Assignments. Math 215. Fall 2009.

1. Assignment Number One.
Due Wednesday, September 2, 2009.
Eccles. Read Chapters 1,2,3.
(a) page 9. 1.4, 1.5
(b) page 19. 2.1, 2.3, 2.4, 2.5, 2.6
(c) Using definitions 2.2.2 and 2.2.3 on page 16, prove that a positive integer n is odd if and only if $n = 2q + 1$ for some positive integer q, or $q = 0$.
(c) Draw a logical conclusion from the premises
"All Dragons are uncanny."
"All Scotchmen are canny."
(d) Explain what is wrong with the following argument.
"All soldiers are brave."
"Some Englishmen are brave."
Therefore "Some Englishmen are soldiers."

2. Assignment Number Two.
Due Wednesday, September 9, 2009.
Eccles. Read Chapters 4, 5.
(a) page 29. 3.1, 3.2, 3.3, 3.5, 3.6, 3.7
(b) page 37. 4.1, 4.2, 4.3, 4.5
(c) If P and Q are two logical expressions, let P = Q mean that *P and Q have the same truth table.*
For example, we have that a^b = ~((~a)v(~b)) and (a ----->b) = (~a)vb.
If an expression P is always true, we will write P = T and if it is always false, we will write P = F. For example.
(a)v(~a) = T while (a)^(~a) = F.
(i) Show that (a <-----> b) = ((~a)^(~b))v(a^b).
(ii) Show that (P <-----> Q) is true exactly when P = Q.

3. Assignment Number Three.
Due Wednesday, September 16, 2009.
Read Chapter 6.
(a) page 51. 5.1, 5.2, 5.4
(b) page 53. 5, 6, 14, 16, 20.

4. Assignment Number Four
Due Wednesday, September 23, 2009.
(a) Prove by induction that

$1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$.

(b) Page 56, problem 19.

(c) Page 57, problem 26: Choose a partner and work on this problem together. Write an account of how far you get with the problem.

(d) Page 72. Problems 6.1 and 6.4.

5. Assignment Number Five.
Due Wednesday, September, 30, 2009.
Read Chapter 7.
(a) Prove by induction that
$1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$.
(b) Page 87. problems 7.5 and 7.6.
(c) Use Venn Diagrams to show that
$(A^c) \vee B = (A^c \wedge B^c) \vee (A \wedge B) \vee (A^c \wedge B)$.
Here v is union, A^c is the complement of the set A, and ^ is intersection of sets.
(d) Let S(n) = {1,2,3,...,n} (n is a natural number).
Prove, by induction on n, that the power set of S(n) has 2^n elements.

6. Assignment Number Six
Due Wednesday, October 7,2009
Read Chapter 8.
p. 86-87, problems 7.2, 7.3, 7.7.
p. 117, problem 12.
Review for Exam on Friday, October 9,2009.
The exam will cover Chapters 1-7.

7. Assignment Number Seven
Due Wednesday, October 14, 2009.
Read Chapter 9.
p. 99, problems 8.1,8.2,8.3.
p. 115 – 116. Problems 3,4,5,6,7.

8. Assignment Number Eight
Due Wednesday, October 21, 2009.
(a) page 113. problems 9.1,9.2,.93.9.4,9.5.
(b) page 116. problems 8., 9, 11, 13.

9. **Assignment Number Nine**
Read Chapter 10. Read also Chapter 14.
Due Wednesday, October 28, 2009.

(a) Re-do problem 6 on page 116, using the following Boolean algebra notation:
Write AB for A ^ B or equivalently for A (Intersect) B.
Write A + B for A v B or A (Union) B. This notation makes the distributive law easy to spot:
(A + B)C = AC + BC
and
(AB) + C = (A +C)(B+C).
(Remember that intersection distributes over union AND union distributes over intersection.) Note how this works
(A + C)(B +C) =
AB + AC + CB + CC =
AB + (A + B)C + C =
AB + C
since (A+B)C is contained in C and CC = C since the intersection of C with itself is just C. This additive and multiplicative notation for Boolean algebra is due to George Boole who invented the idea of using algebraic notation to work with logic in his book "An Investigation of The Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities" published by Macmillan in 1854, and still available as a Dover paperback to this day.

(b) page 118. problems 17, 18, 19.

(c) Two sets X and Y are said to have the *same cardinality* if there exists a *bijection* F:X ---> Y. Recall that a map from one set to another is a bijection if it is an injection and it is onto Y. We write |X| = |Y| when X and Y have the same cardinality. This definition applies to both infinite and finite sets, but infinite sets have different properties in regard to their cardinality. For finite cardinality we take specific finite sets as models. Thus we take
0 = { }
1 = {0}
2 = {0,1}
3 = {0,1,2}

and inductively
n+1 = {0,1,2,...,n}.
More precisely, we have the inductive definition

$$n+1 = n \cup \{n\}$$

with

$$0 = \{\ \}.$$

This inductive definition of numbers as specific sets lets us say that any finite set has cardinality n for some natural number in the set {0,1,2,3,4,...} = N U {0} where N is the natural numbers.
For example {a,b,c} can be put in 1-1 correspondence with 3 = {0,1,2}, and so we say that {a,b,c} has cardinality 3.
Using this set-theoretic inductive definition of natural numbers (and zero), prove the following facts about these sets:
(i) n + 1 is not equal to n (as sets).
(i) If n +1 = m + 1 as sets, then n = m as sets.

(d) Recall that the power set P(X) of a set X is the collection of all the subsets of X. We proved in an earlier problem that the cardinality of P(X) is 2 raised to the number of members of X for X a finite set. That is, we have for X finite that

$$|P(X)| = 2^{|X|}.$$

This means that P(X) has more members than X, and thus no mapping F: X -----> P(X) can be surjective. This means that given such a mapping, we should be able to locate a subset W of X so that W is not equal to F(x) for any x. Georg Cantor gave a beautiful proof of this fact by constructing W as follows:

$$W = \{ y \text{ in } X \mid y \text{ is not a member of } F(y)\}.$$

This problem asks you to prove that Cantor's set W does indeed fulfill this promise.
Prove that for any set X and a mapping F:X ---> P(X), W (defined above) is not equal to F(x) for any x in X.

Here is an example.
Let F: {a,b,c} -----> P({a,b,c}) be given by
F(a) = {a,b}
F(b) = {a,c}
F(c) = {a}.
Then W = {b,c} since a is member of F(a)={a,b}, but b is not a member of F(b) and c is not a member of F(c). You should try some examples of this Cantorian construction yourself.

The amazing thing about Cantor's construction of the set W is that it works for infinite sets as well as finite sets. We shall have more to say about this after you have worked on the present problem.

10. Assignment Number 10.
Due Friday, November 6, 2009.
Read Chapters 10,11,12 and the "Supplement on Finite and Infinite Sets" that is placed on our website.
(a) page 132. problems 10.1, 10.3, 10.4.
(b) Prove Theorem 6 in the "Supplement on Finite and Infinite Sets" available on our website.
(c) Consider the following version of Cantor's diagonal argument:
Let S be the set of all inifinite sequences of letters where the individual letters are either L or R. Thus the following sequences are examples of elements of S:
LLLLL...
LRLRLRLR...
LRLLRLLLRLLLLRLLLLLR...
and many more.
An element of S has the form
$A_1 A_2 A_3 A_4...$
where each A_n equals either L or R.
$S = \{A_1 A_2 A_3 A_4... \mid A_n$ equals either L or R$\}$.
We prove that the cardinality of S is greater than the cardinality of $N = \{1,2,3,4,...\}$ as follows.
Suppose that F:N -----> S is any mapping. We will construct a sequence in S that is not of the form F(n) for any n in N.
This sequence C s defined as follows:
Thus C_n denotes the n-th element in the sequence C.
Define

$$C_n = L \text{ when } F(n)_n = R$$

and

$$C_n = R \text{ when } F(n)_n = L.$$

You can see by construction that C cannot be equal to F(n) for any n. And so the mapping F is not surjective.
Write out the rest of the argument, proving that the cardinality of S is greater than the cardinality of N.

11. Assignment Number 11
Due Wednesday, November 11, 2009.
Page 143. Problem 11.2
Page 181. Problems 14.1, 14.2, 14.3.

12. Assignment Number 12.
Due Wednesday, November 18, 2009.
Read the supplements on the website by John Conway and Raymond Smullyan. Read Chapter 12.
(a) Page 143. Problem 11.3
(b) Page 155. Problems 12.1, 12.2, 12.5
(c) A *partition* of a natural number n is a set of natural numbers whose sum is n. For example {1,1,3} is a partition of 5. Let p(n) denote the number of partitions of n for n a natural number.
Let $p_o(n)$ denote the number of partitions of n such that all the parts of the partition are odd numbers. Let $p_d(n)$ denote the number of partitions of n into distinct parts. For example, {1,1,3} is an odd partition of 5 and {2, 3} is a partition of 5 into distinct parts. Note that {1,1,3} is *not* a partition of 5 into distinct parts.
(i) Make a table of all partitions of n = 1, 2, 3, 4, 5, and 6.
Verify for these n that $p_o(n) = p_d(n)$.
(ii) Find a proof of the Theorem $p_o(n) = p_d(n)$ for any natural number n by finding a bijection between the set of odd partitions of n and the set of distinct part partitions of n. (Hint: When an odd partition has repeated parts, collect some of these parts by adding them together.)

13. Assignment Number 13.
Due Wednesday, November 25, 2009.
Read Chapter 13.
(a) p. 182. problem 4. (Hint: Generalize the formula that counts the number of elements in the union of two sets.)
(b) p. 185. problem s 17 and 19.
(c) Professor Squarepunkt has been lecturing his class about Cantor's diagonal argument. His lecture is transcribed below. Read the lecture and comment on the problem that it raises.
##

Cantor's Demise

by Professor Hilbert Squarepunkt

We have seen how Cantor's diagonal argument can be used to produce new elements that are not on a listing of elements of a certain type. For example there is no complete list of all Left-Right sequences of the form A1A2A3... where An = L or R and two such sequences A and B are said to be equal when An = Bn for all n=1,2,.... We proved this by assuming that we had a list of such sequences A(1),A(2),... such that A(n)m denotes the m-th element of the sequence A(n). Then we constructed the *diagonal sequence* D defined by Dn = A(n)n. And we made the *flipped diagonal sequence* Flip(D) from this by defining Flip(D)n = L when Dn = R and Flip(D)n = R when Dn = L. Cantor argues that Flip(D) is necessarily a new sequence not equal to any Dn that is on our list. The proof is clear, since Flip(D) is constructed to differ from each sequence in the list in at least the n-th place for D(n).

Now Cantor's real intent was to prove that the real numbers are uncountable (not listable) and I have discovered a fatal flaw in his argument! Let me explain. I will use the binary notation for real numbers between 0 and 1. Thus
.0= 0
.1 = 1/2
.01 = 1/4
.001 = 1/8
.0001 = 1/16 and generally
.0000...01 = $1/2^n$ where there are n-1 zeros before the 1.
Then real numbers between zero and one are represented by binary analogues of decimals like .101001000100001...
Note that .0111111... = .1000....
since 1/4 + 1/8 + 1/16 + ... = 1/2.

We apply the Cantor argument to lists of binary numbers in the same way as for L and R. In fact L and R are analogous to 0 and 1. For example if we had the list
(1) .**1**0111...
(2) .1**0**101010...
(3) 0.11**1**0110011...
...
Then we would make the diagonal sequence
D = .101 ...
and flip it to form
Flip(D) = 0.010...
Just as we argued before Flip(D) is not on the list, and so the list is incomplete.

Here is a counterexample to Cantor's argument!
Consider the following list:
(1) .10000000....
(2) .00100...
(3) .0001000...
(4) .00001000...
(5) .000001000...
...
As you can see this is a very definite list. The first element of the list is .10000... and subsequent members of the list consist in n-zeros, a 1 and then zeros forever. Ok? Now the diagonal element is
D = .1000... and
Flip(D) = .01111...
But .0111... = .10000
and so Flip(D) = D = .1000... and so
Flip(D) = the first element of our list!
Flip(D) is not a new number outside the list.
This is a counterexample to Cantor!

Gentleman, Ladies: Cantor lies in ruin before us. His theory is a shambles. No longer can we speak of higher infinities. Mathematics must be revised from the ground up and built anew. I exhort you to join me in this grand project of reconstruction of the truth.
##

Please comment on Professor Squarepunkt's lecture. Do you agree with him? What has he actually shown? Can you find a proof that the real numbers are uncountable in base-2, or is Squarepunkt right and we will have to start all over again with the foundations. (The idea behind Squarepunkt's lecture is due to a real mathematician - Nathaniel Hellerstein.)

14. Assignment Number 14.
Due Friday, December 4, 2009.
Read Chapters 15, 16, 17, 23.
page 198. Problems 15.5,15.6
page 206. Problem 16.1.
page 215. Problems 17.1, 17.2
page 287. Problems 23.1, 23.2, 23.3, 23.7.