Final Exam - Math 215 - Spring 2009 – With Selected Solutions

Do problems $1, 2, 3, 4, 5$ and choose *one* more problem from the remaining seven problems on the exam. Write all your proofs with care, using full sentences and correct reasoning.

1. Prove $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$ for all $n = 1, 2, 3, \cdots$.
Solution. This is a straightforward induction argument. Solution omitted.

2. Prove that the following two statements are equivalent:

$$A \Rightarrow (B \Rightarrow C)$$

and

$$(A \wedge B) \Rightarrow C.$$

In your proof, do *not* use truth tables. Use the facts that

$$A \Rightarrow B \ = (\sim A) \vee B$$

and

$$\sim (A \wedge B) = \ (\sim A) \vee (\sim B),$$

and give a completely algebraic proof.

Solution.

$$A \Rightarrow (B \Rightarrow C) = (\sim A) \vee ((\sim B) \vee C)$$
$$= ((\sim A) \vee (\sim B)) \vee C) = (\sim (A \wedge B)) \vee C = (A \wedge B) \Rightarrow C.$$

3. Define the composition of the function $f : X \longrightarrow Y$ and the function $g : Y \longrightarrow Z$ to be the function $g \circ f : X \longrightarrow Z$ with $g \circ f(x) = g(f(x))$ for all $x \in X$. Prove that if f is surjective and g is surjective, then $g \circ f$ is surjective.

Solution. Let $z \in Z$. Then $z = g(y)$ for some $y \in Y$ since g is surjective. And then $y = f(x)$ for some $x \in X$ since f is surjective. Therefore $g \circ f(x) = g(f(x)) = g(y) = z$. Hence $g \circ f$ is surjective.

4. Given sets A and B, consider the following two statements about a function $f : A \longrightarrow B$.
(i) $\exists b \in B$ such that $\forall a \in A, f(a) = b$.
(ii) $\forall b \in B, \exists a \in A$ such that $f(a) = b$.

One of these statements is the definition for f to be a surjective mapping from A to B. Which one is it? For the other statement, please explain what it says and give an example of a function from $A = \{1, 2, 3\}$ to $B = \{1, 2\}$ that has this property.

Solution. The correct choice is **(ii)**. The rest of the solution is omitted.

5. (a) Let $N = \{1, 2, 3, \cdots\}$ be the set of natural numbers. Let $P(N)$ denote the set of subsets of N. Let $F : N \longrightarrow P(N)$ be any well-defined mapping from N to its power set $P(N)$. Show that F is not surjective. Your proof should not depend upon any particular choice of F.

Solution. Let $C = \{n \in N | \sim (n \in F(n))\}$. Then it follows at once that C is not of the form $F(n)$ for any $n \in N$.

(b) Prove that there is a $1 - 1$ correspondence between the set

$$O = \{1, 3, 5, 7, 9, 11, \cdots\}$$

of odd natural numbers and the set N of all natural numbers.

Solution. Define $f : N \longrightarrow O$ by the equation $f(n) = 2n - 1$. It is easy to verify that this map is a bijection.

(c) Make the special assumption that if x is any set, then it is not the case that x is a member of x. On the basis of this special assumption prove that there does not exist a set U such that for all sets $y, y \in U$.

Solution. If U were a set, then we would have $U \in U$ (since U is the collection of all sets). This is a contradiction since no set is allowed to be a member of itself. Therefore we have shown by contradiction that U is not a set.

CHOOSE AND SOLVE ONE OF THE REMAINING PROBLEMS TO COMPLETE THE EXAM.

Solutions to the remaining problems are omitted here. Please contact the instructor, if you wish to discuss solutions to these problems.

6. Let C_r^n denote the binomial choice coefficient. Thus C_r^n is equal to the number of r-element subsets of a set with n-elements. This is sometimes phrased as the number of ways to choose r things from n things.

(a) State the binomial theorem for $(x+y)^n$ in terms of the coefficients C^n_r. Give the shortest correct proof of the binomial theorem that you know.

(b) Use the identity $(x+y)^{n+m} = (x+y)^n(x+y)^m$ with $y=1$ to find and prove a formula for C^{n+m}_r in terms of C^n_i and C^m_j $(0 \le i \le n, 0 \le j \le m)$. Your formula will be the consequence of applying the binomial theorem to both sides of the above identity.

7. Let i be given so that $i^2 = -1$. Let C denote the set of all numbers of the form $a+bi$ where a and b are real numbers. These are called complex numbers. Complex numbers add and multiply according to the formulas

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$

and

$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i.$$

(The second formula is just what you would get if you use $i^2 = -1$ and assume that the distributive law for multiplication holds for complex numbers.) Note that addition multiplication of complex numbers is commutative and associative.

Define the *conjugate* of a complex number $z = a+bi$ to be the complex number $\bar{z} = a - bi$.

Prove the following facts:

(a) $z\bar{z} = a^2+b^2$ when $z = a+bi$.

(b) $\overline{zw} = \bar{z}\bar{w}$ for any two complex numbers z and w. That is, the conjugate of a product is the product of the conjugates. To prove this, let $z = a+bi$, $w = c+di$ and multiply out $\bar{z}\bar{w} = (a-bi)(c-di)$, and compare your result with the conjugate of zw. Note that we have given the formula for zw above.

(c) Let $z = a+bi$, $w = c+di$. Using part (b) above we see that

$$z\bar{z}w\bar{w} = zw\bar{z}\bar{w} = zw\overline{zw}.$$

This means that $(a^2+b^2)(c^2+d^2)$ must be equal to the sum of two squares A^2+B^2 where $zw = A+Bi$. Give formulas for A and B in terms of a,b,c,d and state the exact identity by which the product of the sum of two squares is equal to a sum of two squares. Note that the work that you have done in this problem *proves* this identity. Finally, use your result to give integers A and B such that $(1^2+3^2)(3^2+4^2) = A^2+B^2$. In other words, write 250 as a sum of two squares.

8. Let $S \subset \{1, 2, \ldots, 2n\}$ where S has $n+1$ elements. Then S contains two numbers such that one divides the other. [Any number can be written uniquely as an odd number times a power of 2, $m = (2k-1)2^j$. Show that there are exactly n odd numbers in the list $\{1, 2, \ldots, 2n\}$, and use this to conclude that in a selection of $n+1$ numbers there must be an occurence of at least two numbers of the form $(2k-1)2^j$ with the *same* k and *different* values of j.]

9. Consider the game of Brussels Sprouts as described in our notes. Recall the rules: The game starts with n nodes in the plane. Each node is in the form of a cross drawn in the plane. A legal move consists in extending an arm of one cross to connect with an arm of another cross, drawing a curve in the plane that does not cross any previously created part of the game-diagram. After making the connection, one draws a line segment transverse to an interior point of the connecting arc that has just been constructed, creating a new node in the process. Players take turns, and a game of Brussels Srouts is over when one player is unable to make any further moves. The player who cannot move loses the game.

Think of the play of a game of Brussels sprouts as the production of a graph (made of nodes and edges) in the plane. At the start of the game there are n nodes and 0 edges. Each move produces one more node and two more edges. At the beginning of the game there are $4n$ freedoms in the form of ends of crosses. Each move removes two freedoms, but adds two more in the creation of the new node. Use this formulation of the structure of the game and prove that a full game of Brussels sprouts, starting with n nodes, will always have $5n-2$ moves. Use the Euler formula for plane graphs to accomplish your proof.

10. Prove that, for a positive integer n, a $2^n \times 2^n$ square grid with any one square removed can be covered using L-shaped non-overlapping tiles. Each tile consists in three adjacent grid squares in an L-shaped pattern.

11. **(a)** Prove that $\sqrt{2}$ is irrational.

(b) Prove that there exist irrational numbers a and b such that a^b is rational.

12. The following problem is due to the Reverend Charles Lutwidge Dodgson (27 January 1832 to 14 January 1898), also known as Lewis Caroll. He is the author of books on Symbolic Logic and also the books "Alice's Adventures in Wonderland" and "Through the Looking-Glass."

From the following three assertions we are to make whatever deductions are possible.

(i) Nobody who really appreciates Beethoven fails to keep silence while the Moonlight Sonata is being played.

(ii) Guinea-pigs are hopelessly ignorant of music.

(iii) No one who is hopelessly ignorant of music ever keeps silence while the Moonlight Sonata is being played.

These can be interpreted as statements about various sets. Let
G = the set of guinea-pigs.
H = the set of creatures that are hopelessly ignorant of music.
K = the set of creatures who keep silence while the Moonlight Sonata is being played.
R = the set of creatures that really appreciate Beethoven.

Rewrite each of **(i), (ii), (iii)** as a statement about sets in set theoretic notation. For example, statement (i) says that $R \subseteq K$. Use this rewrite to deduce that "Guinea pigs do not really appreciate Beethoven."