

PEANO ARITHMETIC

LOUIS H. KAUFFMAN

ABSTRACT. *These notes describe Peano Arithmetic by using a model based on nested Marks. A Mark is a right-angle bracket in the form $\overline{\quad}$ and we identify*

$$1 = \overline{\quad}, 2 = \overline{\overline{\quad}}, 3 = \overline{\overline{\overline{\quad}}}, \dots$$

We axiomatize arithmetic based on this notation, and relate it to the formalism in Spencer-Brown's "Laws of Form".

1. INTRODUCTION

In these notes, we shall consider an elementary arithmetic for nests of Marks. A Mark is a right-angle bracket in the form $\overline{\quad}$ and we identify

$$1 = \overline{\quad}, 2 = \overline{\overline{\quad}}, 3 = \overline{\overline{\overline{\quad}}}, \dots$$

We axiomatize arithmetic based on this notation as follows.

The Axioms for Peano Arithmetic.

- (1) There is a set N with $\overline{\quad} \in N$, and if $x \in N$, then $\overline{x} \in N$.
- (2) $\overline{x} \neq \overline{\quad}$ for all $x \in N$.
- (3) If $\overline{x} = \overline{y}$, then $x = y$.
- (4) Let A be a subset of N such that $\overline{\quad} \in A$. Suppose that whenever $x \in A$ then $\overline{x} \in A$. Then $A = N$.

The last axiom is the Principle of Mathematical Induction for the set N . We then define addition and multiplication of elements of N inductively as follows.

- (1) $x + 1 = \overline{x}$ for any $x \in N$.
- (2) $x + \overline{y} = \overline{x + y}$ for any $x, y \in N$.
- (3) $x \overline{\quad} = x$ for any $x \in N$.
- (4) $x \overline{y} = xy + x$ for any $x, y \in N$.

Note that we have not made the associative or the commutative laws axioms of the system. We shall prove associativity and commutativity from these axioms. The axioms are designed so that they speak about the elements of N without assuming any methods for constructing these elements. Since we have the mark-notations for the elements, we can make constructions and we can see our own limitations in actually drawing nests of

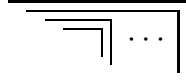
2000 *Mathematics Subject Classification.* 57R40 .

Key words and phrases. Mark, Arithmetic.

The author was partially supported by UIC.

marks. The axioms are about the mathematics of these constructions and do not have to assume arbitrary constructions. Thus the axioms will be an adequate mathematical theory for arithmetic independent of how we choose to represent that arithmetic. For example, a computing machine can handle numbers much larger than the numbers that are available to an unaided human being, but the axioms will apply to the computer just as they apply to our hand calculations.

It is because we want the axioms to be universally available, that we do not begin by saying that the general form of a natural number is



where there are finitely many marks in the nest. This is a correct description of the numbers, but we need to use the principle of mathematical induction to define such forms and to capture the notion of finite. Without that, we are left to think of finite as a temporal sequence of operations that terminates after a (sic) finite number of steps. To avoid all these implicit logical circularities, we take the axioms in the inductive form and work out definitions of the more intuitive views of the numbers. As we progress, methods of set theory will allow us more flexibility in dealing with numbers and we will see how to extend our reach to all the integers, the fractions, and the real and complex numbers.

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE, UNIVERSITY OF ILLINOIS AT CHICAGO, 851 SOUTH MORGAN ST., CHICAGO IL 60607-7045, U.S.A.

E-mail address: kauffman@math.uic.edu

URL: <http://www.math.uic.edu/~kauffman/>