Journal of Knot Theory and Its Ramifications


# MEANDERS, KNOTS, LABYRINTHS AND MAZES

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ABSTRACT

There are strong indications that the history of design may have begun with the concept of a meander. This paper explores the application of meanders to new classes of meander and semi-meander knots, meander friezes, labyrinths and mazes. A combinatorial system is introduced to classify meander knots and labyrinths. Mazes are analyzed with the use of graphs. Meanders are also created with the use of simple proto-tiles upon which a series of lines are etched.

*Keywords*: meander, labyrinth, knots, links, mazes, friezes.

Mathematics Subject Classification 2000: 57M25, 01A07

## 1. Introduction

The meander motif got its name from the river Meander, a river with many twists mentioned by Homer in the Iliad and by Albert Einstein in a classical paper on meanders [1]. The motif is also known as the Greek key or Greek fret shown in Fig. 1 with other Greek meander patterns. The meander symbol was often used in Ancient Greece, symbolizing infinity or the eternal flow of things. Many temples and objects were decorated with this motif. It is also possible to make a connection of meanders with labyrinths since some labyrinths can be drawn using the Greek key. We will refer to any set of twisting and turning lines shaped into a repeated motif as a meander pattern where the turning often occurs at right angles [2]. For applications of meanders, the reader is referred to [3,5]. This paper is in large measure, a reworking of an earlier paper by Jablan and Radovic [6,14].

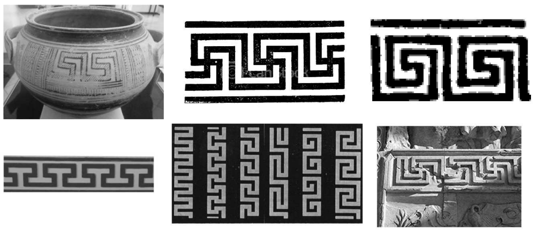

Fig. 1. Greek meanders

Fig. 2. Prototiles

Perhaps the most fundamental meander pattern is the meander spiral which can be found in very early art history. The prototile based on a set of diagonal stripes drawn on a square and a second square in which black and white are reversed (see Fig. 2) called op-tiles are used abundantly in ornamental art going back to Paleolithic times. From these two squares an infinite set of key patterns can be derived. These patterns are commonly found in different cultures (Paleolithic, Neolithic, Chinese, Celtic), and were independently discovered by these cultures. The oldest examples of key-patterns are ornaments from Mezin (Ukraine) about 23 000 B.C. The appearance of meander spirals in prehistoric ornamental art can be traced to archeological findings from Moldavia, Romania, Hungary, Yugoslavia, and Greece, and all of them can be derived as modular structures. In this paper we will study the application of meanders to frieze patterns, labyrinths, mazes and knots.

## 2. Meander Friezes

To create a frieze pattern begin with a basic pattern and translate the pattern along a line in both directions. There are seven classes of frieze patterns employing reflections in a mirror along the line, mirrors perpendicular to the line, and half turns at points along the line. One each of the seven frieze pattern is shown in Fig. 3. To create a meander frieze pattern we use some meander pattern. A subclass of meander friezes can be formed from the initiating pattern formed within a $p \times q$ rectangular grid of points as shown in Fig. 4. A continuous set of line segments is placed in the grid touching each point with no self-intersections. If the grid points are considered to be vertices and the line segments are edges of a graph, then such path through the graph is referred to as a Hamilton path. In this way the pattern has numerous twists and turns inducing a meander configuration resulting in what we refer to as a meander frieze. The pattern has one edge that enters the grid and another leaving the grid at the same level in order to connect to the next translated pattern.

The number of frieze patterns corresponding to each square tends to be quite large. Even a $5 \times 5$ grid gives (up to symmetries) 19 different cases as shown in Fig. 5 and the $7 \times 7$ grid gives more than 2800 possibilities. The variety of mean-

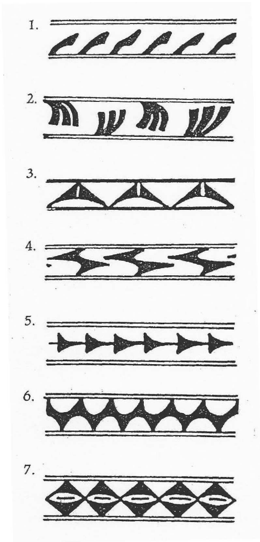


Fig. 3. Examples of the seven frieze patterns

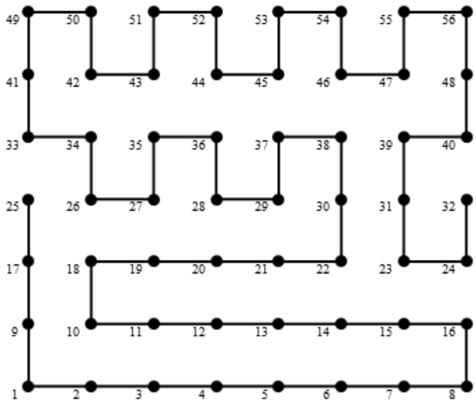


Fig. 4. Pattern for a meander frieze

der friezes can be further enriched by inserting some additional internal elements (intersections), for example, a rosette with a swastika motif as shown in Fig. 6a. It is clear that Ancient Greeks and other cultures created friezes using only a very small portion of the possibilities, restricted only to grids of small dimensions. Hence meander friezes originating from grids of dimension $7 \times 7$ , such as the pattern in Fig. 6b were probably not used at all.

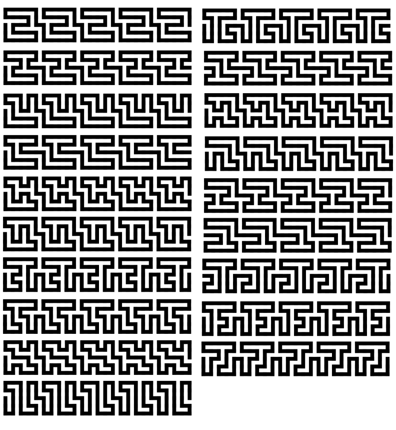

Fig. 5. The nineteen $5 \times 5$ frieze patterns

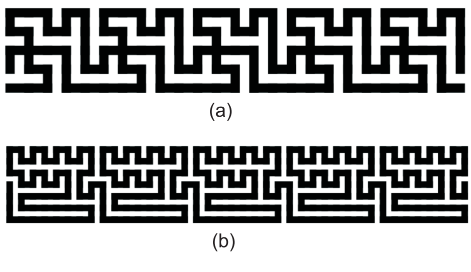


Fig. 6. (a) A meander frieze pattern with a swastika motif, (b) A $7 \times 7$ meander frieze pattern

## 3. Meanders represented by the intersection of two lines

The creation of meander patterns is based on the notion of an open meander [13].

**Definition 3.1.** An open meander is a configuration consisting of an oriented simple curve, and a line in the plane, the axis of the meander, in which the simple curve crosses the axis a finite number of times and intersects only transversally [2]. In this way, open meanders can be represented by systems formed by the intersections of two curves in the plane. Two meanders are equivalent if one can be deformed to the other by redrawing it without changing the number and sequencing of the intersections. In this case the two meanders are said to be homeomorphic. They occur in the physics of polymers, algebraic geometry, mathematical theory of mazes, and planar algebras, in particular, the Temperly-Lieb algebra. One such open meander is shown in Fig.7a. As the main source of the theory of meanders we used the paper [2]. For applications of the theory of meanders, the reader is referred to [2,3,5].

The order of a meander is the number of crossings between the meander curve and the meander axis. For example, in Fig.7a there are ten crossings so the order is 10. Since a line and a simple curve are homeomorphic, their roles can be reversed. However, in the enumeration of meanders we will always distinguish the meander curve from the meander line, the axis. Usually, meanders are classified according to their order. One of the main problems in the mathematical theory of meanders is their enumeration.

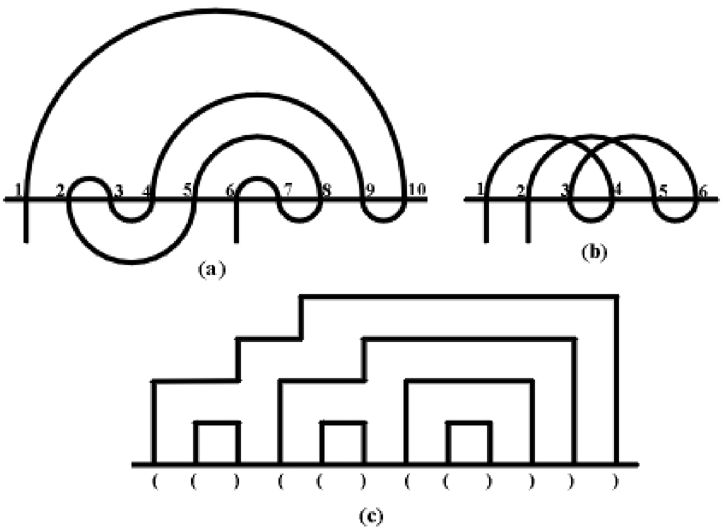


Fig. 7. (a) Open meander given by meander permutation $(1, 10, 9, 4, 3, 2, 5, 8, 7, 6)$; (b) non-realizable sequence $(1, 4, 3, 6, 5, 2)$; (c) piecewise-linear upper arch configuration given by Dyck word (()(()(()))).

An open meander curve and meander axis have two loose ends each. Depending on the number of crossings, the loose ends of the meander curve belong to different half-planes defined by the axis for open meanders with an odd order, and to the same half-plane when the meanders have an even order. For example, in Fig.7a, the loose ends are in the same half-plane since it has an even order. In this case we are able to make a closure of the meander: to join each of the loose ends. We will find that an odd number of crossings results in a knot whereas an even number of crossing results in a link. Knots consist of a single strand whereas links are characterized by the interlocking of multiple strands. We will discuss knots in next section. We will use arch configurations to represent meanders.

**Definition 3.2.** An arch configuration is a planar configuration consisting of pairwise non-intersecting semicircular arches lying on the same side of an oriented line, arranged such that the feet of the arches are a piecewise linear set equally spaced along the line as shown in Fig.7a.

Arch configurations play an essential role in the enumeration of meanders. A meandric system is obtained from the superposition of an ordered pair of arch configurations of the same order, with the first configuration as the upper and the second as the lower configuration. The modern study of this problem was inspired by [3]. If the intersections along the axis are enumerated by $1, 2, 3, ..., n$ every open meander can be described by a meander permutation of order $n$: the sequence of $n$ numbers describing the path of the meander curve. For example, the open meander (Fig. 7a) is coded by the meander permutation $1, 10, 9, 2, 3, 2, 5, 8, 7, 6$. Enumeration of open meanders is based on the derivation of meander permutations. Meander permutations play an important role in the mathematical theory of labyrinths [8]. In every meander permutation odd and even numbers alternate, i.e, parity alternates in the upper and lower configurations. However, this condition does not completely characterize meander permutations. For example, the permutation: $1, 4, 3, 6, 5, 2$ exhibits two crossing arches, (1,4) and (3,6) as shown in Fig. 7b. Therefore, the most important property of meander permutations is that all arches must be nested in order not to produce crossing lines. Among different techniques to achieve this, the fastest algorithms for deriving meanders are based on encoding each configuration as words in the Dyck language [11,12] and the Mathematica program Open meanders by David Bevan (`http://demonstrations.wolfram.com/OpenMeanders/`) [13]. The upper and lower arches are represented by nested parentheses with the loose ends represented by 1. As a result, the upper and lower arches in Fig. 7a are coded by $\{(()((()))), 1(())1()()\}$. The nested curves can also be squared off as shown in Fig. 7c.

## 4. Meander Knots

First we say a few words about knots [4]. A knot can be thought as a knotted loop of string having no thickness. It is a closed curve in space that does not intersect itself. We can deform this curve without permitting it to pass through itself, i.e., no cutting. Although these deformations appear quite different, as shown in Fig.8 , they are considered to be the same knot. If a deformation of the curve results in a simple loop it is referred to as an unknot. To create the shadow of a knot, draw a scribble of lines, with the restriction that at any point of intersection only two lines of the scribble intersect as shown in Fig. 9a. Notice that at each point of intersection of the scribble four edges intersect. By introducing the over/under relation in crossings of the shadow, we get a knot diagram. An alternating knot can be constructed from its shadow by drawing a path through the scribble, entering a point of intersection and taking the middle segment of the three exit choices and then proceeding along the path in an over-under-over-under- pattern as shown in Fig. 9b. Notice that some crossings, such as the crossing at point $P$ can be eliminated by simple twists or movements without cutting. These moves are referred to as Reidemeister moves of which there are three such unknotting rules [4]. After all such movements are made the resulting knot can be reduced to its minimum number of crossings as shown in Fig. 9b. A minimal projection of a knot is one that minimizes the number of crossings. This is called the crossing number, defined to be the least number of crossings that occur in any projection of the knot. It is uniquely defined for any knot. Meander diagrams that have a minimal number of crossings are called minimal meander diagrams.

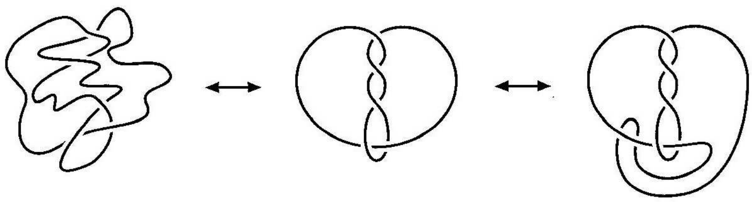

Fig. 8. A knot and its deformations.

As we described in Sec. 3, because when making a closure of meander diagrams we have two possibilities, we choose the one producing a meander knot shadow without crossing lines (loops). After that by introducing under-crossings and over-crossings along the meander knot shadow axis, we can turn it into a knot diagram. When the crossings alternate: under-over-under-over - the knot is said to be alternating. Given a knot, it can be transformed without cutting to eliminate certain crossings. However unless the knot is a loop or unknot there will always remain crossings, specified by the crossing number. Each knot can be classified by its crossing number.

**Definition 4.1.** An alternating knot that has a minimal diagram in the form of a minimal meander diagram is called a *meander knot*.

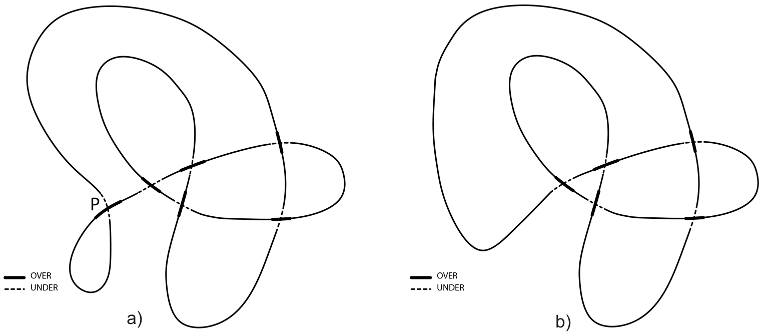


Fig. 9. (a) The shadow of a knot resulting in a knot with one extraneous crossing at P; (b) the knot redrawn with the crossing at P removed.

Another problem is the derivation of meander knots first introduced by S. Jablan. Several meander knots are represented here by their *Gauss codes* and *Conway symbols* [9,10,7]. All computations were obtained by Jablan using the program LinKnot [7]. Gauss codes of alternating meander knot diagrams can be obtained if to the sequence $1, 2, \ldots n$ we add a meander permutation of order $n$ where $n$ is an odd number and in the obtained sequence alternate the signs of successive numbers, e.g., from meander permutation $(1, 8, 5, 6, 7, 4, 3, 2, 9)$ we obtain Gauss code

$$\{-1, 2, -3, 4, -5, 6, -7, 8, -9, 1, -8, 5, -6, 7, -4, 3, -2, 9\}$$

which corresponds to rational alternating knot with nine crossings referred to by $9_7$ and also given by the Conway symbol $3\,4\,2$. The same knot can be obtained from meander permutations $(1, 8, 7, 6, 5, 2, 3, 4, 9)$ and $(1, 8, 7, 4, 5, 6, 3, 2, 9)$, giving Gauss codes

$$\{-1, 2, -3, 4, -5, 6, -7, 8, -9, 1, -8, 7, -6, 5, -2, 3, -4, 9\}$$

and

$$\{-1, 2, -3, 4, -5, 6, -7, 8, -9, 1, -8, 7, -4, 5, -6, 3, -2, 9\}.$$

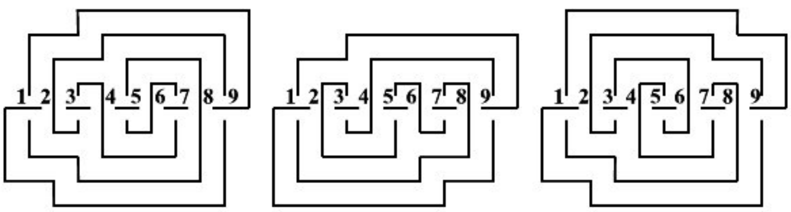


Fig. 10. Non-isomorphic minimal meander diagrams of the knot $9_7 = 3\,4\,2$.

These three representatives of the knot 3 4 2 are shown in Fig.10. It should also be pointed out that if an alternating knot has an alternating minimal meander diagram, all of its minimal diagrams need not be meander diagrams.

The natural question which arises is to find all alternating meander knots with $n$ crossings, where $n$ is an odd number. Alternating meander knots with at most $n = 9$ crossings are illustrated in Fig.11

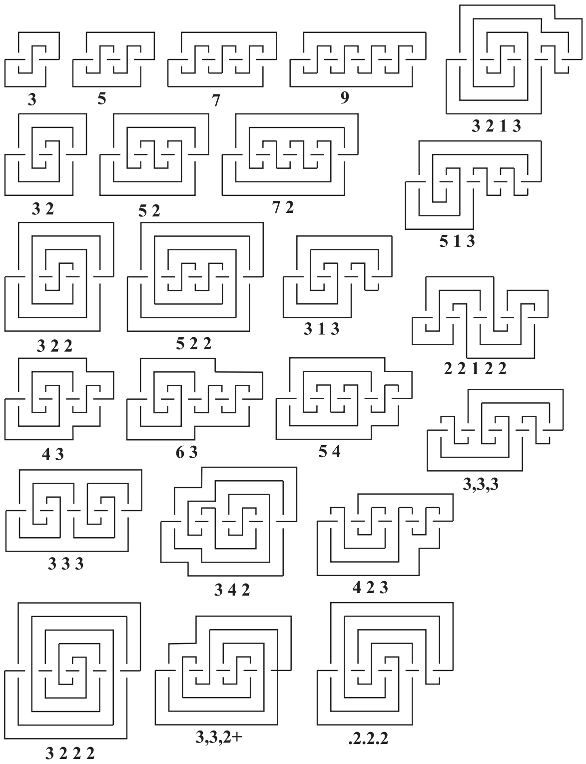


Fig. 11. Alternating meander knots with at most $n = 9$ crossings.

Which knots can be represented by non-minimal meander diagrams? For example the figure-eight knot $4_1$, in Conway notation 2 2 with four crossings cannot be represented by a meander diagram but can be represented by the non-minimal meander diagram given by the Gauss code $\{-1, 2, -3, -4, 5, 3, -2, 1, 4, -5\}$ with $n = 5$ crossings. Knot 2 2 and five additional non-minimal diagrams are shown in Fig. 12. You will also note that the knot is not alternating.

For every knot which is not a meander knot (does not have a minimal meander diagram) but which can be represented by some meander diagram (which is reduced, but has more crossing than the minimal diagram of that knot, i.e., more crossing than the crossing number of that knot ), we can define its meander number, the minimum number of crossings of its meander diagrams where the minimum is taken over over all its meander diagrams. How to find knots which have meander diagram? Alternating meander knots have it, but also non-alternating knots with the same shadows as alternating meander knots also have the meander diagram, with some crossings change from overcrossing to undercrossing and vice versa. The next step is to make all possible crossing changes in alternating minimal meander diagrams, i.e., in Gauss codes of alternating meander knots and see which knots will be obtained.

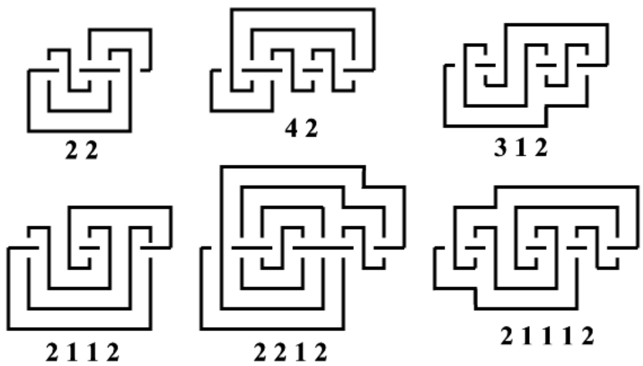


Fig. 12. Non-minimal meander diagrams of knots $4_1$, $6_1$, $6_2$, $6_3$, $7_6$, and $7_7$.

## 5. Two component meander links

Open meanders with an even number of crossings offer the interesting possibility of joining pairs of loose ends of the meander axis, and loose ends of the meander curve. As a result, we obtain the shadow of a 2-component link with one component in the form of a circle and the other component meandering around it. A natural question is which alternating links can be obtained from these shadows, and in general, which 2-component links have meander diagrams. It is clear that components do not self-intersect so the set of 2-component meander links coincides with the set of alternating 2-component links with non-self-intersecting components, and all of their minimum diagrams preserve this property.

As for knots, we pose for 2-component links the natural question as to which 2-component links have meander diagrams. From the definition of meander links it is clear that the answer will be links in which both components will be knots and which components are not self-crossing, i.e., it will be the shadow of a circle. In the case of alternating minimal meander diagrams, all such diagrams of 2-component links will have this property. However, in the case of non-minimal meander diagrams, some links with an odd number of crossings are represented by meander diagrams. Moreover, their minimal diagrams have components with self-intersections, but in their non-minimal meander diagrams none of the components have self-intersections. Meander links up to $n = 10$ crossings are shown in Fig. 13.

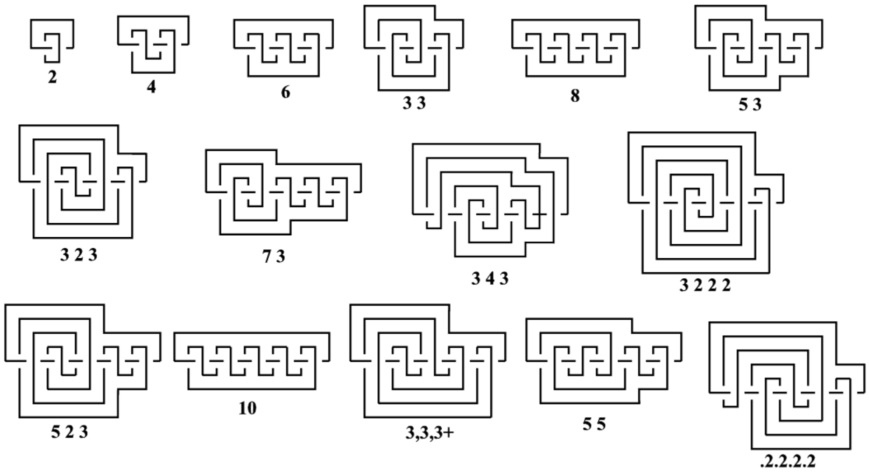


Fig. 13. Meander links up to $n = 10$ crossings.

## 6. Sum of meander knots and links

For two open meander sequences we can define a sum or concatenation: the operation of joining their Dyck words and connecting the second loose end of the first with the first loose end of the second and making a closure in order to obtain a meander knot or link diagram (see Fig. 14). The same definition extends to meander knots and links where we concatenate the meander parts of their Gauss codes. For parity reasons, the sum of two meander knot diagrams or the sum of two meander link diagrams is a meander link diagram, and the sum of a meander knot diagram and meander link diagram or vice versa is a meander knot diagram. The sum of a meander knot diagram and its mirror image is a 2-component unlink.

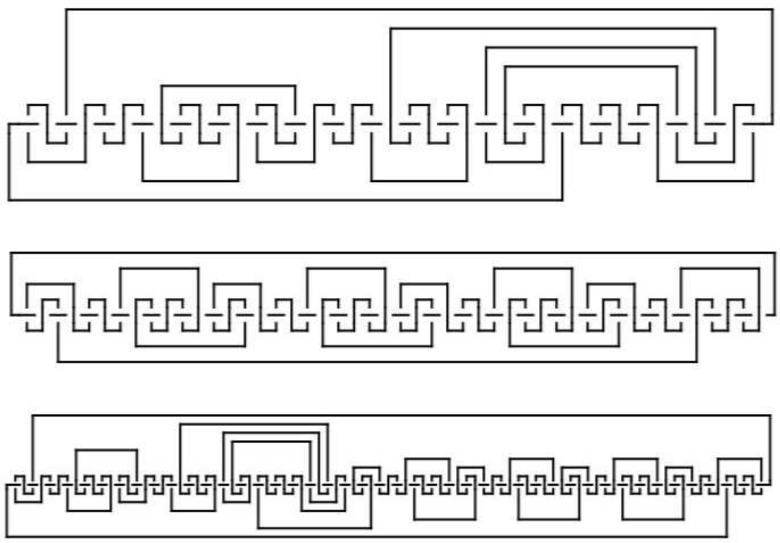

Fig. 14. Sum of a 39-crossing meander knot and 48-crossing meander link giving the 87-crossing meander knot.

## 7. Semi-meander or ordered Gauss code knots

In the case of meanders the axis of a meander is infinite. If the axis is finite, we obtain semi-meanders, where a meander curve can pass from one side of the axis to the other in a region beyond the end(s) of the axis without crossing the axis. Gauss code depends on the choice of the initial (basic) point belonging to some arc and from the orientation of the knot. This means that every rotation or reversal of a sequence of length $2n$ representing the Gauss code of a knot with $n$ crossings represents the same (non-oriented) knot. A Gauss code will be said to be *ordered* if the absolute value of the first part of its Gauss code is the sequence $1, 2, \ldots n$. An alternating knot will be called an *ordered Gauss code* (OCG) or *semi-meander* knot if it has at least one minimal diagram with an ordered Gauss code. The name *semi-meander knot* follows from the fact that the shadow of such a knot represents a meander or semi-meander. It is clear that every meander knot is OGC, and that meander knots represent the proper subset of OGC knots. For OGC knots there is no parity restriction to the number of crossings, so there exist OGC knots which are not meander knots, i.e., OGC knots with an even number of crossings. Moreover, some OGC knots with an odd number of crossings are not meander knots, e.g., knot $7_6 = 2\,2\,1\,2$ which has two minimal diagrams, and among them only one is OGC diagram with ordered Gauss code $\{1, -2, 3, -4, 5, -6, 7, -5, 4, -1, 2, -7, 6, -3\}$. Every OGC diagram is completely determined by the second half of its ordered Gauss code, which will be called short Gauss code. Fig. 15 shows all semi-meander knots with $n \leq 7$ crossings which are not meander knots.

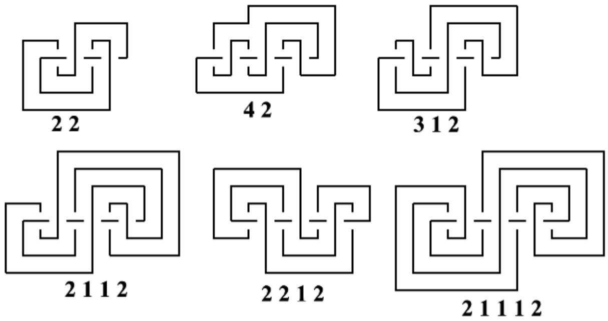


Fig. 15. Semi-meander knots with $n \leq 7$ crossings which are not meander knots.

## 8. Labyrinths

According to the Greek myths, the skillful craftsman Daedalus created the Labyrinth. The purpose of this special architectural structure was to imprison the Minotaur, the son of Pasiphae, the wife of the Cretan King Minos. The myth of the Cretan Labyrinth has been a subject of speculation and archaeological, historical, and anthropological research for a long time just as the visual representations of labyrinthine structures concern not only art historians, but also mathematicians. Karl Kerenyi (1897-1973), the internationally renowned scholar of religion colleague of Carl Jung, and friend and advisor of Thomas Mann, returned time after time to the mythological research of labyrinths and interpreted them both as cultural symbols and specific geometrical structures. Right from the beginning of his labyrinth studies, Kerenyi introduced the labyrinth from three closely interrelated main aspects: 1) as a mythical construction; 2) as a spiral path that was followed by dancers of a specific ritual; and 3) as a structure that was represented by a spiral line. In his 1941 essay series [15], he summarized the most important concepts of previous studies and made several original observations and comparisons, which are still widely quoted and referred to in Labyrinth Studies. With the comparative mythological and morphological analysis of the Babylonian, Indonesian, Australian, Norman, Roman, Scandinavian, Finnish, English, German and medieval and Greek labyrinth tradition, he has proven the global presence of labyrinthine structures and revealed the artistic and architectural impulse behind the creation of them to rituals and cultic dances where participants followed aspiral line and made meandering gestures and dance-movements. In 1963, Kerenyi devoted a lengthy essay to Greek folk dance [16] and pointed out how the movements of the ancient labyrinth dances were transformed into the main components of the Syrtos, a dance that is still performed in Greece today. And in his last book written in 1969 [17], where he explored the Cretan roots of the cult of Dionysis, he discussedin depth the labyrinthine and meander-like patterns of Knossos in dance, art, and architecture. When a dancer follows a spiral whose angular equivalent is precisely the meander, he returns to his starting point, wrote Kerenyi, quoting Socrates from Platos dialogue The Euthydemus. Socrates speaks there of the labyrinth and describes it as a figure whose most easily recognizable feature is an endlessly repeated meander or spiral line: Then it seemed like falling into a labyrinth; we thought we were at the finish, but our way bent round and we found ourselves, as it were, back at the beginning, and just as far from that which we were seeking at first [17].There resulted a classical picture of this procession, which originally led by way of concentric circles and surprising turns to the decisive turn in the center where one was obliged to rotate on one own axis in order to continue the circuit [17]. The labyrinths surprising turns and the decisive turn in their center is responsible for their symbolic meaning as well. Kerenyi sees the labyrinth as a depiction of Hades, the underworld, and interprets the structures as narrative symbols which express the existential connection between life and death, between the oblivion of the dead and the return of the eternal living. From a morphological perspective, Kerenyi presupposes the transformation of the spiral to the meander pattern because straight lines were easier to draw and so the rounded form was early changed into the angular form. For Kerenyi the meander is the figure of a labyrinth in linear form. In the third to second centuries BC, as he explains, we find the figure and the word unmistakably related: in the Middle Ages labyrinths were also called meanders [16]. We find a detailed connection between meanders and labyrinths in Matthews' book, Mazes and Labyrinths [18]. Although both Matthews and Kerenyi made the connection between labyrinths and meanders clear, the ornamental evolution of angular labyrinths were not discussed by any of them in a way that could explain the geometrical development process underlying them. Our approach seeks to remedy this. Before proceeding I would like to make clear the difference between labyrinths and mazes since these words are often used interchangeably. Both labyrinths and mazes can be described by graphs. However, in the case of labyrinths, there is a single path leading from the entrance to the center, whereas for mazes there are at various points bifurcations in the path, with some choices of continuance leading to dead ends and others leading on to the center. So in a sense labyrinths can be thought of as being subsets of mazes in which there is a unicursal path through the graph.

## 9. Labyrinth studies and visual arts

We have found that the oldest examples of geometrical ornamentation in Paleolithic art were from Mezin (Ukraine) dated to 23,000 B.C. (see Fig. 16).

Among the set of ornaments found at Mezin is the first known meander frieze under the well-known name Greek key. Take a set of parallel lines, cut a square or

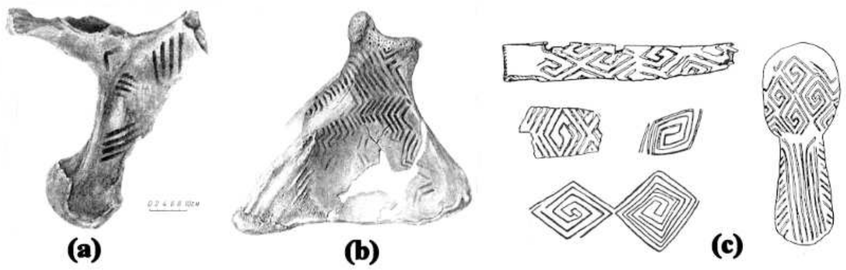

Fig. 16. Ornaments from Mezin.

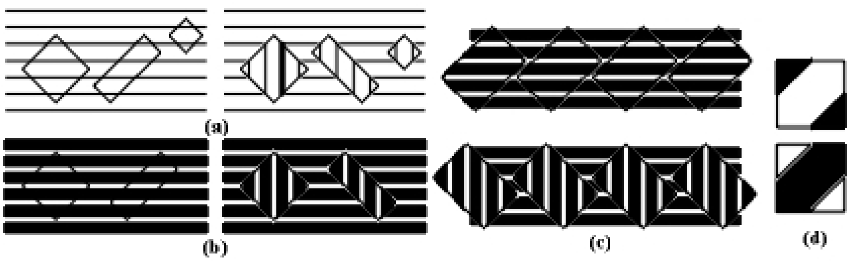

Fig. 17. (a,b) "Cut and paste" construction; (c) Kufic tiles.

rectangular piece with the set of diagonal parallel lines incident to the first ones, rotate by 90°, and if necessary translate it in order to fit with the initial set (See Fig. 17). More aesthetically pleasing results will be obtained by using the initial set of black and white strips of equal thickness.

## 10. From meanders to labyrinths

The word labyrinth is derived from the Latin word labris, making a two-sided axe, the motif related to the Minos palace in Knossos. The walls of the palace were decorated by these ornaments while the interior featured actual bronze double axes. This is the origin of the name labyrinth and the famous legend about Theseus, Ariadne, and the Minotaur. The Cretan labyrinth is shown on the silver coin from Knossos (400 B.C.) as shown in Fig. 18.

To create the Cretan labyrinth, first consider a Simple Alternating Transit labyrinth, or SAT labyrinths [3,8]. An SAT labyrinth is laid out on a certain number of concentric or parallel levels. The labyrinth is simple if the path makes essentially a complete loop at each level, in particular, it travels on each level exactly once. It is alternating if the labyrinth -path changes direction whenever it changes level, and transit if the path progresses without bifurcation from the outside of the maze to the center. Most SAT labyrinths occur in a spiral meander form with the path leading from the outside to the center. Each such labyrinth can be sliced down its axis and unrolled into an open meander form. Now the path enters at the top of the form and exits at the bottom: the top level (center) of the labyrinth becomes the space below the open meander form. This process is illustrated in Fig.10 for the Cretan labyrinth. The topology of an SAT labyrinth is entirely determined by its level sequence, i.e., its open meander permutation as described in Sec. 3,for example, the meander permutation 3,2,1,4,7,6,5. Hence the enumeration of open meanders and their corresponding SAT labyrinths is based on the derivation of meander permutations. For the derivation of open meanders one can use the Mathematica program open meanders by David Bevan [13] which we modified in order to compute open meander permutations.

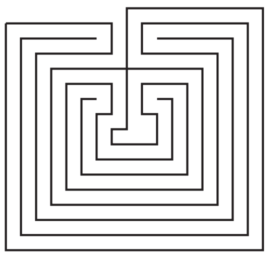

Fig. 18. The Cretan labyrinth.

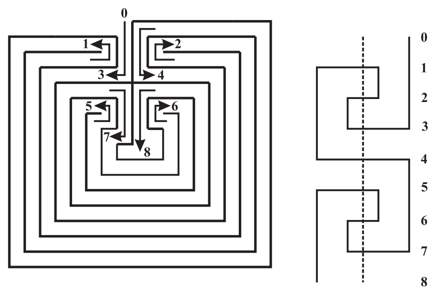


Fig. 19. Meander permutation and the unrolling procedure to create the labyrinth.

How does one construct a unicursal path without knowledge of computer programs and topological transformations? The simplest natural labyrinth is a spiral meander: a piecewise-linear equidistant spiral. It is defined by a simple algorithm: central point and after every step turn by 90°, and continue with the next step, where the sequence of step distances is $1, 1, 2, 2, 3, 3, 4, 4 \ldots$. Tracing this sequence we have a labyrinth path: a simple curve connecting the beginning point (the entrance) with the end point (Minotaur room) . Fig.20 shows an elegant way to construct a Cretan maze. Draw a black spiral meander (Fig.20a), cut out several rectangles or squares, rotate each of them around its center by 90°, and place it back to obtain a labyrinth (Fig.20b). Even very complex labyrinths can be constructed in this way (Fig.21).

It is interesting to notice that even the Knossos dancing pattern, using the shape of a double axe, can be reconstructed in a similar way (Fig.22). So, a simple pattern (Fig 23), an optile [19], can be considered as the logo of a Paleolithic designer from which Mezin ornaments can be created. These tiles were also discovered by Ben Nicholson who referred to them as Versatiles.

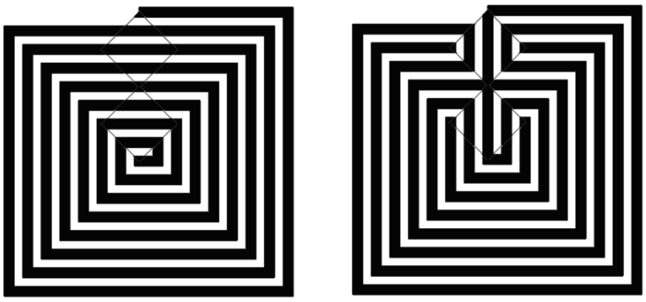


Fig. 20. Cut and paste construction of a spiral labyrinth.

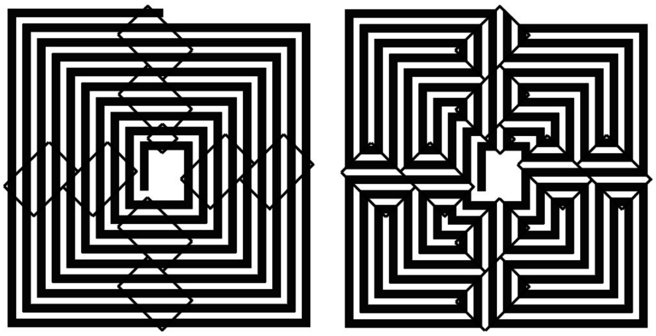

Fig. 21. Cut and paste construction of a complex labyrinth.

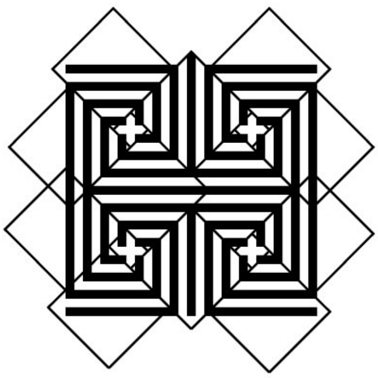

Fig. 22. The Knossos dancing pattern.

Fig. 23. Prototiles.

## 11. A Labyrinth Workshop

(1) From linoleum squares of the dimensions $40 \times 40$ cm and self-adhesive tape of two colors (e.g., black and silver) make basic tiles (Fig. 24);
(2) Make a meander spiral (Fig. 25, left);
(3) By rotating only two tiles by $90^o$ you obtain Cretean labyrinth (Fig. 25, right) as was done by S. Jablan at the Bridges Conference in Pecs 2010 (Fig. 26).

Notice that only tiles 5) and 6) are black Optiles, and the other Optiles are colored by two colors in order to use them for the border art of the spiral and labyrinth.

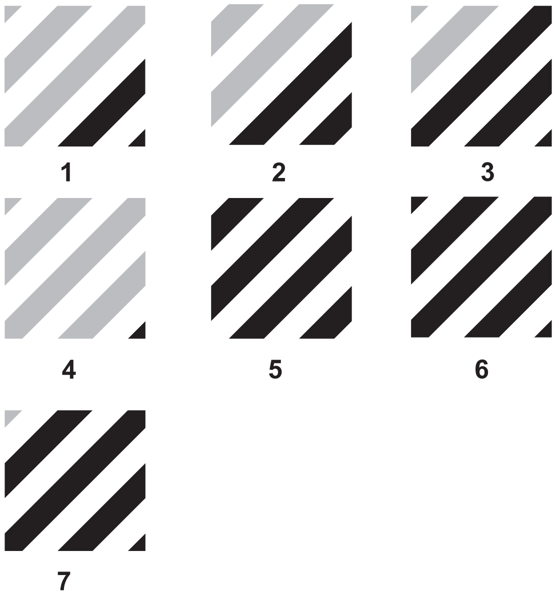


Fig. 24. Basictiles.

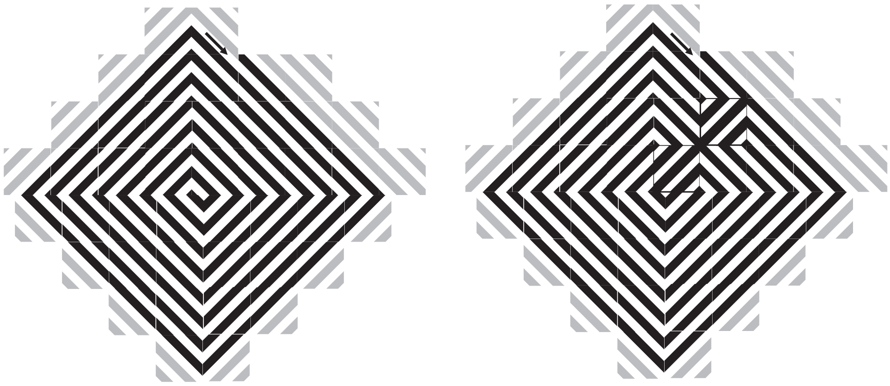

Fig. 25. Meander spiral and Cretean labyrinth.

Fig. 26. Modular labyrinth from Bridges conference in Pecs 2010 (photos by Norbert Horvath.).

## 12. Mazes

If the path from entrance to center is not unicursal then we have a maze. For example, a fun house is pictured as a maze in Fig. 27. Paths from room to room are denoted in the accompanying graph. You will notice that certain rooms have access to several other rooms along the pathways. The Hampton Court maze was commissioned around 1700 by William III. It covers a third of an acre, is trapezoidal in shape, it is planted with high hedges, and is the oldest surviving hedge maze. Its patterns shown in Fig. 28 along with its access graph whose vertices are the points at which the path bifurcates.

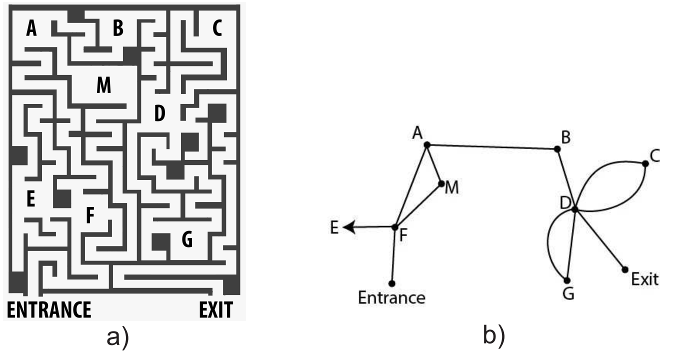


Fig. 27. (a) A fun house pictured as a maze. (b) Graph of the maze [N. Friedman, 2003].

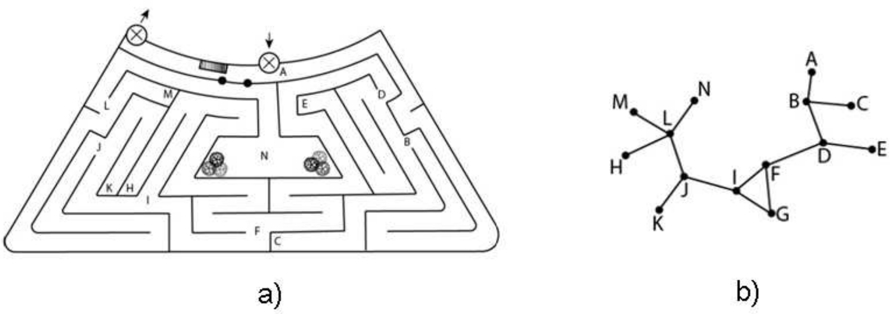


Fig. 28. The Hampton Court maze and its graph [N. Friedman, 2003].

I invite you to draw the graph from entrance to center for the maze in Fig. 29.

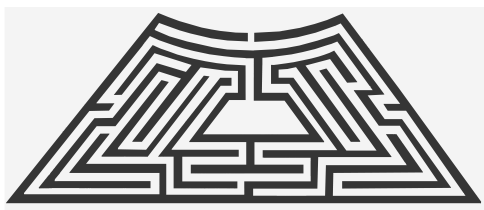

Fig. 29. Example of another maze [N. Friedman, 2003].

## 13. Conclusion

We have seen how a wide range of mathematics and design has emerged from the concept of a meander. Knots, frieze patterns, labyrinths and mazes all owe their existence to the concept of a meander. Future work in these fields must take this provenance into account.

## Acknowledgements

We are grateful to Louis Kauffman, Sofia Lambropoulou and Radmila Sazdanovic for editing the special issue of JKTR dedicated to our dear colleague and friend. This paper was written over the course of few years during which Lj. Radovic was partially supported by the Ministry of Science and Technological Development of Serbia (Grant No. 174012)

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