Non-Commutative Worlds

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We explore how a discrete viewpoint about physics is related to non-commutativity, gauge theory and differential geometry.

Discrete Measurement is Intrisically Non-commutative.

Time Series: $X, X', X'', \ldots$

Derivative: $\dot{X} = (X' - X)/dt$

Here dt is a finite time increment.

$X\dot{X}$: Observe $\dot{X}$, then observe X.

$\dot{X}X$: Observe X, then observe $\dot{X}$.

$$X\dot{X} = X'(X'-X)/dt$$

$$\dot{X}X = (X'-X)X/dt$$

$$X\dot{X} - \dot{X}X = (X'-X)(X'-X)/dt$$

$$[X,\dot{X}] = (dX)^2/dt$$

$$X\dot{X} - \dot{X}X = (X'-X)(X'-X)/dt$$

$$[X,\dot{X}] = k \quad \text{then} \quad k = (dx)(dx)/dt$$

$$X' = X \pm dx$$

The discrete analog of Heisenberg's equation yields a Brownian walk with diffusion constant k.

Discrete calculus is embedded in commutator calculus.

$$X\dot{X} = X'(X'-X)/dt$$

$$\dot{X}X = (X'-X)X/dt$$

$\dot{X}$ is a signal to time-shift the algebra to its left.

Make algebraic by defining new operator J with

$$XJ = JX'.$$

Redefine

$$\dot{X} = J(X' - X)/dt.$$

Then $\dot{X} = (XJ - JX)/dt = [X, J/dt].$

Constructions will be performed in an (abstract) Lie algebra A.

On $\mathcal{A}$, a variant of calculus is built by defining derivations as commutators (or more generally as Lie products). For a fixed N in $\mathcal{A}$ one defines

$$\nabla_N : \mathcal{A} \longrightarrow \mathcal{A}$$

by the formula

$$\nabla_N F = [F, N] = FN - NF.$$

∇_N is a derivation satisfying the Leibniz rule.

$$\nabla_N(FG) = \nabla_N(F)G + F\nabla_N(G).$$

The Liebniz rule does not hold in discrete calculus. We regain it by embedding Discete Calculus in non-commutative Calculus.

There are many motivations for replacing derivatives by commutators. If $f(x)$ denotes (say) a function of a real variable x, and $\tilde{f}(x) = f(x+h)$ for a fixed increment h, define the *discrete derivative* Df by the formula $Df = (\tilde{f} - f)/h$, and find that the Leibniz rule is not satisfied. One has the basic formula for the discrete derivative of a product:

$$D(fg) = D(f)g + \tilde{f}D(g).$$

Correct this deviation from the Leibniz rule by introducing a new non-commutative operator J with the property that

$$fJ = J\tilde{f}.$$

Define a new discrete derivative in an extended non-commutative algebra by the formula

$$\nabla(f) = JD(f).$$

It follows at once that

$$\nabla(fg) = JD(f)g + J\tilde{f}D(g) = JD(f)g + fJD(g) = \nabla(f)g + f\nabla(g).$$

Note that

$$\nabla(f) = (J\tilde{f} - Jf)/h = (fJ - Jf)/h = [f, J/h].$$

Let's build a non-commutative world of flat coordinates suitable for advanced calculus.

The flat coordinates X_i satisfy the equations below with the P_j chosen to represent differentiation with respect to X_j:

$$[X_i, X_j] = 0, \quad \text{Coordinates Commute.}$$
$$[P_i, P_j] = 0, \quad \text{Partials commute.}$$
$$[X_i, P_j] = \delta_{ij}. \quad \text{Derivative formula.}$$

Derivatives are represented by commutators.

$$\partial_i F = \partial F/\partial X_i = [F, P_i],$$
$$\hat{\partial}_i F = \partial F/\partial P_i = [X_i, F].$$

Temporal derivative is represented by commutation with a special (Hamiltonian) element H of the algebra:

$$dF/dt = [F, H].$$

(For quantum mechanics, take $i\hbar dA/dt = [A, H]$.)

Hamilton's Equations express the Mathematics of a Non-Commutative Flat World

$$dP_i/dt = [P_i, H] = -[H, P_i] = -\partial H/\partial X_i$$

$$dX_i/dt = [X_i, H] = \partial H/\partial P_i.$$

These are exactly Hamilton's equations of motion. The pattern of Hamilton's equations is built into the system.

Note that we have derived
Hamilton's equations
from nothing but
the assumption of a
flat non-commutative world.

Remark: Hamilton's formulation was in terms of Poisson brackets.

$$\{A, B\} = (\partial A/\partial q)(\partial B/\partial p) - (\partial A/\partial p)(\partial B/\partial q)$$

Poisson brackets do not obey the Leibniz rule.

$$(d/dt)\{A, B\} = \{dA/dt, B\} + \{A, dB/dt\} - \{A, B\}(\partial \dot{q}/\partial q + \partial \dot{p}/\partial p)$$

Unless $(\partial \dot{q}/\partial q + \partial \dot{p}/\partial p) = 0$.

This is an integrability condition for Hamilton's equations:

$$\dot{q} = \partial H/\partial p,$$

$$\dot{p} = -\partial H/\partial q.$$

Discrete Measurement. Consider a time series $\{X, X', X'', \cdots\}$ with commuting scalar values. Let

$$\dot{X} = \nabla X = JDX = J(X' - X)/\tau$$

where τ is an elementary time step (If X denotes a times series value at time t, then X' denotes the value of the series at time $t + \tau$.). The shift operator J is defined by the equation $XJ = JX'$ where this refers to any point in the time series so that $X^{(n)}J = JX^{(n+1)}$ for any non-negative integer n. Moving J across a variable from left to right, corresponds to one tick of the clock. This discrete, non-commutative time derivative satisfies the Leibniz rule.

1. Let $\dot{X}X$ denote the sequence: observe X, then obtain $\dot{X}$.
2. Let $X\dot{X}$ denote the sequence: obtain $\dot{X}$, then observe X.

The commutator $[X, \dot{X}]$ expresses the difference between these two orders of discrete measurement. In the simplest case, where the elements of the time series are commuting scalars, one has

$$[X, \dot{X}] = X\dot{X} - \dot{X}X = J(X' - X)^2/\tau.$$

Proof. $X\dot{X} = XJ(X' - X)/\tau = JX'(X' - X)/\tau$

$$\dot{X}X = J(X' - X)X/\tau.$$

Emergence of the Diffusion Constant

Thus we can interpret the equation

$$[X, \dot{X}] = Jk$$

(k a constant scalar) as

$$(X' - X)^2/\tau = k.$$

This means that the process is a Brownian walk with spatial step

$$\Delta = \pm\sqrt{k\tau}$$

where k is a constant. In other words, we have

$$k = \Delta^2/\tau.$$

We have shown that a Brownian walk with spatial step size Δ and time step τ will satisfy the commutator equation above exactly when the square of the spatial step divided by the time step remains constant. This means that *a given commutator equation can be satisfied by walks with arbitrarily small spatial step and time step, just so long as these steps are in this fixed ratio.*

Classical Point of View on the Diffusion Constant

$$x(t+\tau) = x(t) \pm \Delta$$

so that the time step is τ and the space step is of absolute value Δ. We regard the probability of left or right steps as equal, so that if $P(x,t)$ denotes the probability that the Brownian particle is at point x at time t then

$$P(x, t+\tau) = P(x-\Delta, t)/2 + P(x+\Delta, t)/2.$$

From this equation for the probability we can write a difference equation for the partial derivative of the probability with respect to time:

$$(P(x, t+\tau) - P(x,t))/\tau = (h^2/2\tau)[(P(x-\Delta, t) - 2P(x,t) + P(x+\Delta))/\Delta^2]$$

The expression in brackets on the right hand side is a discrete approximation to the second partial of $P(x,t)$ with respect to x. Thus if the ratio $C = \Delta^2/2\tau$ remains constant as the space and time intervals approach zero, then this equation goes in the limit to the diffusion equation

$$\partial P(x,t)/\partial t = C\partial^2 P(x,t)/\partial x^2.$$

C is called the diffusion constant for the Brownian process.

The appearance of the diffusion constant from the observational commutator shows that this ratio is fundamental to the structure of the Brownian process itself, and not just to the probabilistic analysis of that process.

Heisenberg/Schrödinger Equation. Here is how the Heisenberg form of Schrödinger's equation fits in this context. Let the time shift operator be given by the equation $J = (1 + H\Delta t/i\hbar)$. Then the non-commutative version of the discrete time derivative is expressed by the commutator

$$\nabla\psi = [\psi, J/\Delta t],$$

and we calculate

$$\nabla\psi = \psi[(1 + H\Delta t/i\hbar)/\Delta t] - [(1 + H\Delta t/i\hbar)/\Delta t]\psi = [\psi, H]/i\hbar,$$

$$i\hbar\nabla\psi = [\psi, H].$$

This is exactly the Heisenberg version of the Schrödinger equation.

Dynamics and Gauge Theory. One can take the general dynamical equation in the form

$$dX_i/dt = \mathcal{G}_i$$

where $\{\mathcal{G}_1, \cdots, \mathcal{G}_d\}$ is a collection of elements of $\mathcal{A}$. Write $\mathcal{G}_i$ relative to the flat coordinates via $\mathcal{G}_i = P_i - A_i$.

$$R_{ij} = [\mathcal{G}_i, \mathcal{G}_j]$$

$$= [P_i - A_i, P_j - A_j]$$

$$= -[P_i, A_j] - [A_i, P_j] + [A_i, A_j]$$

$$= \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

This is the well-known formula for the curvature of a gauge connection.

Curvature as Commutator

$$\nabla_i(F) = [F, \mathcal{G}_i],$$

then one has the curvature

$$[\nabla_i, \nabla_j]F = [R_{ij}, F]$$

and

$$R_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

Metric

Suppose we have elements g_{ij} such that

$$[g_{ij}, X_k] = 0$$

and

$$g_{ij} = g_{ji}.$$

We choose

$$H = \frac{(g_{ij}P_iP_j + P_iP_jg_{ij})}{4}.$$

This is the non-commutative analog of the classical $H = (1/2)g_{ij}P_iP_j$.

Then one calculates that

$$[X_i, \dot{X}_j] = g_{ij}.$$

Lemma 3. *Let* g_{ij} *be given such that* $[g_{ij}, X_k] = 0$ *and* $g_{ij} = g_{ji}$*. Define*

$$H = \frac{(g_{ij} P_i P_j + P_i P_j g_{ij})}{4}$$

(where we sum over the repeated indices) and

$$\dot{F} = [F, H].$$

Then

$$[X_i, \dot{X}_j] = g_{ij}.$$

Proof: Consider

$$\begin{aligned}[X_k, g_{ij} P_i P_j] &= g_{ij}[X_k, P_i P_j] \\ &= g_{ij}([X_k, P_i]P_j + P_i[X_k, P_j]) \\ &= g_{ij}(\delta_{ki} P_j + P_i \delta_{kj}) = g_{kj} P_j + g_{ik} P_i = 2g_{kj} P_j.\end{aligned}$$

Then

$$\begin{aligned}[X_r, \dot{X}_k] = [X_r, [X_k, H]] &= \left[X_r, \left[X_k, \frac{(g_{ij} P_i P_j + P_i P_j g_{ij})}{4}\right]\right] \\ &= \left[X_r, \left[X_k, \frac{(g_{ij} P_i P_j)}{4}\right]\right] + [X_r, [X_k, (P_i P_j g_{ij})/4]] \\ &= 2[X_r, 2g_{kj} P_j/4] = [X_r, g_{kj} P_j] = g_{kj}[X_r, P_j] = g_{kj}\delta_{rj} \\ &= g_{kr} = g_{rk}.\end{aligned}$$

This calculation actually shows that
the Hamiltonian H obeys the
constraint that

$$\dot{F} = [F,H] = (1/2)(\dot{X}_i [F , P_i] + [F ,P_i]\dot{X}_i).$$

Asking for higher order constraints
of this type gives deeper relationships.
For example, if we ask for a second order
constraint, then the metric must obey equations
that are a fourth-order version of Einstein's
equations.
(Joint work in preparation with
Tony Deakin and Clive Kilmister.)

Summary

$$\frac{dX_i}{dt} = \dot{X}_i = P_i - A_i = \mathcal{G}_i .$$

$$[\dot{X}_i, \dot{X}_j] = R_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$$

$$[X_i, \dot{X}_j] = [X_i, P_j] - [X_i, A_j] = \delta_{ij} - \frac{\partial A_j}{\partial P_i} = g_{ij}$$

Feynman-Dyson is case where
metric is Kronecker delta.

$$\nabla_i F = [F, P_i - A_i] = \partial_i(F) - [F, A_i] = [F, \dot{X}_i]$$

$$\hat{\partial}_i F = [X_i, F]$$

We assume $[X_i, g_{jk}] = 0$.

Levi-Civita Connection and Dynamics.

$$[X_i, \dot{X}_j] = g_{ij}.$$

Lemma. *Let* $\Gamma_{ijk} = (1/2)(\nabla_i g_{jk} + \nabla_j g_{ik} - \nabla_k g_{ij})$. *Then*

$$\Gamma_{ijk} = (1/2)\hat{\partial}_i \hat{\partial}_j \ddot{X}_k.$$

Proof.

$$\dot{g_{jk}} = [\dot{X}_j, \dot{X}_k] + [X_j, \ddot{X}_k]$$

$$\begin{aligned}
\hat{\partial}_i \hat{\partial}_j \ddot{X}_k &= [X_i, [X_j, \ddot{X}_k]] \\
&= [X_i, \dot{g_{jk}} - [\dot{X}_j, \dot{X}_k]] \\
&= [X_i, \dot{g_{jk}}] - [X_i, [\dot{X}_j, \dot{X}_k]] \\
&= [X_i, \dot{g_{jk}}] + [\dot{X}_k, [X_i, \dot{X}_j]] + [\dot{X}_j, [\dot{X}_k, X_i]] \\
&= -[\dot{X}_i, g_{jk}] + [\dot{X}_k, [X_i, \dot{X}_j]] + [\dot{X}_j, [\dot{X}_k, X_i]] \\
&= \nabla_i g_{jk} - \nabla_k g_{ij} + \nabla_j g_{ik} \\
&= 2\Gamma_{kij}.
\end{aligned}$$

One finds that

$$\ddot{X}_r = G_r + F_{rs}\dot{X}^s + \Gamma_{rst}\dot{X}^s\dot{X}^t,$$

where G_r is the analogue of a scalar field, F_{rs} is the analogue of a gauge field and Γ_{rst} is the Levi-Civita connection associated with g_{ij}.

The Levi-Civita Connection
appears as a direct consequence
of the Lebniz rule and the
Jacobi identity.

Classical physics contains part of
the explanation, since a
particle moving in general
coordinates and obeying Hamilton's equations
moves in a geodesic described by the
Levi-Civita connection.

This derivation of the
Levi-Civita connection
suggests a reformulation
of
differential geometry
where the notion of parallel translation
is secondary to the
dynamics of non-commutativity.

Generalized Feynman Dyson Derivation

In this section we assume that specific time-varying coordinate elements X_1, X_2, X_3 of the algebra $\mathcal{A}$ are given. *We do not assume any commutation relations about* X_1, X_2, X_3.

We define fields B and E by the equations

$$B = \dot{X} \times \dot{X} \text{ and } E = \partial_t \dot{X}.$$

Here $A \times B$ is the non-commutative vector cross product:

$$(A \times B)_k = \Sigma_{i,j=1}^{3} \epsilon_{ijk} A_i B_j.$$

We show that E and B satisfy a generalization of the Maxwell equations.

We take

$$\partial_i(F) = [F, \dot{X}_i],$$

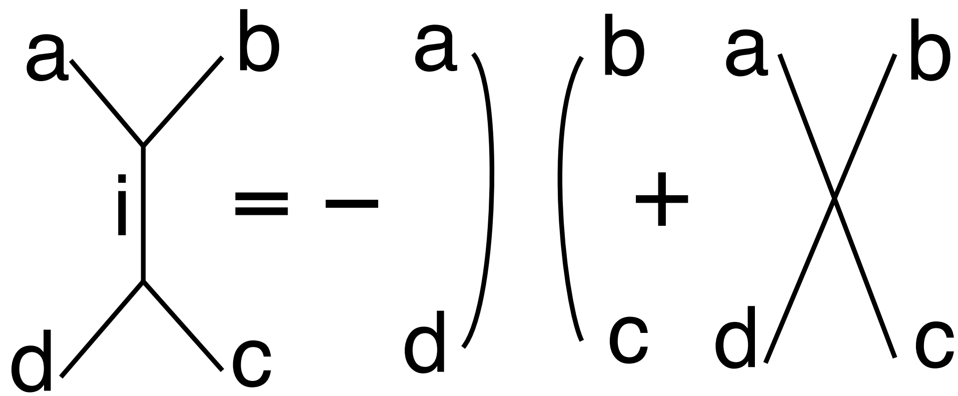
a covariant derivative.

In defining

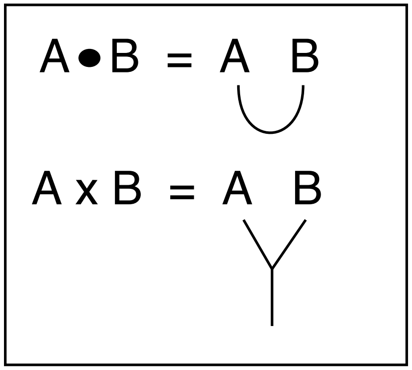
$$\partial_t F = \dot{F} - \Sigma_i \dot{X}_i \partial_i(F),$$

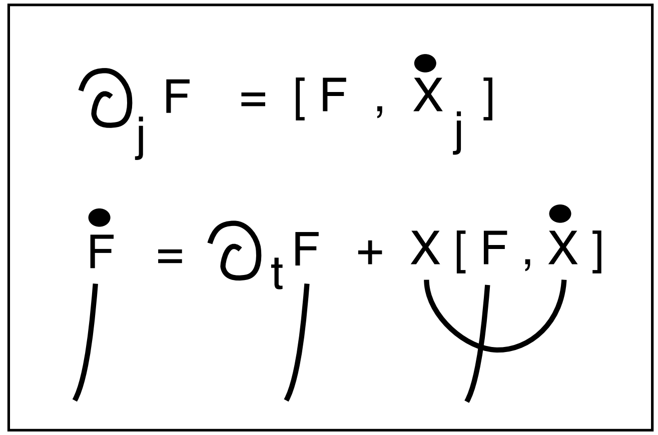
we are using the definition itself to obtain a notion of the variation of F with respect to time. The definition itself creates a distinction between space and time in the non-commutative world.

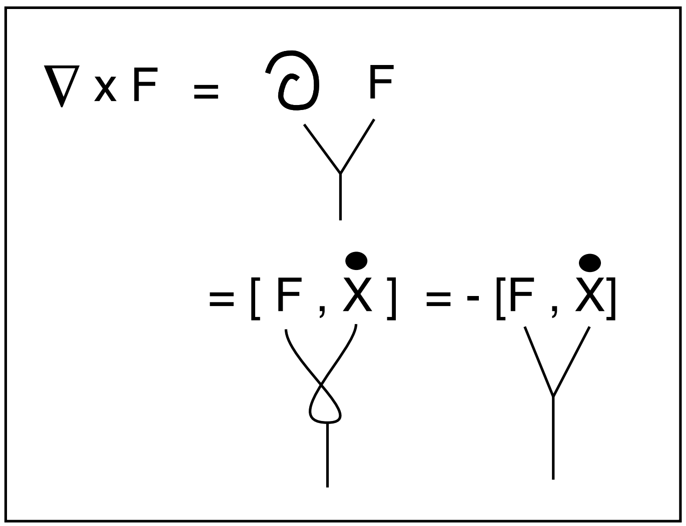
The Epsilon Identity



$$\Sigma_i \, \epsilon_{abi}\epsilon_{cdi} = -\delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd}.$$


A • B = A B
A x B = A B


∂j F = [F , Ẋ j]
Ḟ = ∂t F + X [F , Ẋ]


∇ x F = ∂ F
= [F , Ẋ] = - [F , Ẋ]

$$\dot{F} = \partial_t F + \dot{X}[F, X]$$

$$\dot{F} = \partial_t F + \dot{X} F X - \dot{X} X F$$

$$\ddot{X} = \partial_t \dot{X} + \dot{X}\dot{X}\dot{X} - \dot{X}\dot{X}\dot{X}$$

$$= \partial_t \dot{X} + \dot{X}\dot{X}\dot{X}$$

$$\boxed{\ddot{X} = \partial_t \dot{X} + \dot{X} \times (\dot{X} \times \dot{X})}$$

$$E = \partial_t \dot{X} \qquad B = \dot{X} \times \dot{X}$$

$$\boxed{\dot{X} = E + \dot{X} \times B}$$

$$\nabla \bullet B = [B, \dot{X}]$$

$$= B\dot{X} - \dot{X}B = \dot{X}\dot{X}\dot{X} - \dot{X}\dot{X}\dot{X} = 0$$

$$\nabla \bullet B = 0$$

$$\partial_t B = \dot{B} + \dot{X}\,[\,\dot{X}\,,\,B\,]$$

$$\boxed{\begin{aligned} \dot{B} &= (1/2)[\,\dot{X}\,,\,\dot{X}\,]^{\bullet} = [\,\ddot{X}\,,\,\dot{X}\,] \\ &= [\,E\,,\,\dot{X}\,] + [\,\dot{X} \times B\,,\,\dot{X}\,] \\ &= -\nabla \times E + [\,\dot{X}\,B\,,\,\dot{X}\,] \end{aligned}}$$

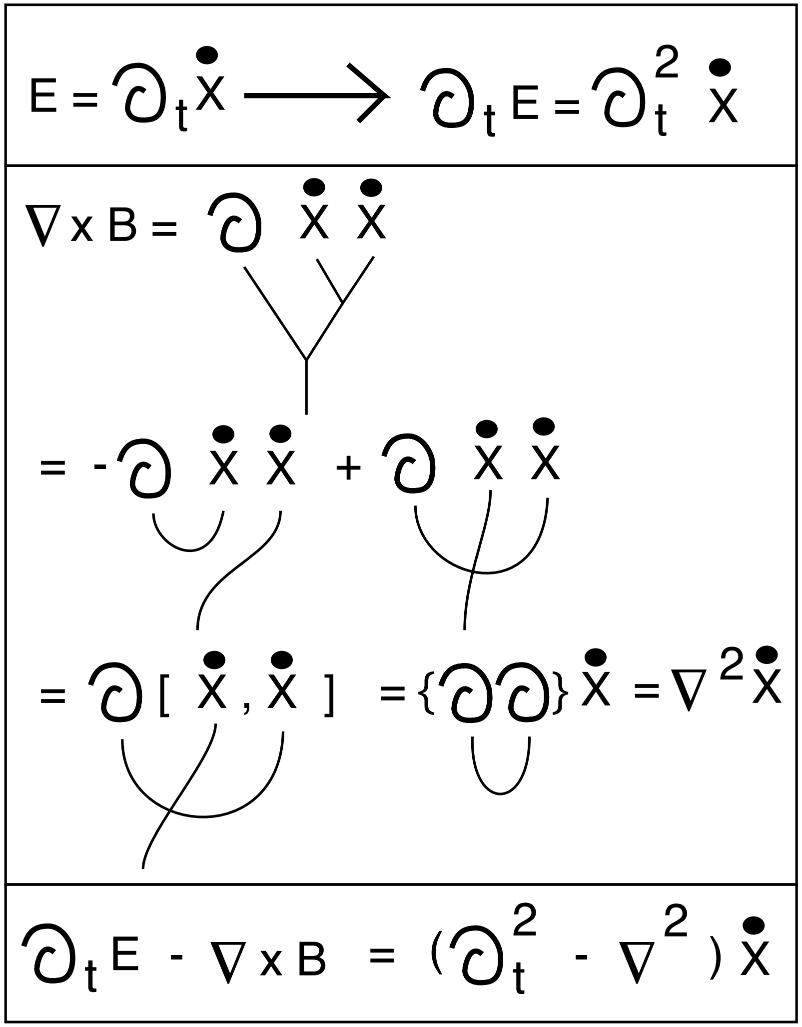
$$\partial_t B + \nabla \times E = \dot{X}\,[\dot{X}, B] + [\dot{X}\,B\,,\,\dot{X}\,]$$

$$= \dot{X}\,[\dot{X}, B] + [\dot{X}\,B\,,\,\dot{X}\,] + [\dot{X}\,B\,,\,\dot{X}\,]$$

$$= -\dot{X}\dot{X}B + \dot{X}\dot{X}B$$ (Note that $\dot{X}\,B = B\,\dot{X}$)

$$= \dot{X}\,\dot{X}\,B = B \times B$$

$$\boxed{\partial_t B + \nabla \times E = B \times B}$$



$E = \partial_t \dot{X} \longrightarrow \partial_t E = \partial_t^2 \dot{X}$
$\nabla \times B = \partial \dot{X} \dot{X}$
$= -\partial \dot{X} \dot{X} + \partial \dot{X} \dot{X}$
$= \partial [\dot{X}, \dot{X}] = \{\partial \partial\} \dot{X} = \nabla^2 \dot{X}$
$\partial_t E - \nabla \times B = (\partial_t^2 - \nabla^2) \dot{X}$

Electromagnetic Theorem With the above definitions of the operators, and taking

$\nabla^2 = \partial_1^2 + \partial_2^2 + \partial_3^2$, $B = \dot{X} \times \dot{X}$ and $E = \partial_t \dot{X}$ we have

1. $\ddot{X} = E + \dot{X} \times B$
2. $\nabla \bullet B = 0$
3. $\partial_t B + \nabla \times E = B \times B$
4. $\partial_t E - \nabla \times B = (\partial_t^2 - \nabla^2)\dot{X}$

(B x B is not always zero in discrete models.)

Discrete Models.

X is a vector of a three dimensional time series.

$$\dot{F} = J(F' - F) = [F, J]$$

$$\Delta(F) = F' - F.$$

$$\dot{F} = J\Delta(F) \;\; , \;\; \Delta_i = X_i' - X_i$$

$$\partial_i(F) = [F, \dot{X}_i] = [F, J\Delta_i] = FJ\Delta_i - J\Delta_i F$$

$$= J(F'\Delta_i - \Delta_i F)$$

$$\partial_t F = J[1 - J\Delta' \bullet \Delta]\Delta(F)$$

$$R_{ij} = [\dot{X}_i, \dot{X}_j] = X_i J\Delta_j - J\Delta_j X_i$$

$$= J(X_i'\Delta_j - \Delta_j X_i) = J\Delta_i\Delta_j$$

$$B = \dot{X} \times \dot{X} = J^2\Delta(X') \times \Delta(X)$$

$$E = \ddot{X} - \dot{X} \times (\dot{X} \times \dot{X}) = J^2\Delta^2(X) - J^3\Delta(X'') \times (\Delta(X') \times \Delta(X))$$

Next:
The Non-Commutative
World of a Knot in Three - Space