Assignments. Math 215. Fall 2010.

1. Assignment Number One.
Due Monday, August 30, 2010.
Eccles. Read Chapters 1,2,3.
Read the article on Even/Odd by David Joyce
(available from our website).
Download the article on Peano Axioms from our website
(you do not have to do these problems yet).
Page 8. 1.1, 1.2,1.3, 1.4.
Page 19. 2.1, 2.3, 2.4, 2.5, 2.6.
Page 29. 3.1, 3.2, 3.3.

2. Assignment Number Two
Due Friday, September 10, 2010
Eccles, Read Chapters 4,5.

p. 37. 4.1,4.2
p. 51, 5.1, 5.4, 5.6
p. 53, 1, 2, 5, 10, 12, 16, 20.

Numbers by Notation

We shall define non-negative integers n by using a special notation:
0= [ ]
1 = [*]
2 = [* *]
3 = [* * *]
and in general we consider elements
n = [* * * ... * *]
where there are a finite number of *'s in between the brackets.
We say that n = [s] and m = [t] are equal, and we write
n = m if and only if there is a 1-1 correspondence between the set of stars for n and the set of stars for m. (* is a single star). We are allowed to move the patterns of stars inside the brackets in any pattern we please as long as the change in arrangement is a 1-1 correspondence. Define addition of numbers by
[s] + [t] = [s t]
where s t denotes the juxtapostion of the stars of n with the stars of m. For example,
[* *] + [* * *] = [* *   * * *] = [* * * * *],
2 + 3 = 5.
Define multiplication by

[ ] x [s] = [ ]
{ *] x [s] = [s]
[* *] x [s] = [s s]
[* * *] x [s] = [s s s]

In general

[t] x [s] = [s s s ... s]

where there are as many copies of s in [t] x [s] as there are stars in t. In other words, **each star in t is replaced by a copy of s.**

For example, 2 x 3 = [* *] x [* * *] = [* * *   * * *] = 6.

Prove the following statements about non-negative integers.

(a) 0 + n = n for all n.
(b) n + 1 is not equal to n for any n.
(c) n + m = m + n for all m and n.
(d) n x m = m x n for all m and n.
(e) (n + m) + p = n + (m + p) for all n, m ,p.
(f) (n x m) x p = n x (m x p) for all n, m, p.
(g) Define a natural number to be any n with a non-zero set of stars. Thus the set N of natural numbers consists in N = {1,2,3,4,...}. Call a natural number n even if n = 2 x m for some m. Prove that if n is not even then n = 2 x m + 1 for some m, or n = 1.
(h) Note that
1^3 = 1,
1^3 + 2^3 = 9 = (1 + 2)^2,
1^3 + 2^3 + 3^3 = 36 = (1 + 2 + 3)^2.
Make a conjecture on the basis of these patterns and prove your conjecture by using mathematical induction.

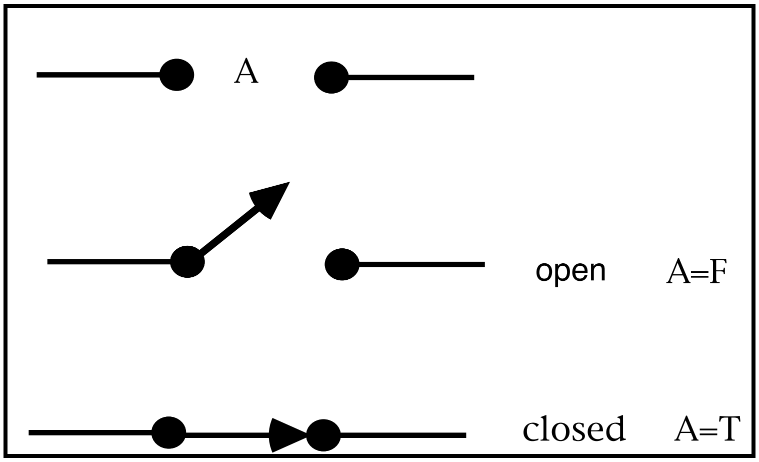
Problem on Switching Circuits

Design a switching circuit with four switches such that each switch independently controls a given light. In other words solve the problem of controlling an entrance light to a building from each of four separate floors of the building.

Notes on Switching Circuits

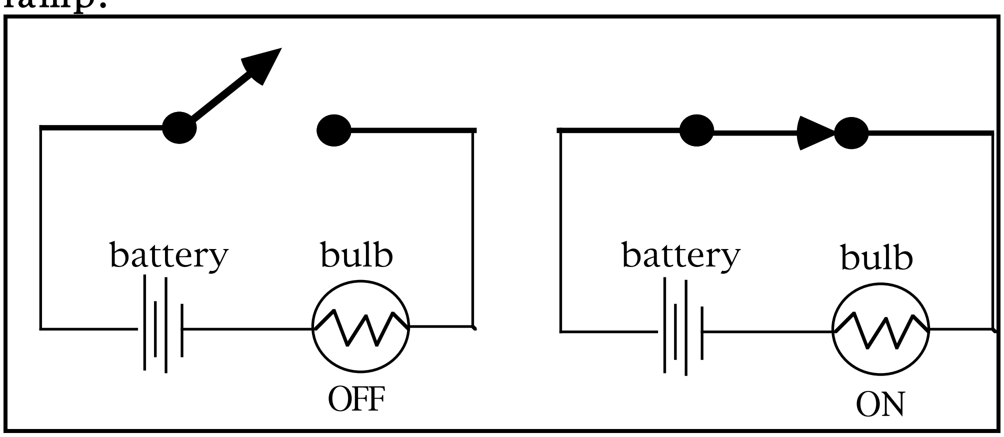
This application of logic to switching circuits is the discovery of Claude Shannon

The Journal The Journal of Symbolic Logic, Vol. 18, No. 4 (Dec., 1953), p. 347
and forms the basis for the design of computers to this day. Here is an abstract switch:



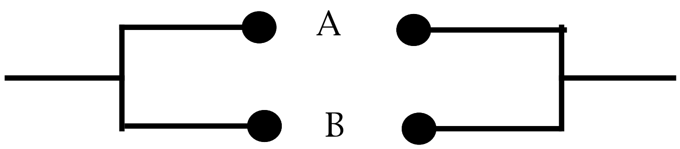
A signal can go from left to right through the switch when it is closed, and no signal can go through when the switch is open. We choose to designate a switch by a label such as A above and we let A = T correspond to the closed switch position (T for "transmit" if you like!) and we let A = F when the switch is in the open position (F = ~ T = not transmit).

The position of a switch can control a device such as a lamp.



Two basic ways to put switches together are Series and Parallel Connection:

Series Connection: A ^ B

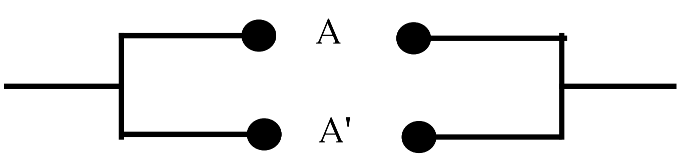


Parallel Connection: A v B

As you can see, the only way for a signal to get from left to right in the series connection of A and B is if both A and B are T (closed). Thus the series connection corresponds to A and B which we write in logic notation as A ^ B. Similarly, the only way for a signal to get from left to right in a parallel connection is if one of A or B is closed. Hence this corresponds to A or B, which we write as A v B. Thus our two basic logical operations are mirrored in the behaviour of networks that carry signals.

What about negation? An example will show you how we handle this. What we do is, we allow multiple appearances of a given label or its negation. We will write either ~A or A' for the negation of A. Here we will use A' ok? Then in the mutiple appearances, all A's will be either closed or open and if you have A' and A is closed, then A' will be open. If A is open then A' will be closed.
Look at this example:

A ^ A' = F



A v A' = T

Each of these circuits has an A and an A'. In the series connection, this means that one switch is always open and so the value of the whole circuit is the same as a simple open circuit. On the other hand in the parallel connection of A with A' either one line will transmit, or the other line will transmit. So the circuit as a whole behaves like a single switch that is closed. Thus we see that the series connecction corresponds to the identity A ^ A' = F, while the parallel connection corresponds to the identity
A v A' = T.

Terminology: Since a label A in one of our circuits can appear in many places we will still refer to A as a 'switch' even though it may be composed of a number of elementary switches. In such a multiple switch, if you make one, there has to be a mechanical way to make sure that all the different parts of the switch work together. Also these switching patterns may, in practice, happen in elecronic circuitry that syncronizes actions at separate locations. For our purposes, we shall imagine simple mechanical switching devices.

Sample Problem. Design a switching circuit with three switch labels a,b,c such that each of a, b and c individually control transmission of the signal. That is if the circuit is open, then changing the state of any one of a,b,c will close the circuit and if the circuit is closed, then changing the state of any one of a,b,c will open the circuit.
(We discussed how to do this with in analogous case of two labels and how it is related to (a ^ b) v (a' ^ b') in class.)

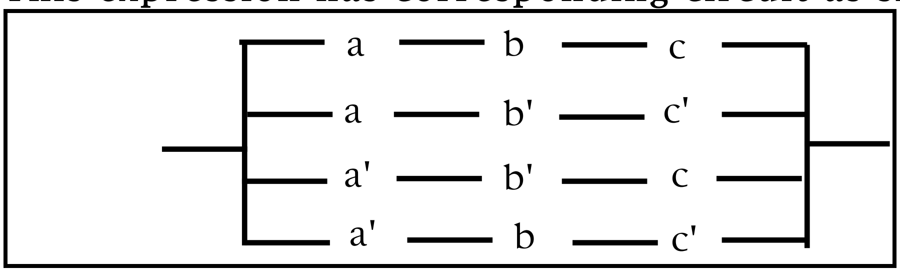
Solution to Sample Problem:
The logical condition for the three switches corresonds to the following settings

a	b	c
T	T	T
T	F	F
F	T	F
F	F	T

That is, we will have the line transmitting when all the switches are in the T-state, and whenever any two of them are flipped to the F state. This is expressed by the symbolic expression

E =(a^b^c)v(a^b'^c')v(a'^b'^c)v(a'^b^c').

This expression has corresponding circuit as shown below.

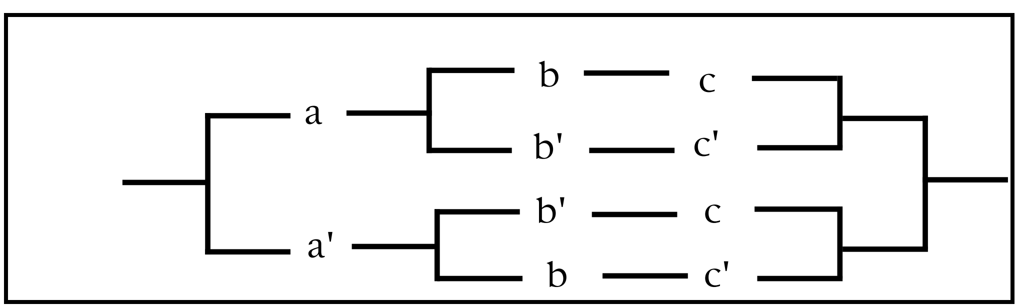


Now we face an interesting problem of simplification. The switches in this solution are a bit complicated, and it is clear that if we want to generalize to four or more switches that control one light, then this method will give us increasingly complex designs.

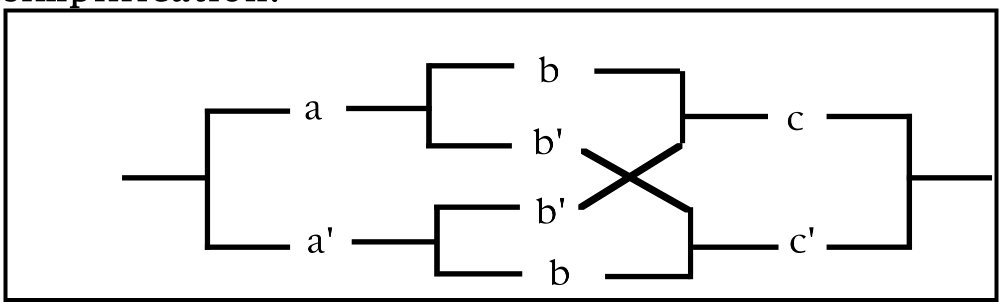
One way to search for simplification is to algebraically simplify the expression E. We can apply the distributive law (a^X)v(a^Y) = a^(XvY) and reduce the number appearances of a and of a' in E. Then

E = a^((b^c)v(b'vc')) v a'^((b'^c)v(b^c')).

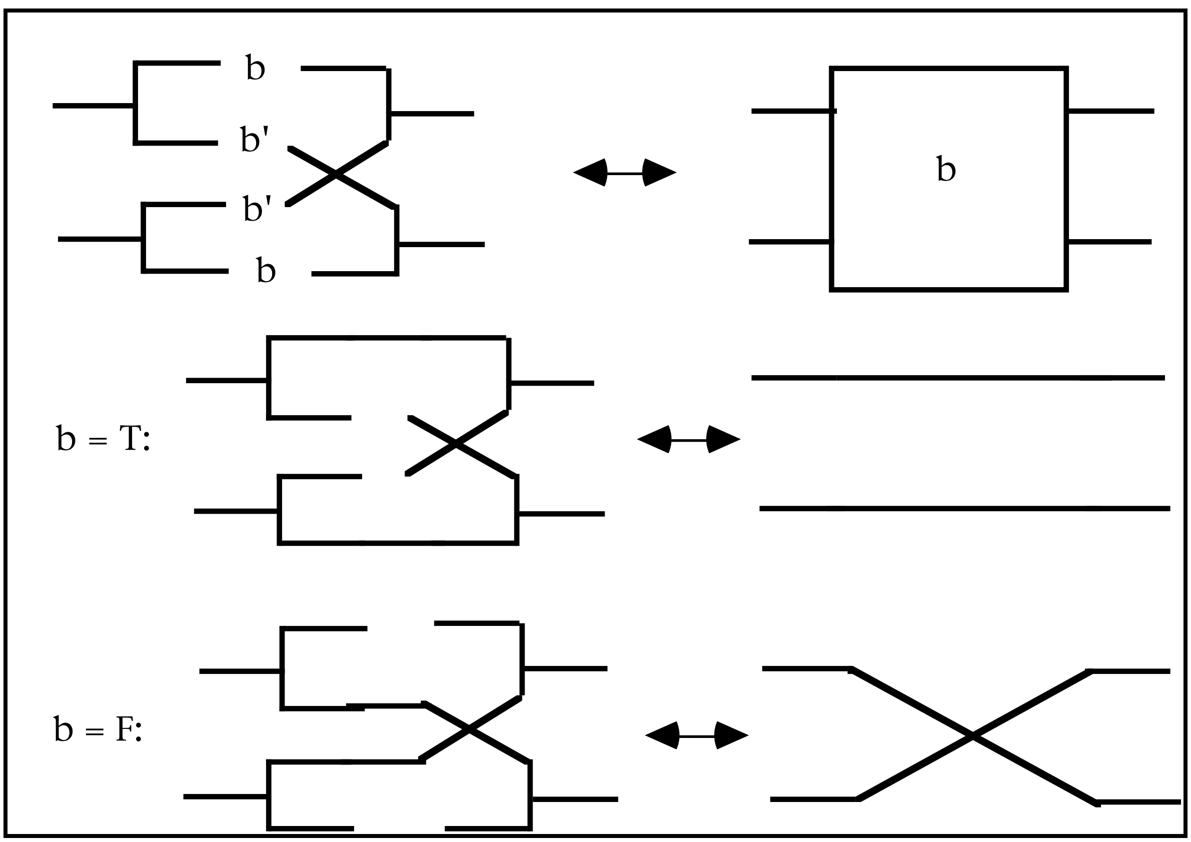
The circuit for this version of E is shown below.



Now comes the subtle part! What we have just done suggests a corresponding modfication on the right hand side of the circuit graph. We can clearly reduce to just one c and one c' on the right in same pattern by which we did it through the distributive law on the left. In order to do this, we will have to cross some wires in the circuit, and we shall understand that this does not mean that these crossed wires interact in any way as far as how they carry current. Here is a drawing of the result of the next simplification.



We have simplified the left and right switches as much as possible, and the middle switch has a really nice description if you look at it. When b = T it corressponds to two parallel lines and when b=F it corresponds to two crossed lines.



3. Assignment Number Three

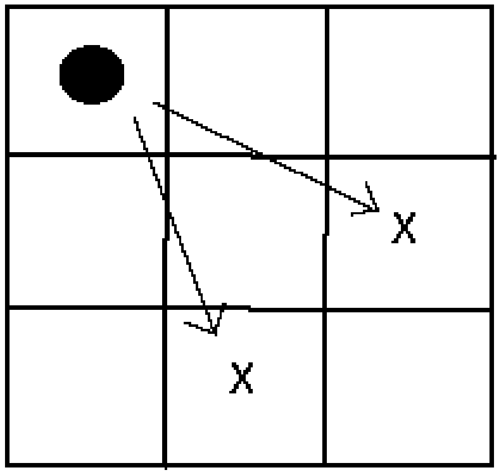
(A) Eccles page 53. Problems 3., 9., 10., 11., 13., 14., 25.

(B) Consider a placement of n straight lines in the plane, such that any two lines intersect AND not more than two lines intersect at any given point. Call such a placement of n lines an "n-arrangement" of the lines. It is easy to see that a 0-arrangement has no lines and divides the plane into a single region. A 1-arrangement has one line and divides the plane into two regions. A 2-arrangement has two lines and divides the plane into four regions. Let $F(n)$ denote the number of regions formed in the plane by an n-arrangement.
Show directly that $F(0)=1$, $F(1)=2$, $F(2)= 4$, $F(3) = 7$, $F(4) = 11$.
Prove by induction that $F(n+1) = F(n) + n +1$. Can you find a specific Formula for $F(n)$? Investigate the corresponding problem for planes arranged in three dimensional space.

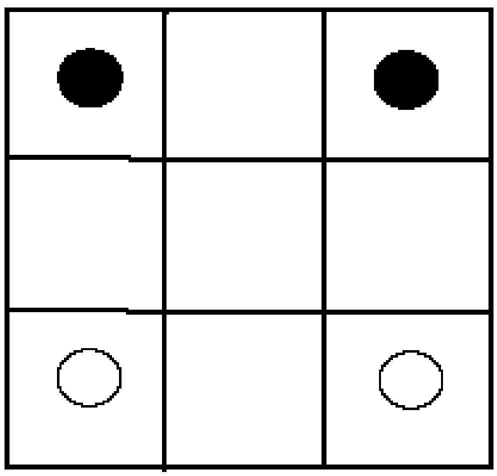
(C) Consider knights on a 3 x 3 chessboard as illustrated below.
The knights are represented by black and white circles.
The first part of the illustration shows how a knight moves on the board. It is taken to a square that is two squares in one direction and one square in a perpendicular direction removed from its starting square.

In playing a game, two players alternate moving their knights. Black moves the black knights, and White moves the white knights. No player is allowed to move his knight to a square that is already occupied by either himself or his opponent.

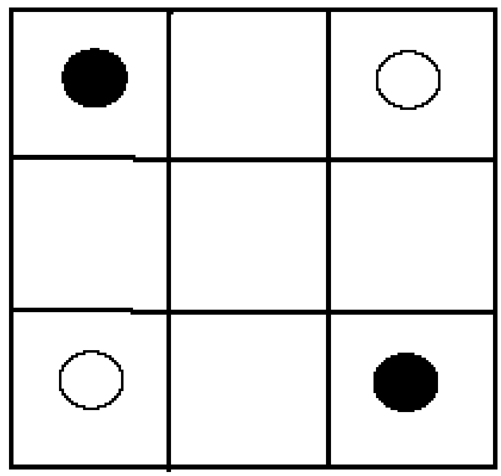
Two positions (Position 1 and Position 2) are shown in the diagram below. Is it possible for the players to start in Position 1 and end up in Position 2? If it is possible, please give a sequence of moves showing how this is done. If it is not possible, please give a proof that the transition from Position 1 to Position 2 cannot be accomplished.



Knight moves
on a three by three
chess board.



Position 1



Position 2

Assignment 4. Read Chapters 6,7,8,9 for the main ideas.

(a) p. 53. problem 19.

(b) P. 72. Problems 6.4, 6.5.

(c) Prove, by induction that if X[n] = {1,2,3,...,n}, |S| denotes the number of elements of a set S, and P(S) denotes the set of subsets of a set S, then $|P(X[n])| = 2^n$. That is, show that the number of subsets of X[n] is two raised to the nth power.

(d) P. 115. Problems 3 and 6.

(e) Consider the following discussion. Suppose that F:X --→ P(X) is a well-defined function from a set X (any set) to its power set P(X). Recall that P(X) is the set of subsets of X. Thus for each x in X, we have F(x) is a subset of X. This means that x might or might not be a member of F(x).
Let C(F) = {x in X| x is not a member of F(x)}. Prove that C(F) is not of the form F(z) for any z in X.

(f) Problem (e) is rather abstract, and so you might like to do this problem first. Before proving that C(F) is not of the form F(z) in general, take the following special case:
X = {1,2,3} and
F(1) = {2,3},
F(2) = {1,3},
F(3) = {1,2,3}.
Then what is C(F)? Do you see that C(F) is not of the form F(z) for any z in X? Fine. Now make up some more examples of your own, of maps from a set to its power set, and construct the corresponding set C(F). You can use either finite or infinite sets in your examples. Here are a couple of examples to work with:
(1.) X = N = {1,2,...}, F(n) = {p | p is a prime number that divides n} (note that a number p is prime if it is a natural number not equal to 1, such that the only divisors of p are 1 and itself.)
(2.) X = N, F(1) = { }, F(2)= {1}, F(3) = {2}, F(4) = {1,2}, F{5} = {3}, F{6} = {1,3}, F{7} = {1,2,3}, ... the intent here is the F should be defined so that F makes a list of all the **finite** subsets of N. Can you devise an inductive definition of F? Find C(F).