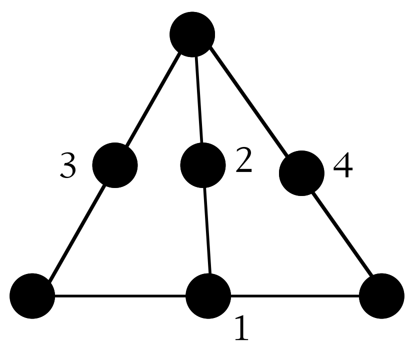

Notice that you can tell that the second new spot is due to the second player since it is labeled by 2. The real point, if we continue labeling in this way, is that second player will always label with even numbers and first player will always label with odd numbers. Lets call the players **First** and **Second.**

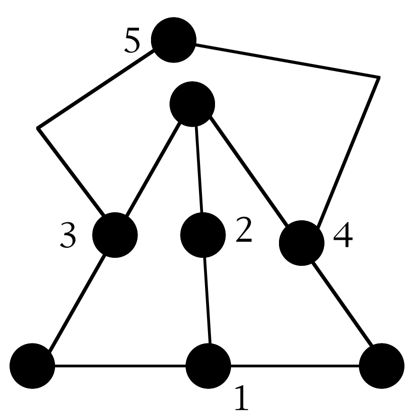
**An Important Rule:**
*When a spot has three edges attached to it (locally) then it is COMPLETE. If a spot is complete, you can no longer connect any further edges to that spot. Thus the spot labeled 1 in the above figure is now complete.*

**The goal in this game:** *The first person to make a move so that his opponent can not make a reply is the winner.*

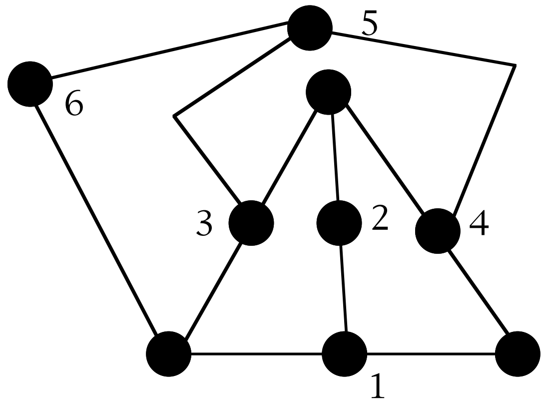
**Lets follow this game for a bit and see who wins.**

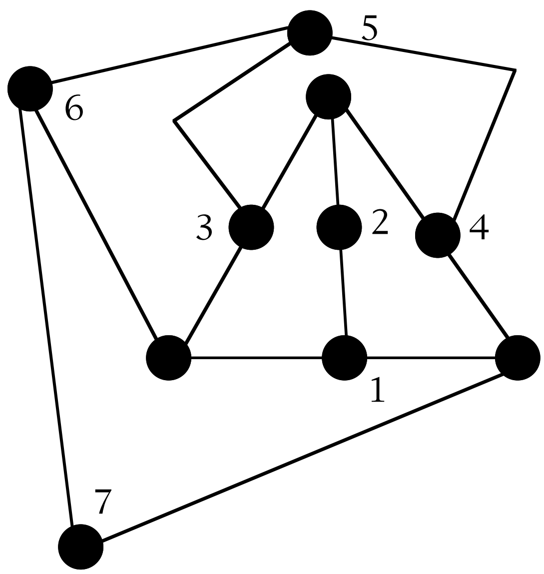



In this last position we have drawn an "edge with angles" between 3 and 4 and placed spot 5 on the middle of the angular edge. If you were drawing with a pen or a pencil you would just draw a curved edge connecting 3 and 4, but I have been using a simple drawing program that only makes straight segments. The restrictions on the edges is that the **new edges touch the original graph only at the end-spots. You are not allowed to have one edge cross through another one.**



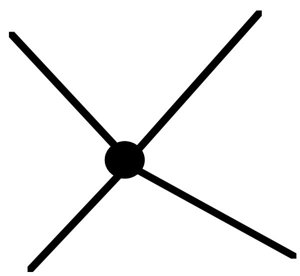

**The game is over.** Even though spot s 2 and 7 are incomplete, all the other spots are complete, and there are no moves left. The game is won by the player who put in spot 7. This was the First Player since 7 is odd. Notice that the last diagram contains a complete record of the game.

**Problem 4.1.** Find an opponent and play a number of games of 2-spot and 3-spot sprouts. Find out the best strategy that you can for 2-spot sprouts.

**Problem 4.2.** Prove that every game of sprouts starting with any finite number of spots (say with N spots) must eventually end, no matter how the players play. (Of course they must obey the rules.) If the game starts with N spots, give an upper bound on the number of moves in any game.

**Problem 4.3.** Prove that for every natural numbrer n (n = 1, 2, 3, 4, ...) there is a sprouts game, starting with n sprouts, that ends in exaactly 3n -1 moves. You can use mathematical induction in your proof.

**Problem 4.4.** In *4-sprouts* we use the same rules as in ordinary sprouts, but we allow 4 lines to touch a spot as in the diagram below.

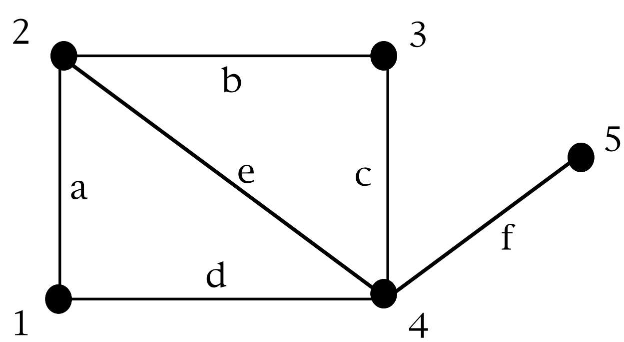


A move has the same form as in regular spots, but notice that the new spot, having two lines going into it, has two freedoms in the game of 4-spots. Show that some games of 4-spots can go on forever. Give specific examples. Think about the question of how to modify the rules of 4-spots so that it will become a game that always ends in a finite number of moves.

##########################################################

**Graph Theory and Euler's Formula.**

A *graph* is a collection of *nodes* or *vertices*, usually depicted as dark spots or points, and a collection of *edges* that can connect two nodes or connect a node with itself. For example, the graph below has five nodes and six edges. It is a c*onnected* graph in the sense that there is a pathway along the edges between any two nodes.



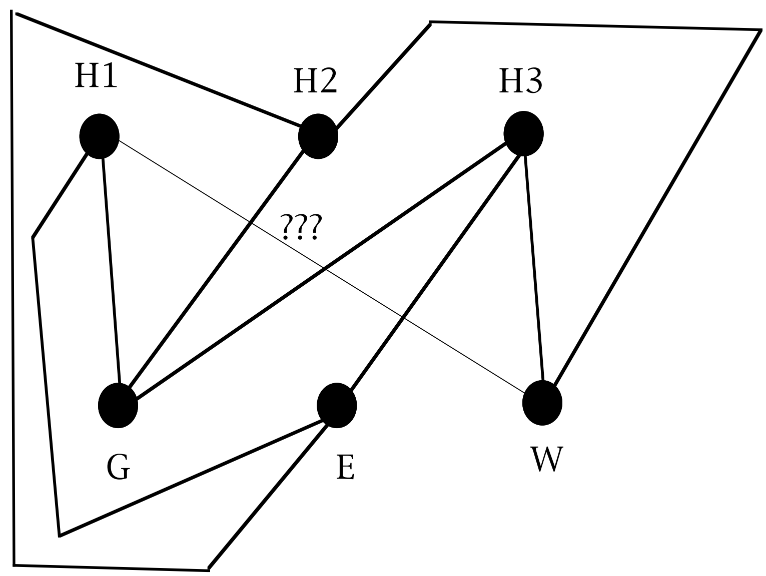
A connected graph G.

Graphs are fundamental mathematical structures and they have lots of applications. We are all familiar with the graphical notation for electrical circuits. Subway system maps are graphs with special decorations. In general, when we want to desribe engineering systems, economic systems, and other systems of relationship, we can start with a collection of definite entities (the nodes) and the information about how they are connected with one another (the edges).

As you can see, the game of sprouts is a game that is played by constructing a graph. The graph constructed in sprouts is special in that it cannot have more than three edges touching any node, and the sprouts graph is drawn in the plane in such a way that no two edges of it touch except at the nodes of th graph. We say that a graph that can be drawn in the plane in this way is a **plane graph**. The graph in the figure above is also a plane graph.

Not every graph is planar! That is if you specify a set of nodes and a set of connections to be made among these nodes, it may not be possible to accomplish these connections in the plane without having some edges cross over one another. Here is a problem that will show you how that can happen.

**Problem 5. (The Gas-Electricity-Water Problem)**
Three companies, the gas company, the electricity compay and the water company want to make connections from the gas main (G), the electrical source (E) and the water main (W) to three houses (H1, H2 and H3). They wish to lay their lines so that no two lines meet except at the sources (G,E and W) and at the houses (H1, H2 and H3). Can you find a solution to this design problem? If not, then why not?

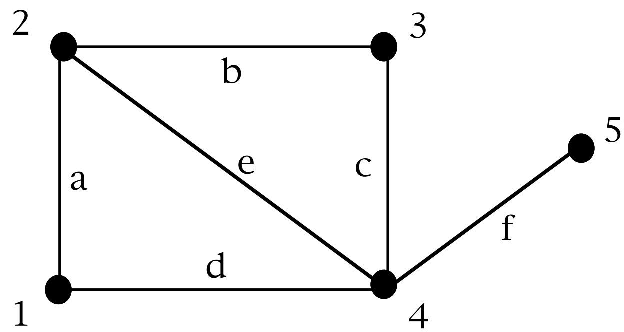



In the  illustration above the city planners have drawn a graph to help them design the connections but they have run into a difficulty with making a water line from W to H1. Everything went fine with the design up to that point, but then there does not seem to be any way to conncet from W to H1 without crossing previously created lines.  It will cost the city a great deal to dig tunnels to make lines cross over one another. So these designers really need to know whether the job can be done with no crossovers, and if it cannot be done that way, then they want to know the least number of crossovers that are needed to do the job.
##############################################

We will now discuss a formula about plane graphs that was discoverd around 1750 by the Swiss mathematician Leonhard Euler.
<http://en.wikipedia.org/wiki/Leonhard_Euler>
Euler was one of the greatest mathematicians of all time, and his formula about plane graphs is the beginning of the subjects of graph theory and topology (topology is the study of mathematical spaces and includes and generalizes classical geometry). Here is Euler's result:

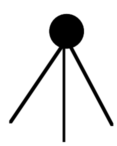
**Theorem.** Let G be a connected finite plane graph with V nodes, E edges and F faces (a face is a region in the plane that is delineated by the graph in the plane). Then $V - E + F = 2$.

Here is an example of Euler's formula for a specific graph in the plane.




Here we have V = 5, E = 6 and F = 3. The regions we count are the interiors of the two triangles and the outer region consisting in the rest of the plane. Note that V - E + F = 5 - 6 + 3 = 2 as promised by Euler's Theorem.

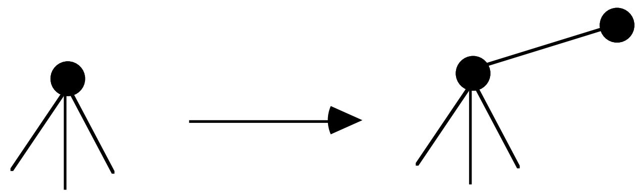
**Problem 6.** Construct a proof of Euler's formula by induction on the total number of edges and vertices in the graph **G**. You should consider how the graph can be built up from simpler graphs by adding edges to them. In fact, any connected graph can be built from a single vertex graph by adding new edges in two ways that I will now explain, but first we introduce an abbreviation: The diagram below stands for s*ome vertex in a larger graph*.



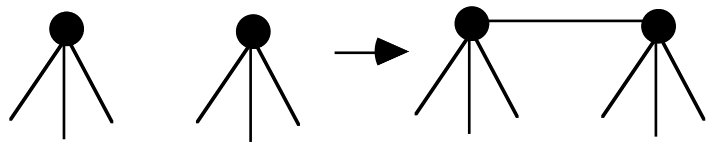
You can tell when I am using this abbreviation because the edges that go out of this vertex are not meeting any other vertices in the picuture. The picture is a shorthand for a possibly larger and more complete picture.  In the abbreviation we show three edges touching the vertex. In a real situation some edges touch the vertex, but the number is not necessarily equal to three. Ok?

Now lets use this and illustrate two ways to make a larger graph.

I.

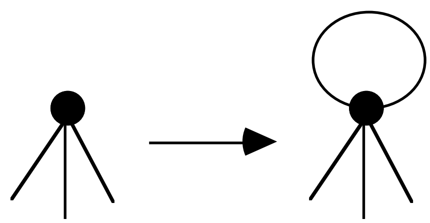


II.



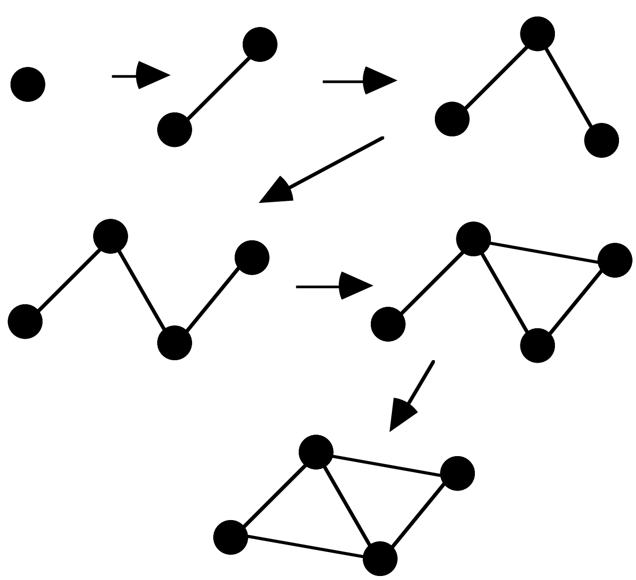
In method number I we add a new edge and a new vertex by attaching the new edge to an already existing vertex. In method number II we connect two vertices with a new edge.

**Remark.** We regard the move



as a special case of II.

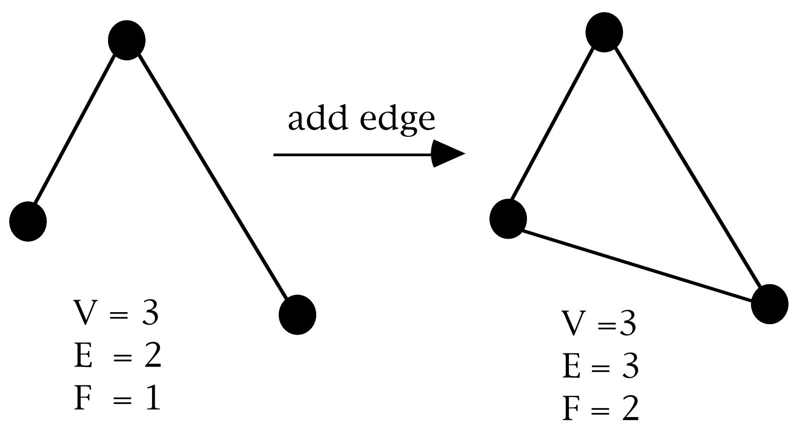
I claim that *any connected graph can be built up by performing a sequence of operations of these two types*. Here is an example.



You can use this claim in your proof, and if you want, you can also make a proof of the claim. We will discuss why the claim is true in class.

Now, to prove the Euler Theorem, you can proceed by induction, showing that V - E + F does not change its value when you perform a move of type I or type II. You will find that it is very easy to see this for type I, and that in order to see it for type II you need to start with a connected graph. If the graph is connected, then a move of type II will create a new region in the graph. Look at the example above and see how this works. You can use this fact also in your proof (that a move of type II will create a new region). You should then be able to construct an inductive proof of the Euler formula.

**Here is an example:**
**We create a triangle graph by adding an edge to a tree.**



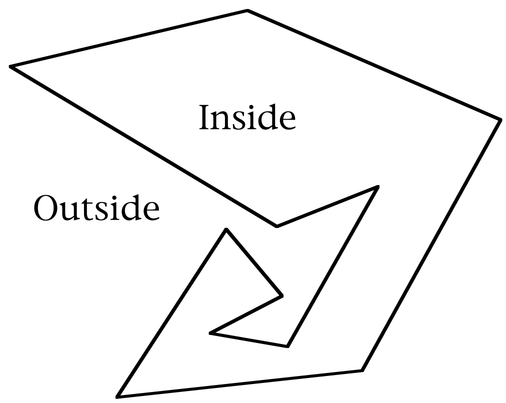

Note that adding the edge creates a new region, and V-E + F does not change from before to after the addition of the new edge.

(c) Discuss your proof of the Euler formula with another student in the class. Do you both feel that the proof is complete? What might be missing? In this problem, it worth having the discussion.

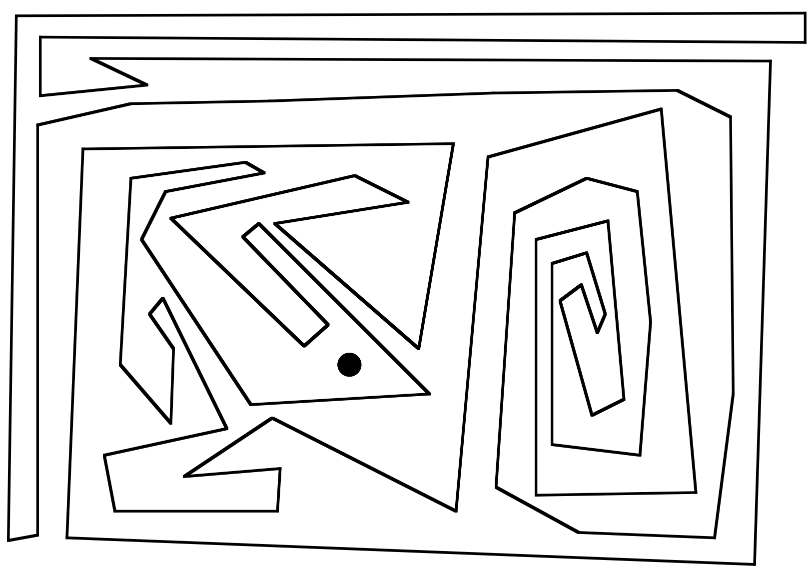
**Supplement.**
There is a fact about curves in the plane that you can use in thinking about regions that are created when graphs are drawn in the plane.
This fact is called the

**Jordan Curve Theorem:** *A closed curve in the plane without any self-intersections divides the plane into exactly two regions.*

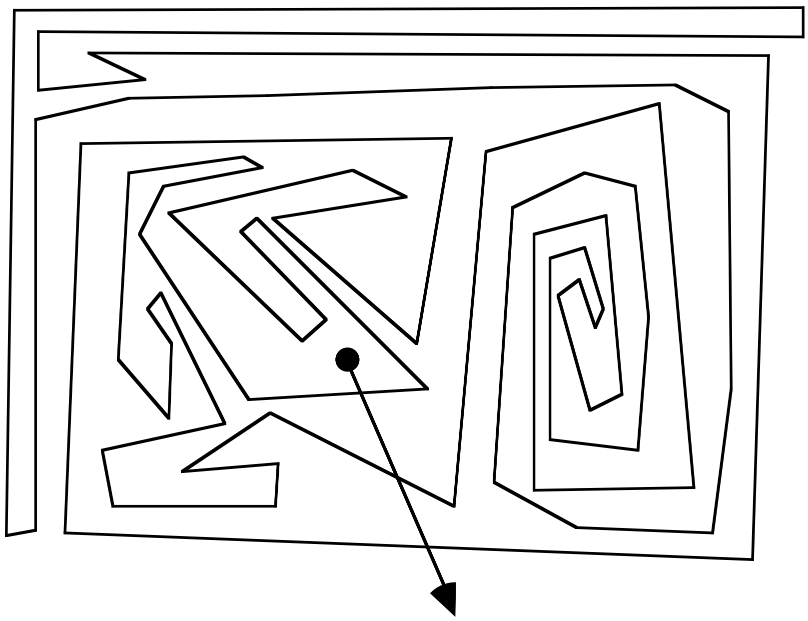
Here is an example:




You are not required to prove this result, but you can use it and it is interesting to see how complex examples can look!



Is the black dot inside or outside this curve?
Of course you can solve this like solving a maze, but look!



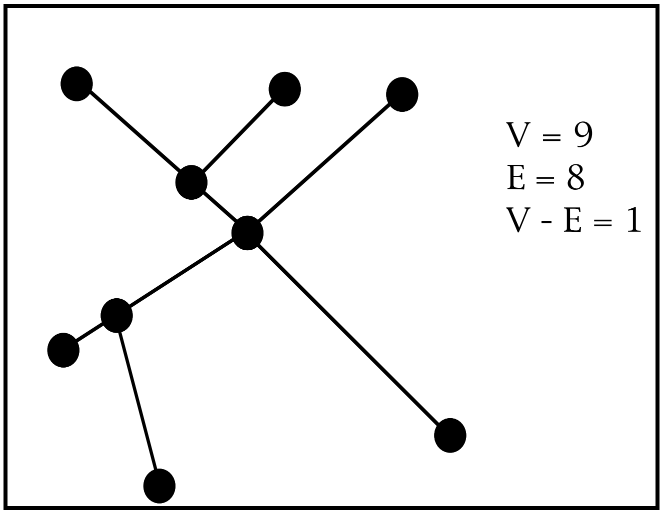
An arrow from the dot intesects the curve in an ODD number of points. I claim that this tells you that the point must be INSIDE. If the intersection number were EVEN, then the point would be outside. Can you explain why this works? (I say explain, and of course I am hoping that your explanation will turn into a mathematical proof. But lets explore.)

We will discuss in class why and how the Jordan Curve Theorem is relevant to proving Euler's Formula.

**Remark.** Another approach to the Euler formula uses the concept of a tree: A graph is said to be a *tree* if it does *not* contain any cycles (a cycle is a sequence of distinct edges such that the each edge shares its endpoints with the edges before and after it in the sequence. For example in the graph above, bce is a cycle and abcd is a cycle. When a plane graph has no cycles then the only region it can delineate is the rest of the plane other than itself, and so a tree has $F = 1$.

**Show that for a connected tree, $V - E = 1$.**

From this is follows that for connected plane trees $V - E + F = 2$, and so we know the Euler formula already for trees.




**The picture above illustrates this result for trees. You can prove that V - E = 1 for a connected tree by induction on the number of edges in the tree.**

You can then prove the Euler formula for an arbitrary connected plane graph by just making that graph by adding edges by our type II move to a tree. Think about this and try some examples.