Fourier Knots

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1 Introduction

This paper introduces the concept of Fourier knot. A Fourier knot is a knot that is represented by a parametrized curve in three dimensional space such that the three coordinate functions of the curve are each finite Fourier series in the parameter. That is, the knot can be regarded as the result of independent vibrations in each of the coordinate directions with each of these vibrations being a linear combination of a finite number of pure frequencies.

The previously studied Lissajous knots [1] constitute the case of a single frequency in each coordinate direction. Not all knots are Lissajous knots, and in fact the trefoil knot and the figure eight knot are the first examples of non-Lissajous knots. The first section of this paper sketches the proof that every tame knot is a Fourier knot. Subsequent sections give robust examples of Fourier representations for the trefoil, the figure eight and a class of knots that we call Fibonacci knots. In the case of the trefoil we have given a minimal Fourier representation in the sense that it has single frequencies in two of the coordinate directions and a combination of frequencies in the third direction. The paper ends by pointing out the usual compact non-linear trigonometric formula for torus knots, and raises the question of the finite Fourier series representations for these knots.

On completing an early draft of this paper, we learned that an extensive study of Fourier knots (there called Harmonic Knots) has been carried out by Aaron Trautwein in his 1995 PhD Thesis [7] at the University of Iowa under the direction of Jon Simon. While the independently obtained results of the present paper are primarily illustrative of the idea of Fourier knots, Trautwein's pioneering work establishes relationships between the complexity of the harmonic representation and knot theoretic indices such as crossing number and superbridge index. The interested reader should consult this work.

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2 Every Knot is a Fourier Knot

In considering problems about knots it is interesting to have an equation for the knot or link under consideration. By an equation for a knot I mean the specification of a parametrized curve in three dimensional space of the form

$$X(t) = A(t)$$
$$Y(t) = B(t)$$
$$Z(t) = C(t)$$

where A, B, C are smooth functions of the variable t with (for a knot) a total period of $P > 0$, specifying an embedding of the circle into three dimensional space. Here the circle is the quotient space of the interval from 0 to P , $[0, P]$, obtained by identifying the ends of the interval to each other with the quotient topology.

A function $F(t)$ is said to be a finite Fourier series if it has the form

$$F(t) = A_1 Cos(K_1 T + L_1) + ... + A_N Cos(K_N T + L_N)$$

where A_1,A_2,...A_N and K_1,K_2,...K_N and L_1,L_2,L_3,...L_N are given constants and K_1,K_2,...K_N are each rational numbers. Of course, $F(t)$ can also be expressed in terms of the Sin function or as a combination of Sin and Cos functions, since $Sin(X + \pi/2) = Cos(X)$.

Definition. A knot K embedded in three dimensional space will be called a *Fourier knot* if it has an equation (as described above) with each of the functions A,B,C a finite Fourier series.

Definition. A knot K is said to be *tame* if every point p of K in R^3 (Euclidean three-space) has a neighborhood such that the intersection of K with that neighborhood is equivalent to a standard pair of three-dimensional ball and diameter-arc of that ball.

It is well known that every tame knot can be represented topologically by equations where A,B and C are smooth (i.e. infinitely differentiable) functions. It then follows by standard approximation theorems for Fourier series that A, B and C can be taken to be finite Fourier series. Thus we have proved the

Theorem. Every tame knot is (topologically equivalent to) a Fourier knot.

3 Lissajous Knots and the Arf Invariant

One class of Fourier knots that have been studied are the Lissajous knots [1]. In a Lissajous knot there is one term in each of the Fourier series. Thus a Lissajous knot has the form

$$X(t) = A_1 Cos(K_1 t + L_1)$$
$$Y(t) = A_2 Cos(K_2 t + L_2)$$
$$Z(t) = A_3 Cos(K_3 t + L_3)$$

In [1] it is proved that the Arf invariant of a Lissajous knot is neccessarily equal to zero. This means that many knots are not Lissajous. In particular the trefoil knot and the figure eight knot are not Lissajous knots. This leads to the question: If a tame knot K is not Lissajous, what is the "simplest" representation of K in terms of finite Fourier series?

In the next section we shall give a definite answer to this question in the case of the trefoil knot, and a conjecture in the case of the figure eight knot. In all cases, when one answers this question there is also the parallel question of obtaining Fourier equations for the knot K that are robust in the sense that plots of these equations yield pleasing images that can be explored geometrically.

Since knots of non-zero Arf invariant are neccessarily not Lissajous, it will be useful for us to recall one definition of the Arf invariant. An interested reader can apply this definition to find examples of Fourier knots that are not Lissajous knots. There are algebraic definitions of the Arf invariant and a fundamental geometric definition as well. See [2] or [3] for more details.

We recall an algebraic definition of the Arf invariant by first defining an integer valued invariant, $a(K)$, associated to any oriented knot K. The invariant $a(K)$ is defined by the (recursive) equation (*)

$$a(K_+) = a(K_-) = Lk(K_0)$$

where K_+, K_- and K_0 are three diagrams that differ at a single crossing as shown in Figure 1, and $Lk(K_0)$ denotes the linking number of the link of two components K_0. K_+ and K_- are each knot diagrams, differing from each other by a single switched crossing. K_0 is obtained from either K_+ or K_- by smoothing that crossing. A *smoothing* is accomplished by reconnecting the strands at the crossing so that the arcs no longer cross over one another (as shown in Figure 1). Both the switching and the smoothing operations can change the topological type of the diagram. Smoothing always replaces a knot by a link of two components. Thus K_0 is a such a link. By definition, $a(K)$ is equal to zero if K is topologically equivalent to an unknotted circle.

Note that in Figure 1 we have implicitly assigned signs of $+1$ or -1 to the two types of oriented crossings. This number, $+1$ or -1, is called the sign of the crossing. The linking number of a link L is defined by the equation

$$Lk(L) = \Sigma_p e(p)/2$$

where the summation runs over all the crossings in K that are between two components of K. Crossings of any given component with itself are not counted in this summation.

It is a (non-obvious) fact that the recursive equation (*) defines a topological invariant of knots and links. It is, in fact part of a much larger scheme

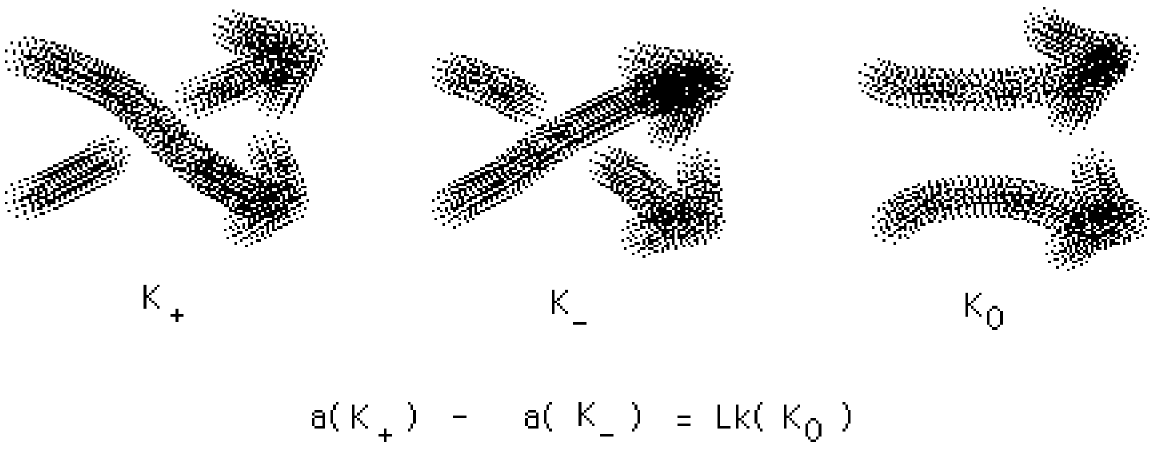


Figure 1: Switching Relation

of things. For example it is the second coefficient of the Conway polynomial. One way to define the Arf invariant, $Arf(K)$, is by the equation

$$Arf(K) = a(K)(mod 2).$$

Thus the Arf invariant of K is either 0 or 1 depending upon the parity of $a(K)$. See Figure 2 for a sample calculation of the Arf invariant of the trefoil knot.

It is a remarkable fact that Lissajous knots have Arf invariant zero. I do not know if every knot of vanishing Arf invariant is a Lissajous knot.

4 A Fourier Trefoil Knot

Consider the following equations

$$x = Cos(2T),$$
$$y = Cos(3T + (1/2)),$$

$$a(T) - a(U) = Lk(L)$$
$$a(T) - 0 = 1$$
$$a(T) = 1$$

Figure 2: a(Trefoil)

$$z = (1/2)Cos(5T + (1/2)) + (1/2)Sin(3T + (1/2)).$$

These equations define a trefoil knot, showing that the trefoil knot is a Fourier knot where only one coordinate needs to be a combination of frequencies. The proof that these equations give a trefoil knot is left to the reader. One way to verify this is to use a computer to draw the pictures in three dimensions and then examine the results. Figure 3 illustrates a computer drawing of this Fourier trefoil. The drawing illustrates what I mean by a robust representation of the knot. The knot does not come ambiguously close to itself, and the form of the drawing is aesthetically pleasing.

The author wishes to acknowledge Lynnclaire Dennis [4] for inspiring him to search for the Fourier trefoil. In her book "The Pattern" Ms. Dennis draws a picture of a knot (the Pattern knot) that closely resembles our Fourier trefoil. In projection the Pattern knot looks like a Lissajous figure with frequencies 2 and 3 and the Pattern knot is a trefoil knot. This led to trying combinations of frequencies for the third coordinate, and eventually to the equations above with pure frequencies 2 and 3 in two directions and the combination of frequencies 5 and 3 in the third direction. The Pattern knot

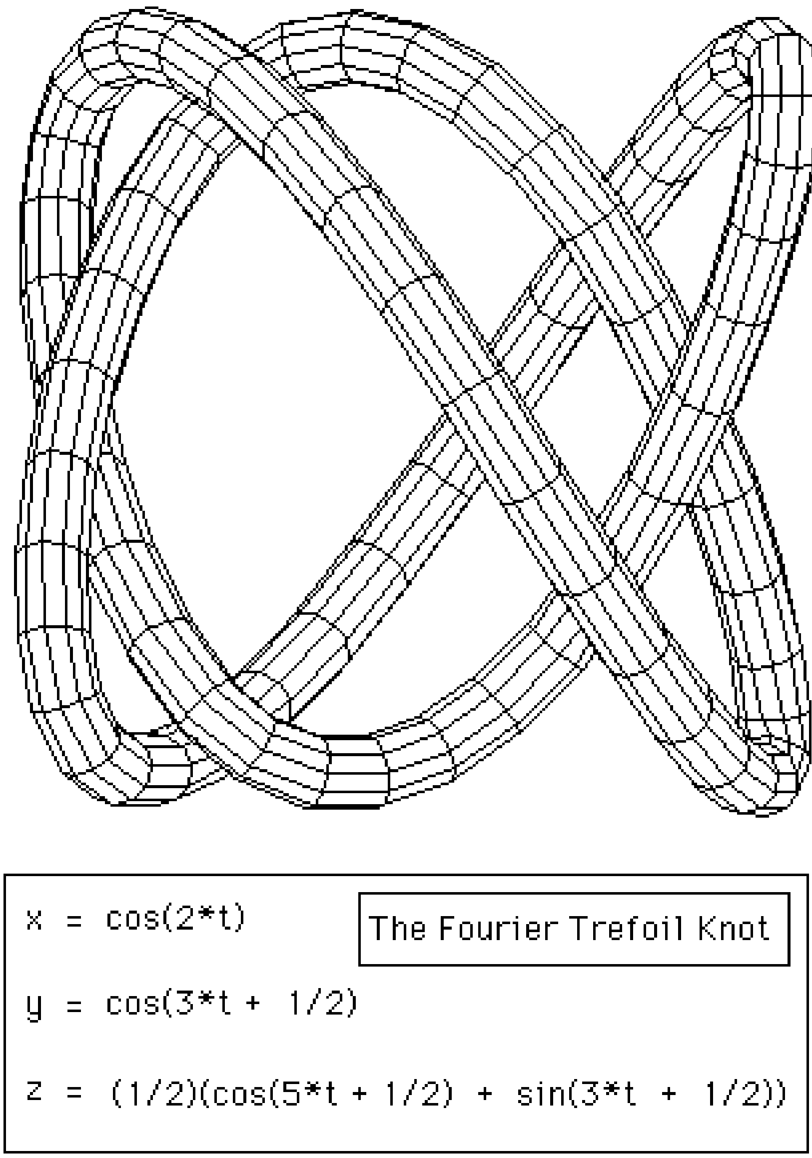


Figure 3: The Fourier Trefoil

is more spherically symmetrical than the Fourier trefoil, and does not have an obvious equation.

I would also like to mention an experiment that I performed with the Fourier trefoil in the form of a (hand-drawn) diagram corresponding to the knot in Figure 3. I gave this diagram as input to Ming, a knot energy program written by Ying-Quing Wu at the University of Iowa. (A diagrammatic interface for Ming was written by Milana Huang at the Electronic Visualisation Lab at the University of Illinois). Ming sets the knot on a descending energy trajectory, following Jon Simon's energy [5] for the knot. The result of this experiment is that the flat knot diagram quickly unfurls into a three dimensional geometry very similar to the Fourier trefoil and nearly stabilizes in this form. Then slowly the knot moves off this slightly higher energy level and settles into the familiar symmetry of the (empirically) known energy minimum for the trefoil knot. Thus there appears to be a "point of inflection" in this particular way of descending to minimum energy for the trefoil knot. This experiment points to a wide range of possible explorations, investigating the gradient descent for knot energy from particular starting configurations for a knot.

5 A Fourier Figure Eight Knot

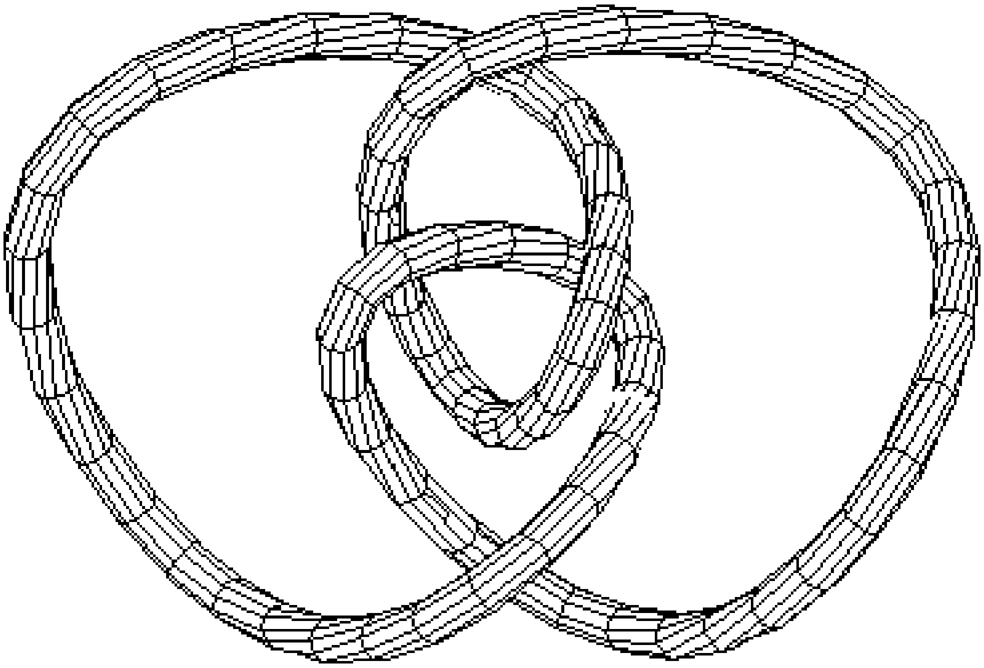
The following equations describe a figure eight knot.

$$x = Cos(t) + Cos(3t),$$
$$y = .6Sin(t) + Sin(3t),$$
$$z = .4Sin(3t) - Sin(6t).$$

See Figure 4.
I do not know if there is a simpler Fourier representation for this knot.

6 A Series of Fibonacci Fourier Knots

Recall the Fibonacci series



X = 10*COS(T) + 10*COS(3*T)

Y = 6*SIN(T) + 10*SIN(3*T)

Z = 4*SIN(3*T) - 10*SIN(6*T)

Figure Eight Knot

Figure 4: The Fourier Figure Eight Knot

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...$$

The n-th term, f_n, of the series is equal to the sum of the previous two terms, and $f_1 = f_2 = 1$. Consider the equations

$$x = Cos(8T),$$
$$y = Cos(13T + .5),$$
$$z = .5Cos(21T + .5) + .5Sin(13T + .5),$$

and more generally

$$x = Cos(f_n T),$$
$$y = Cos(f_{n+1}T + .5),$$
$$z = .5Cos(f_{n+2}T + .5) + .5Sin(f_{n+1}T + .5).$$

The last set of equations defines a knot that we shall dub $F(n)$, the n-th Fibonacci knot. Thus $F(3)$ is the Fourier trefoil of Section 3, and the first equations we have written in this section denote $F(6)$. In Figure 5 we illustrate a computer drawing of $F(6)$.

The Fibonacci Fourier knots provide a strong class of knots for investigation using computer graphics.

7 Torus Knots

Recall that a knot that winds P times around a torus in one direction and Q times in the other direction - a torus knot of type (P, Q) - has the equation

$$x = Cos(T)(1 + .5Cos((Q/P)T)),$$
$$y = Sin(T)(1 + .5Cos((Q/P)T)),$$
$$z = .5Sin((Q/P)T).$$

Now use the trigonometric identities

$$cos(a)cos(b) = .5(cos(a + b) + cos(a - b)),$$

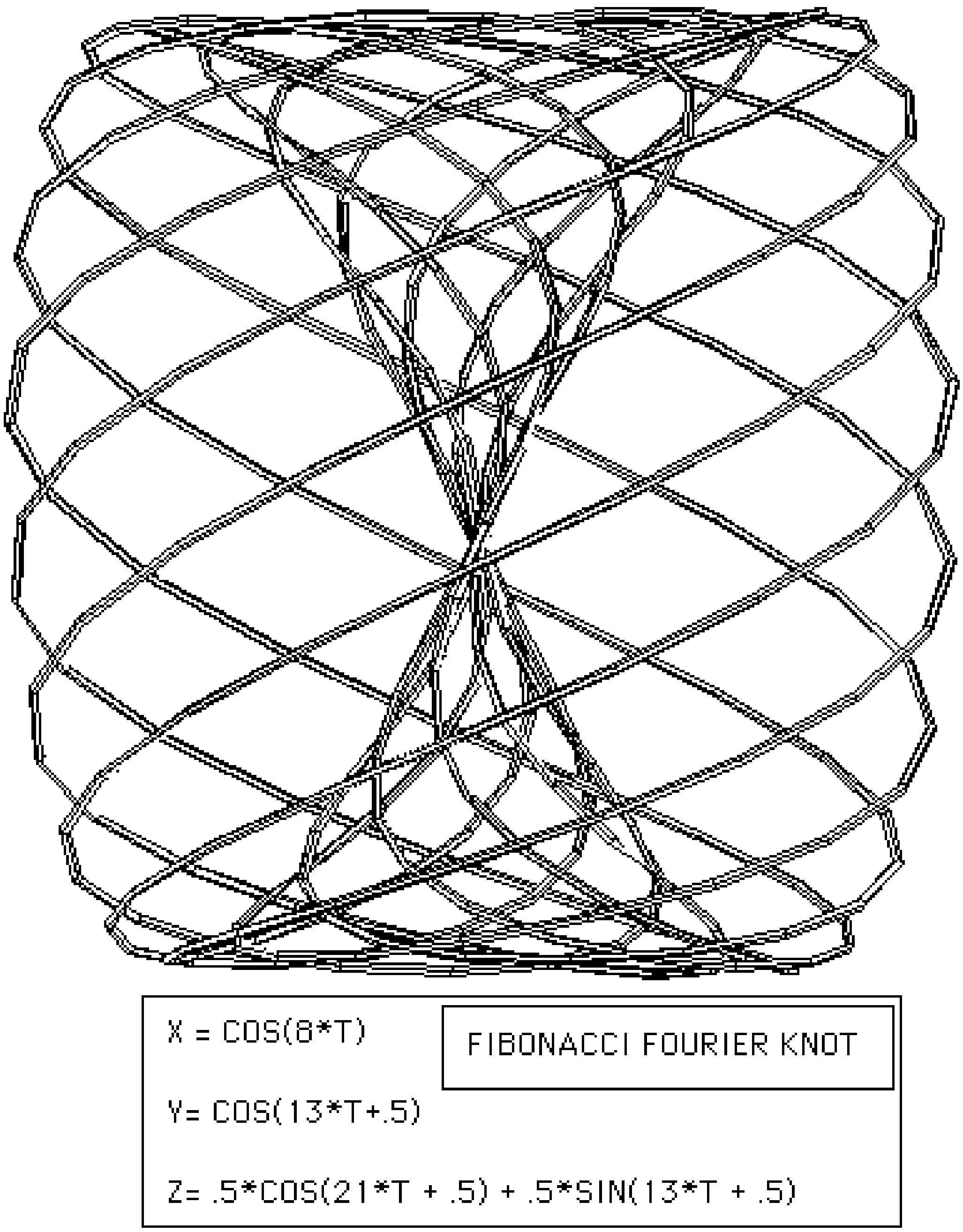


Figure 5: A Fibonacci Fourier Knot

$$sin(a)cos(b) = .5(sin(a+b) + sin(a-b)).$$

The equations above then become

$$x = cos(T) + .25cos((1+Q/P)T) + .25cos((1-Q/P)T)$$
$$y = sin(T) + .25 * sin((1+Q/P)T) + .25 * sin((1-Q/P)T)$$
$$z = .5sin((Q/P)T)$$

Thus torus knots are Fourier knots, and we can ask if these are simplest Fourier representations for torus knots. The parametrization shown above appears in [6]. I am indebted to Peter Roegen for pointing this out.

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