Project Math 215 - Due Wednesday, April 29,2009.

Instead of a third hour exam, I am asking you to write a paper (hand-written or typed, as you wish). The paper will have as its title "Logic, Sets and Mathematics" and you can use any materials or problems that you have encountered in the course in creating the paper. Write an exposition that starts with the notion of a set as a collection and the notion that two sets A and B are equal exactly when they have the same members. You can include any ideas and results related to sets, logic and mathematics that you like.
You are asked to include the following points in your paper:

1. Explain the meaning of membership and the definition of equality of sets.

2. Prove that there is a unique empty set.

3. Explain the meaning of the terms union, intersection, complement for sets.

4. Give a concise explanation of Venn diagrams and how they work and how they are used to show identities about sets.

5. Explain how sets and logic are related to one another when one defines sets via $\{x|P(x)\}$ where $P(x)$ is a proposition about x that is either true or false. Give an example of a logical identity and how it corresponds to an identity about sets. Use Venn diagrams to illustrate your example.

6. Suppose that a set X is made up from sets so that its members are also sets. Then it is possible to have A is a member of B and B is a member of C, but A is not a member of C. For example,
C = {B} and B = {A} , A = { } so that C = { { A } }. We say that a set X is well-founded if it does not have any infinite descending chains of membership. By an infinite descending chain of membership for X, we mean that there are sets $X_1, X_2, X_3, \ldots$ such that
X_1 is a member of X,
X_2 is a member of X_1,
X_3 is a member of X_2,
...
(going on forever).
In a finite chain of membership we would have, for some n,

X_1 is a member of X
X_2 is a member of X_1
...
X_{n-1} is a member of X_n
and X_n is empty.
For example C = { { { } } } has a chain of length three and is well-founded. But Z = { { { { { ... } } } } } has an infinite descending chain of membership and is not well-founded. In fact we have that Z = { Z } and so when you look for a member of Z you find Z! And this process does not stop. *Prove that if W is a set of sets and each member of W is well-founded, then W is well-founded.*

7. Give other examples of sets whose members are sets and consider the sequence
S_0 = { }
S_1 = { { } }
S_2 = { { }, { { } } }
S_3 = { { }, { { } } , { { }, { { } } } }
...
$S_{n+1} = S_n \cup \{ S_n \}$.
(Here A U B denotes the union of the sets A and B.)
Prove by induction that for all n= 0,1,...
(a) $S_{n+1} =\{S_0,S_1,...,S_n\}$,
(b) S_{n+1} is not equal to S_i for any i between 0 and n.
(Hint: Use the result of the previous problem.)

8. Explain the notions of injective, surjective and bijective mappings of sets in your own words, and give examples of mappings that are
(a) injective but not surjective
(b) surjective but not injective.
Recall that a bijective mapping of sets is called a 1-1 correspondence between them.

9. Write a proof in your own words of Cantor's Theorem that states there does not exist a bijection between a set X and its power set P(X). (P(X) is the set of subsets of X.)

10. Choose a topic either from the book or via your reading or from the internet (e.g. from the Wikipedia) related to set theory and write a very concise report on some aspect of this topic that you find of interest. The point is that all aspects of mathematics are related to

set theory, and you can explore this by looking at some specific mathematics such as calculus, algebra, geometry, graph theory, game theory and so on, and also by looking at logic and set theory themselves. Take the time to browse in some books and on the internet and see if you find connections with what you have already learned or even see if you find something that is entirely new to you.