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Applications of Negative Dimensional Tensors

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I wish to describe a theory of "abstract tensor systems" (abbreviated ATS) and indicate some applications. Unfortunately I shall only be able to give a very brief outline of the general theory here.†

I take as my model, the conventional tensor index notation with Einstein's summation convention, which has become so familiar in physics and in what is now referred to as "old fashioned" differential geometry. The elements of an ATS may be denoted by kernel symbols with indices in a way formally identical with the tensor index notation, but now the meanings of the indices are to be quite different. This will enable more general types of object than ordinary tensors to be considered. Some of these (for example, "negative dimensional" tensors) will not be representable in terms of components in the ordinary way.

Each index is to be simply a *label* and does not stand for, say, 1, 2, ..., n. Thus an element ξ^a (a "vector") of an ATS is not a set of components, but a single element of a vector space (or module) $\mathscr{T}^a$ over a field (or ring) $\mathscr{T}$. Since I wish to mirror the ordinary index notation and allow expressions such as $\xi^a \xi^b$ or $\xi^a \eta^b - \eta^a \xi^b$, for example, I shall also need an element ξ^b distinct from ξ^a, and so on. Thus we need another vector space (or module) $\mathscr{T}^b$ which will be canonically isomorphic with $\mathscr{T}^a$, etc. etc. Let me define the labelling set $\mathscr{L}$:

$$\mathscr{L} = (a, b, c, \ldots, z, a_0, b_0, \ldots, a_1, \ldots),$$

supposed infinite. The elements of $\mathscr{L}$ are to be the allowed "abstract indices". We shall then have an infinite class of canonically isomorphic modules

$$\mathscr{T}^a \cong \mathscr{T}^b \cong \ldots\ldots$$

where corresponding elements are denoted by $\xi^a, \xi^b, \ldots$. Thus $\lambda\xi^a + \mu\eta^a = \zeta^a$ iff $\lambda\xi^b + \mu\eta^b = \zeta^b$, etc. where $\lambda, \mu \in \mathscr{T}$; $\xi^a, \eta^a, \zeta^a \in \mathscr{T}^a$; $\xi^b, \eta^b, \zeta^b \in \mathscr{T}^b$; etc. But

† A more extended account is to be found in [4]. However, the theory is there made unnecessarily complicated, because of an inconvenient (but apparently natural) choice having been made in connection with the notation. The point is avoided here by the use of infinitely many canonically isomorphic copies of each vector space or module. See [5] in connection with the approach used here. Also, compare [3] as regards diagrammatic notation.

in addition we shall need to mirror the other tensor operations (e.g. outer product, contraction) and thus to consider many-indexed quantities which look ($a, b, \ldots, d, f, \ldots, h$ being all distinct) like

$$\chi^{ab\ldots d}_{f\ldots h}.$$

Such quantities will be elements of sets $\mathscr{T}^{ab\ldots d}_{f\ldots h}$, each of these sets again being a module over $\mathscr{T}$. (The ordering of $a, b, \ldots d$ and of $f, \ldots, h$ is of significance for each $\chi^{ab\ldots d}_{f\ldots h}$ but not for the *sets* $\mathscr{T}^{a\ldots d}_{f\ldots h}$. That is: $\mathscr{T}^{ab\ldots d}_{f\ldots h} = \mathscr{T}^{db\ldots a}_{h\ldots f}$, for example, but generally $\chi^{ab\ldots d}_{f\ldots h} \neq \chi^{db\ldots a}_{h\ldots f}$).

The entire tensor system $\{\mathscr{T}\}$ will have, as elements, the members of the sets

$$(\mathscr{T}, \mathscr{T}^a, \mathscr{T}^b, \ldots, \mathscr{T}^{a_0}, \ldots, \mathscr{T}_a, \mathscr{T}_b, \ldots, \mathscr{T}^a_b, \ldots, \mathscr{T}^{xm}_p, \ldots).$$

There are four basic operations on $\{\mathscr{T}\}$. These are

addition: $\quad \mathscr{T}^{x\ldots z}_{u\ldots w} + \mathscr{T}^{x\ldots z}_{u\ldots w} \to \mathscr{T}^{x\ldots z}_{u\ldots w}$

(outer) multiplication: $\quad \mathscr{T}^{a\ldots d}_{p\ldots r} \times \mathscr{T}^{x\ldots z}_{u\ldots w} \to \mathscr{T}^{a\ldots dx\ldots z}_{p\ldots ru\ldots w}$

contraction, e.g. $\mathscr{C}^p_q$: $\quad \mathscr{T}^{px\ldots z}_{qu\ldots w} \to \mathscr{T}^{x\ldots z}_{u\ldots w}$

index substitution: $\quad \mathscr{T}^{x\ldots z}_{u\ldots w} \to \mathscr{T}^{f\ldots h}_{k\ldots m}$.

In the first three of these operations, the differently denoted index letters appearing are all assumed to be different elements of $\mathscr{L}$. In the final operation of index substitution, $x, \ldots, z, u, \ldots, w$ of $\mathscr{L}$ are all distinct and so are $f, \ldots, h, k, \ldots, m$. Otherwise they are unrestricted except that $x, \ldots, z$ and $f, \ldots, h$ are equal in number and that $u, \ldots, w$ and $k, \ldots, m$ are equal in number.

The axioms are: *First* that addition defines an Abelian group structure for each $\mathscr{T}^{\cdots}_{\cdots}$. *Second*, multiplication is distributive over addition and commutative.† (Non-commutative systems can also be considered, but I shall not concern myself with them here.) *Third*, contraction is distributive over addition and it appropriately commutes with multiplication and with other contractions, e.g.

$$\mathscr{C}^a_b(\chi^{\cdot\cdot a\cdots}_{\cdot\cdot b\cdots} + \phi^{\cdot\cdot a\cdots}_{\cdots\cdot b\cdot}) = \mathscr{C}^a_b(\chi^{\cdot\cdot a\cdots}_{\cdot\cdot b\cdots}) + \mathscr{C}^a_b(\phi^{\cdot\cdot a\cdots}_{\cdots\cdot b\cdot}),$$

$$\mathscr{C}^a_b(\chi^{\cdot\cdot a\cdots}_{\cdot\cdot b\cdots}\psi^{\cdots\cdots}_{\cdots\cdots}) = \mathscr{C}^a_b(\chi^{\cdot\cdot a\cdots}_{\cdot\cdot b\cdots})\psi^{\cdots\cdots}_{\cdots\cdots},$$

$$\mathscr{C}^a_b(\mathscr{C}^c_d(\chi^{\cdot\cdot a\ldots c\cdots}_{\cdot\cdot d\cdots\cdots b\cdot})) = \mathscr{C}^c_d(\mathscr{C}^a_b(\chi^{\cdot\cdot a\ldots c\cdots}_{\cdot\cdot d\cdots\cdots b\cdot})).$$

† This is not really inconsistent with the usual algebraic definition of tensor product, which is formally non-commutative. Here, the *labels* cope with the ordering in a tensor product. Thus, whereas $\xi^a\eta^b = \eta^b\xi^a$, we have $\xi^a\eta^b \neq \eta^a\xi^b$.

Fourth, an index substitution is effected by any permutation of $\mathscr{L}$ and leaves the validity of any formula unaltered, e.g. if

$$\chi_c^{ab} = \phi_c^a \eta^b + \mathscr{C}_q^p (\theta^{bpa} \zeta_{cq}),$$

then

$$\chi_b^{xn} = \phi_b^x \eta^n + \mathscr{C}_a^q (\theta^{nqx} \zeta_{ba}).$$

etc. The "dummy index" notation will be used for contraction. Thus we can write the above

$$\chi_c^{ab} = \phi_c^a \eta^b + \theta^{bpa} \zeta_{cp},$$

for example, so that repeated indices (one upper and one lower) do not contribute to the total "label" $\{{}^{ab}_{c}\}$ of the tensor. These operations and axioms define an *abstract tensor system* or ATS.

Let me remark, in passing, one way of generating new tensor systems from old. This is by means of the device of "clumping indices together". Thus, if we consider a new labelling system $\widetilde{\mathscr{L}} = (\mathscr{A}, \mathscr{C}, \mathscr{E}, \ldots, \mathscr{A}_0, \ldots\ldots)$ where, say, $\mathscr{A} = (a, b), \mathscr{C} = (c, d), \mathscr{E} = (e, f), \ldots, \mathscr{A}_0 = (a_0, b_0), \ldots,$ then we can construct a new ATS $\{\widetilde{\mathscr{T}}\}$ whose elements belong to a subset of $\{\mathscr{T}\}$ namely $\widetilde{\mathscr{T}} = \mathscr{T}$, $\widetilde{\mathscr{T}}^{\mathscr{A}} = \mathscr{T}^{ab}, \ldots$. More generally, we can also clump together upper *and* lower indices as a single compound index. (For example $\widetilde{\mathscr{T}}^{\mathscr{A}} = \mathscr{T}^{a}_{bc}$, $\mathscr{T}^{\mathscr{D}} = \mathscr{T}^{d}_{ef}, \ldots, \widetilde{\mathscr{T}}_{\mathscr{A}} = \mathscr{T}^{bc}_{a}, \ldots$. Here, each script letter stands for three elements of $\mathscr{L}$ two of which are in reverse position.) The axioms for the new ATS are satisfied in consequence for those for the old. We can also consider ATS's with indices of differing types (e.g. $\chi^{ab}{}_{\sigma}{}^{\alpha AB}$) where contractions and index substitutions are not permitted between indices of different types. If we consider different kinds of clumping operations applied simultaneously to an ATS, then each different kind of clumping plays the role of a different type of index in the resulting new ATS.

If desired, we may require of an ATS that it possesses "unit elements". For example, an element $1 \in \mathscr{T}$ such that

$$1\chi^{::::}_{\cdots\cdot} = \chi^{::::}_{\cdots\cdot}$$

for all $\chi^{::::}_{\cdots\cdot} \in \mathscr{T}^{::::}_{\cdots\cdot}$ and an element $\delta_b^a \in \mathscr{T}_b^a$ such that

$$\chi^{\cdot\cdot p\cdot\cdot}_{\cdots\cdots}\, \delta_p^q = \chi^{\cdot\cdot q\cdot\cdot}_{\cdots\cdots}, \quad \chi^{\cdots\cdots}_{\cdot\cdot x\cdot\cdot}\, \delta_y^x = \chi^{\cdots\cdots}_{\cdot\cdot y\cdot\cdot}$$

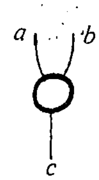
(Thus δ^a_b has the formal properties of a Kronecker delta.) These elements must necessarily be unique. The systems $\{\mathscr{Z}\}$ and $\{\mathscr{I}\}$ generated by δ^b_a and 1, and by δ^b_a and the elements of $\mathscr{T}$, respectively, have a special interest. (These mirror those tensors in an "ordinary" system whose components are absolutely invariant under the full linear group of coordinate transformations, these components being integers in the case of $\{\mathscr{Z}\}$.) We define the "dimension" of $\mathscr{T}^a$ to be the scalar

$$\delta^a_a = \nu.$$

In "reasonable" systems, ν is a non-negative integer. We shall also consider "unreasonable" systems in which ν is a *negative* integer.

A motivation for the above notation is that even in the case of ordinary finite dimensional systems we can retain the full flexibility and simplicity of the tensor index notation while eliminating the undesirable basis dependence of the usual notation. However the above is still subject to the other criticism which is sometimes levelled at such index notations, namely the fact that with many indices, expressions may become cumbersome and all-important index connections are easily misread. I shall therefore introduce a diagrammatic notation for tensors which in most instances allows connections between indices to be discerned at a glance.

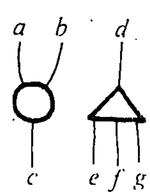
Let $a, b, \ldots$ label points in a plane. We denote an object $\chi^{a\ldots c}_{d\ldots f}$ by a symbol with an "arm" corresponding to each of $a, \ldots, c$ and a "leg" corresponding to each of $d, \ldots, f$. For example, if $\theta^{ab}_c \in \mathscr{T}^{ab}_c$, $\chi^a_{bcd} \in \mathscr{T}^a_{bcd}$ we could write



$= \theta^{ab}_c,$

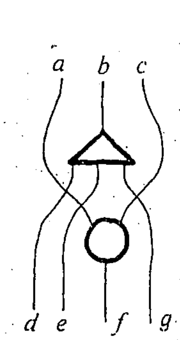
$= \chi^a_{bcd}.$

Normally, the label symbols $a, b, c, \ldots$ would be omitted in a diagram. In fact, we may regard the labels simply *as* points of the plane on which the figure is depicted. Outer products are drawn simply as a juxtaposition of the individual symbols:



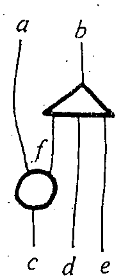
$= \theta^{ab}_c\, \chi^d_{efg} \in \mathscr{T}^{abd}_{cefg}$

or



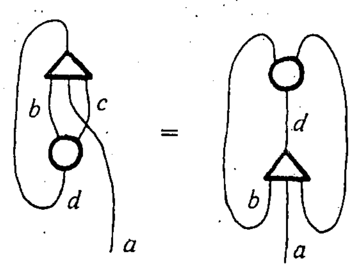
$$=\theta^{ac}_{f}\,\chi^{b}_{deg}$$

for example. We do not require products to be ordered horizontally. (We should not have had this freedom if we had been concerned with non-commutative ATS's.) To depict a contraction, we simply join the corresponding arm and leg:



$$=\theta^{af}_{c}\,\chi^{b}_{fde}\in\mathscr{T}^{ab}_{cde}$$

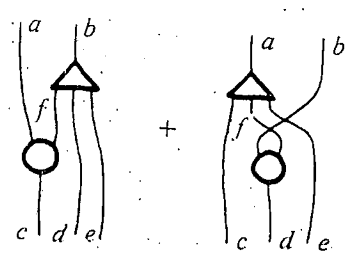
or



$$=\theta^{bc}_{d}\,\chi^{d}_{bac}\in\mathscr{T}_{a}.$$

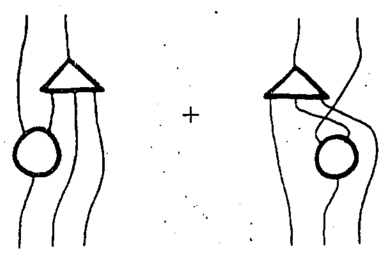
For clarity it will be usual to draw all the free arms (non-dummy upper indices) as emerging at the top of the picture and free legs (non-dummy lower indices) emerging at the bottom. In any case each arm should emerge as finally directed upwards and each leg as finally directed downwards. (As far as possible we try to keep the lines proceeding essentially vertically.) This will cease to apply when we consider Cartesian systems shortly, however.

Normally, to add two expressions it will be convenient simply to draw the diagram for each term and put a "+" sign between them, e.g.

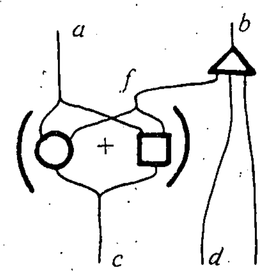


$$= \chi^{b}_{fde}\,\theta^{af}_{c} + \chi^{a}_{cfe}\,\theta^{bf}_{d}$$

We may omit the labels and draw this



making sure that it is clear, from the arrangements of free arms and legs, which are corresponding to which in the different terms. Occasionally it is convenient to employ a notation



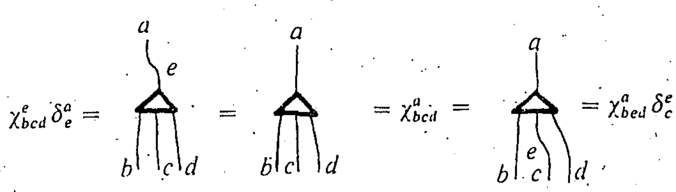
$$= (\theta^{af}_{c} + \gamma^{af}_{c})\,\chi^{b}_{fde}$$

when sums in parenthesis are involved.

The notation for the unit δ^a_b is simply a "disembodied" line:

$$= \delta^a_b$$

since then the rules

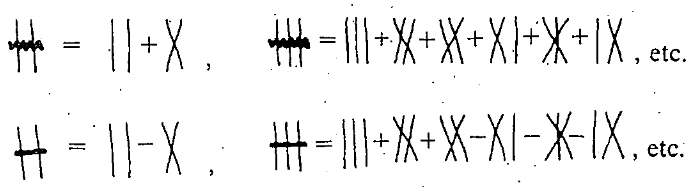


$$\chi^{e}_{bcd}\,\delta^{a}_{e} = \qquad = \qquad = \chi^{a}_{bcd} = \qquad = \chi^{a}_{bed}\,\delta^{e}_{c}$$

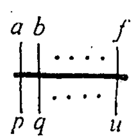
remain notationally consistent when the labels are omitted from the diagrams. The "dimension" ν is depicted simply as a closed loop:

$$= \delta^{a}_{a} = \nu.$$

The utility of the diagrammatic notation is much enhanced when a special notation is employed for certain important elements of $\{\mathscr{Z}\}$. Write



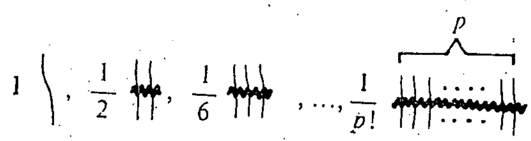
The general pattern should be clear. For each integer p we define two elements of $\{\mathscr{Z}\}$. These have p arms and p legs and are sums (or differences) of $p!$ terms, one corresponding to each permutation of the legs (or, equivalently of the arms). Each term is an outer product of p "Kronecker deltas". The negative signs occur, in the second case, whenever the permutation is odd. (These are really Aitken's diagrams [1]. The permutation is odd if the number of intersection points between the lines is odd, counted correctly if multiple intersections are present.) In the case of ordinary tensors, the object



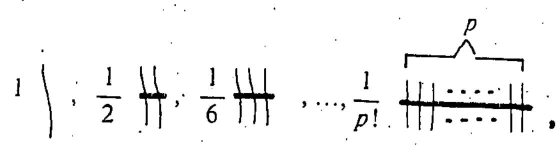
$$= \begin{vmatrix} \delta^{a}_{p} & \delta^{b}_{p} & \cdots & \delta^{f}_{p} \\ \delta^{a}_{q} & \delta^{b}_{q} & \cdots & \delta^{f}_{q} \\ \vdots & \vdots & & \vdots \\ \delta^{a}_{u} & \delta^{b}_{u} & \cdots & \delta^{f}_{u} \end{vmatrix} = \delta^{ab\ldots f}_{pq\ldots u}$$

will be recognized as the "generalized Kronecker delta".

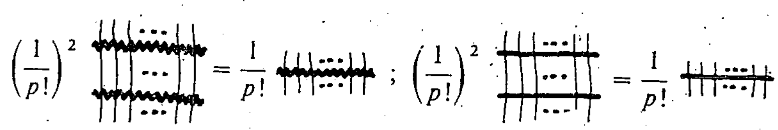
If the ring $\mathscr{T}$ of scalars contains (a subring isomorphic to) the rationals, then we can form the symmetrizers



and skew-symmetrizers

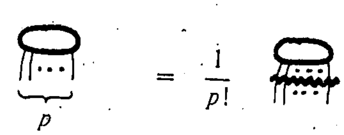


which have the idempotence:

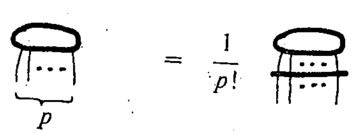


(As an alternative notation, we could absorb the $1/p!$ factors into the definitions of the diagrams. On the whole I prefer not to. This is partly because one would like to be able to handle the case when $\mathscr{T}$ has finite characteristic. But, in addition, it turns out that the factors $1/p!$ are rather a nuisance when expansions of complicated diagrams are carried out.) We can form symmetric and skew parts of tensors as follows:

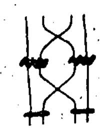
symmetric part of



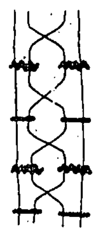
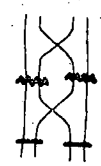
skew part of



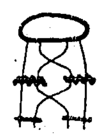
We can also build up Young tableau operators (cf. [7]) from these. For example, the expression

$\frac{1}{12}$ 

is an irreducible Young idempotent:

$\left(\frac{1}{12}\right)^2$  $= \frac{1}{12}$ 

corresponding to the partition (2^2). When applied to a four-legged object:

“ $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$ part” of  $= \frac{1}{12}$ 

This particular operator produces an object with the symmetries of a Riemann–Christoffel tensor.

An ATS will be called *weakly regular* iff for all $\chi^{\mathscr{A}} \in \{\mathscr{T}\}$, there exists $\theta_{\mathscr{A}} \in \mathscr{T}_{\mathscr{A}}$ such that

$$(\chi^{\mathscr{A}}\theta_{\mathscr{A}})\chi^{\mathscr{B}} = \chi^{\mathscr{B}}$$

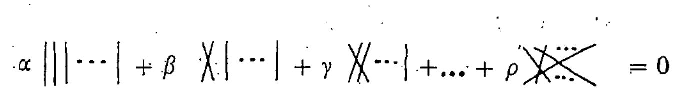
An ATS is *regular* iff for all $\chi^{\mathscr{A}\Gamma} \in \mathscr{T}^{::::}_{::::}$ (with every different possible clumping of its indices into two groups) there exists $\theta_{\Gamma\mathscr{A}}$ such that

$$\chi^{\mathscr{A}\Gamma}\theta_{\Gamma\mathscr{C}}\chi^{\mathscr{C}\Delta} = \chi^{\mathscr{A}\Delta}.$$

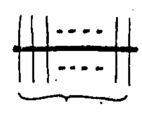
(This terminology is suggested by the concept of a regular ring [8].) If we assume that the scalars $\mathscr{T}$ constitute a field, then clearly $\{\mathscr{T}\}$ is weakly regular iff for every non-zero $\chi^{\mathscr{A}} \in \{\mathscr{T}\}$, there exists $\theta_{\mathscr{A}} \in \mathscr{T}_{\mathscr{A}}$ such that

$$\chi^{\mathscr{A}}\theta_{\mathscr{A}} \neq 0.$$

Furthermore, it can be shown [4] that if there is $\delta_a^b \in \mathscr{T}_a^b$ satisfying some identical relation



which is non-trivial (i.e. with some non-vanishing coefficient) and if $\{\mathscr{T}\}$ is weakly regular, with $\mathscr{T}$ a field, then $\{\mathscr{T}\}$ is regular. In fact, for any ATS whose elements are representable in the ordinary way in terms of (finite) sets of components and which contains δ_a^b, there will be an identical relation of this kind. For, the dimension $\nu = \delta_a^a$ must now be a positive integer, and we have



$= 0$

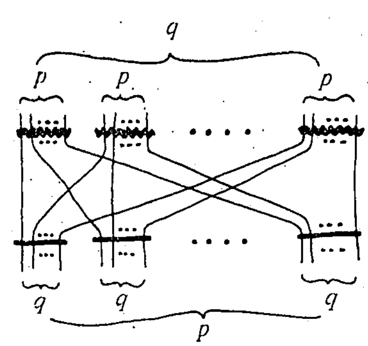
as a well-known simple property of the generalized Kronecker delta.

Simply for comparison with the weak regularity condition above, let us consider another (stronger), condition on an ATS (for which $\mathscr{T}$ is a field). We call an ATS *complete* if: for every non-zero $\chi^{\mathscr{A}\Gamma} \in \{\mathscr{T}\}$ (with each possible clumping into two clumps), there exists $\theta_{\mathscr{A}} \in \mathscr{T}_{\mathscr{A}}$ such that

$$\chi^{\mathscr{A}\Gamma}\theta_{\mathscr{A}} \neq 0.$$

Every such *complete* ATS will be isomorphic to the system of *all* n-dimensional ("ordinary") tensors over the field $\mathscr{T}$.

The possible $\{\mathscr{I}\}$-systems can be classified completely—$\mathscr{T} = \mathscr{I}$ being, for simplicity, a field without characteristic. The result (cf. [4]) is that either $\{\mathscr{I}\}$ is *free* (i.e. no identical relations of the type considered), or the identical relations are such that they can all be obtained from a single relation of the type



$= 0$

corresponding to a *rectangular* partition (p^q). We then have $\nu = q - p$. Call such a δ_a^b of type $[q-1, p-1]$ (p and q must be positive integers). The δ_a^b's which generate regular $\{\mathscr{S}\}$ systems are then just those of type $[n, 0]$ or $[0, n]$. A δ_a^b of general type $[m, n]$ is constructible as a "direct sum" of one of type $[m, 0]$ and one of type $[0, n]$. The "direct product" of δ_a^b of type $[m, n]$ with δ_α^β of type $[p, q]$ is δ_A^B of type $[mp + nq, mq + np]$ $(A = (a, \alpha), B = (b, \beta))$.

A *Cartesian* ATS is one containing a special element $g_{ab} \in \mathscr{T}_{ab}$ which is symmetric:

$$g_{ab} = g_{ba}$$

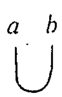
and invertible in the sense that there exists another element $g^{ab} \in \mathscr{T}^{ab}$ with

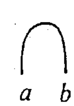
$$g_{ab}\, g^{bc} = \delta_a^c$$

(Since $g_{cd}\, g^{ca}\, g^{db} = g^{ab}$, it follows that g^{ab} is also symmetric: $g^{ab} = g^{ba}$.) The elements g_{ab}, g^{ab} can be used, in the usual way, to translate an upper index into a lower one, or a lower index into an upper one:

$$\chi_{ad}{}^{c} = \chi_a{}^{bc}\, g_{bd}, \quad \chi_a{}^{bc} = \chi_{ad}{}^{c}\, g^{db}.$$

It now ceases to be important to maintain a distinction between upper and lower indices. For, writing

$g^{ab} =$ 

, $\quad g_{ab} =$ 

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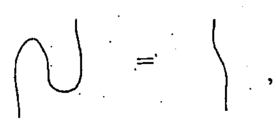
which is consistent with



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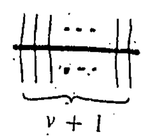
we can turn arms into legs and vice-versa. Since it now makes no difference whether an arm points up or down, we may as well also allow arms to point right or left or, indeed, in any direction in the plane. This will give our diagrams an additional flexibility. (In fact we can attain this flexibility also in the case of a non-Cartesian system if we allow our index lines to be *directed*—as may be indicated, say, by an arrow on the line, rather than simply by the vertical ordering.)

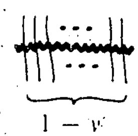
The complete description of Cartesian $\overset{c}{\{\mathscr{S}\}}$-systems (i.e. systems generated by g_{ab}, g^{ab} and scalars) is a little more complicated than that of the non-Cartesian $\{\mathscr{S}\}$-systems. Assume $\overset{c}{\mathscr{S}}$ is a field without characteristic, as before.

Then there is a unique regular $\{\overset{c}{\mathscr{I}}\}$-system for each integral value of ν. When $\nu = 2, 3, 4, \ldots$ (or in the trivial cases $\nu = 0, 1$) we have simply a system isomorphic with the "ordinary" Cartesian system of (numerically) invariant tensors over a ν-dimensional vector space, with component representation in the normal way. We have the basic identical relations:



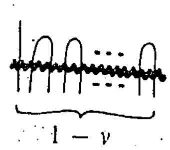
$= 0$

with δ_a^b of type $[\nu, 0]$. When $\nu = -2, -4, -6, \ldots$ we have δ_a^b of type $[0, -\nu]$ with



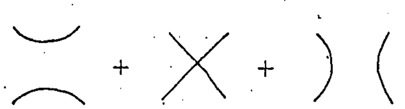
$= 0$

as an identical relation. However, this can be contracted down to give a relation of lower valence:



$= 0$

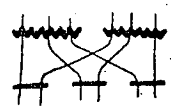
The most interesting system of all is, perhaps, the regular Cartesian system with $\nu = -2$. The elements of this system I call *binors*. The above basic identical relation can be written out as



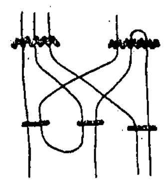
$= 0$

We readily see that every contraction of this expression is identically zero if $v = -2$. This expression must indeed vanish for a (weakly) regular $\{\overset{c}{\mathscr{I}}\}$-system with $\nu = -2$.

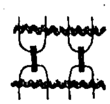
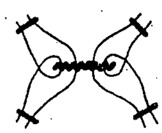
When $\nu = -1, -3, -5, \ldots$, the systems are somewhat more complicated. The type is now $[1, 1-\nu]$ instead of $[0, -\nu]$. Let us illustrate this with the case $\nu = -1$. We have the identical relation

 $= 0$

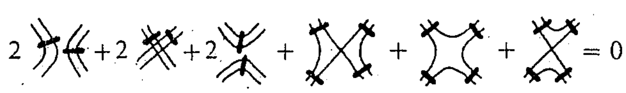
but this contracts down to give (equivalent) relations of smaller valence, e.g.

 $= 0.$

More symmetrical forms, *each* of which is equivalent to either of the above two relations, are

 $= 0,$  $= 0.$

This second expression, partly expanded out, is:



We readily verify that every contraction of this vanishes if $\nu = -1$. The cases $\nu = -3, -5, \ldots$ are essentially similar.

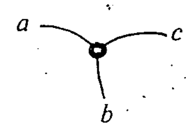
So much for general theory. Now let us examine some applications.† First consider a simple (regular) Cartesian system generated by the complex numbers as scalars and an ordinary three-dimensional Cartesian Kronecker delta δ_{ab} ($= g_{ab}$) and the skew-symmetrical (Levi–Civita) ε_{abc}. We can represent δ_{ab} and ε_{abc} in the normal way in terms of components, with

$$\delta_{11} = \delta_{22} = \delta_{33} = 1,$$
$$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1,$$
$$\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1,$$

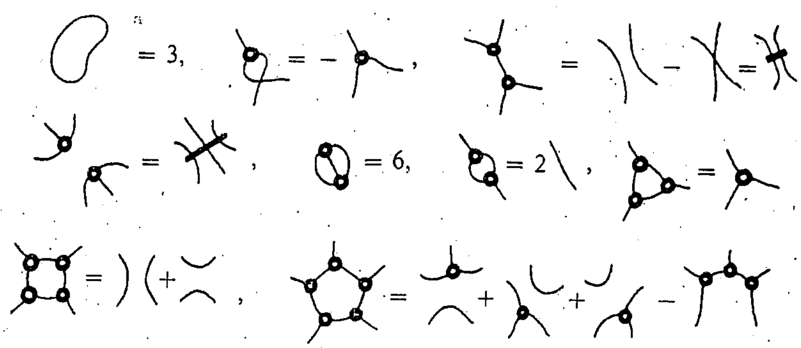
† The main application described here was, for the most part, obtained during the period 1953–55, but has not been published hitherto.

the remaining components being zero. Contraction is represented by the Einstein summation convention in the usual way. Depict:

$\delta_{ab} =$ 

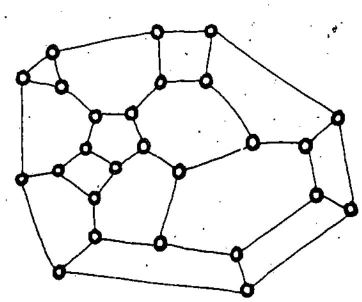
, $i\,\varepsilon_{abc} =$ 

where the factor $i\,(=\sqrt{-1})$ is included for simplicity in the signs of reduction formulae. We have, as well-known formulae of tensor calculus, or by repeated applications, thereof:



etc.†

Now consider a planar graph of degree three. We can associate with it a certain contracted product of ε_{abc}'s, where one $i\varepsilon_{abc}$ is drawn at each vertex, a contraction occurring for each edge of the graph. The result is just some complex number (actually an integer). For example:

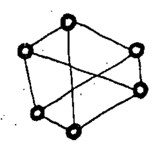
 $= K.$

The number K is, in fact, precisely the number of ways of colouring the edges of the graph with three colours so that three distinct colours occur at each

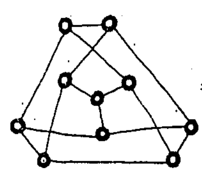
† I am grateful to T. G. Murphy for pointing the last of these formulae out to me.

vertex. This, by a well-known result [2], is just one-quarter of the number of ways of colouring the faces of the graph with four colours so that no two similarly coloured faces have an edge in common.

To see the connection between K and the edge-colouring number, we think of the representation in terms of components. The tensor expression is then simply a sum of terms, each of which arises when one of the numbers 1, 2, 3 is assigned to each edge (corresponding to a summed index). We get a factor i for each vertex where the numbers 1, 2, 3 appear cyclicly, and $-i$ if 1, 2, 3 appear anti-cyclicly. The term is zero if a repetition of 1, 2, 3 appears at some vertex. Thus each non-zero term arises from an allowable colouring of the edges with three colours. Finally we have to check that each non-zero term contributes precisely the value $+1$. It is at this point that the planarity of the graph enters. There are various ways of seeing that the sign comes out correctly here, but I have not been able to think of an argument which can be presented in a nutshell, so I shall just omit it here. It is, however, worthwhile to illustrate the necessity of the planarity condition by an example. We have

 $= 0$

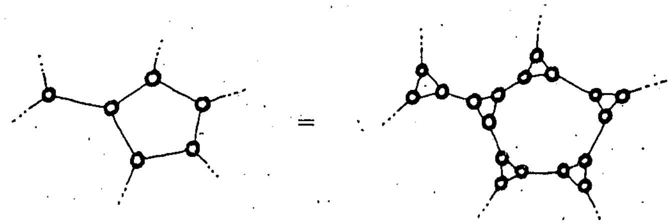
even though the graph's edges *can* be coloured with three colours (in twelve ways). The vanishing of the tensor expression follows from the skew-symmetry of ε_{abc}. For, a pair of vertices can be interchanged while leaving the configuration invariant, but with three signs changed in the process. Of course any non-planar graph of degree three whose edges are *not* colourable with three colours *must* have a vanishing tensor expression, e.g. the Petersen graph gives:

 $= 0$

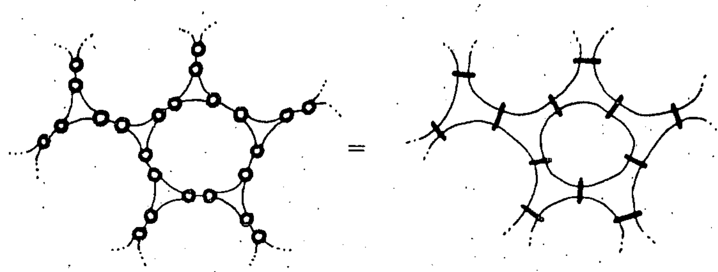
(the vanishing is also a consequence of skew-symmetry again in this case).

There is an alternative way of establishing the equality of the tensor expression K and the number of colourings in the case of a planar graph. This is simply to observe that the last four of our reduction formulae for ε_{abc} expressions can be used to eliminate the faces one by one. We merely have to verify that these four reduction formulae are equally valid if they refer to numbers of colourings or to ε_{abc} expressions. Then we finally end up with a single closed loop. This "edge" can be coloured in three ways and the value is 3, thus completing the induction.

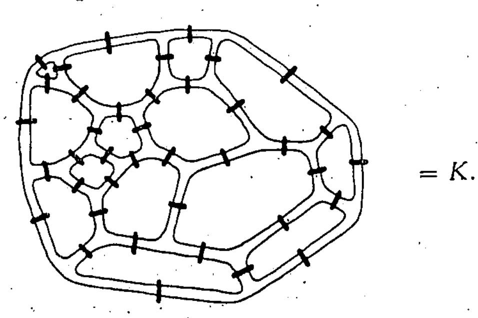
Now let us try something further. We can replace each vertex by a little triangle, without changing the value (because of one of our reduction formulae). Thus



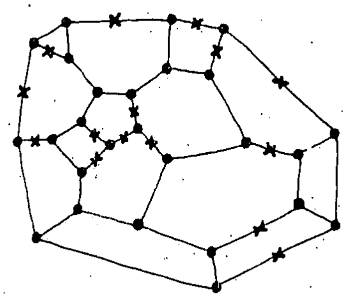
The two new vertices at the ends of each original edge can now be slid together and another of our reduction formulae employed to give:



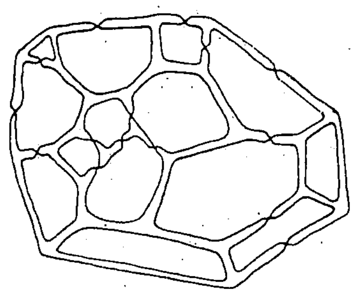
Thus, we have a new expression for K:



We can reinterpret this combinatorially. Expanding out each of the "skew-symmetrizers" we get a sum of 2^E terms, where E is the number of edges in the graph. We can describe each term as follows. Select a subset of edges of the graph and call these the crossed edges. Then the term is positive or negative according as the number of crossed edges is even or odd. We build up a number of closed loops by following along just inside the boundary of each face and then crossing over to the other side of the edge at its mid-point whenever the edge is a crossed edge. Let the total number of closed loops thus formed be C. Then the absolute value of the term is 3^C, (since a closed loop takes the value $\nu = 3$). For example, in the graph just considered, a typical term is given by an assignment of crossed edges as follows:



The corresponding term is

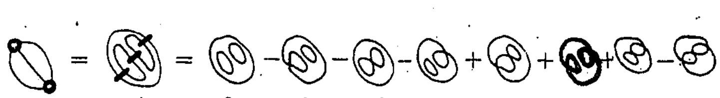
$-$  $= -3^5$.

since there are five closed loops, the minus sign occurring because there are 15 crossed edges. Thus we have

$$K = \sum (-1)^X 3^C$$

where X is the number of crossed edges and C the resulting number of closed loops.

A simple worked example is the following:



$$= 3^3 \quad - 3^2 \quad - 3^2 \quad - 3^2 \quad + 3 \quad + 3 \quad + 3 \quad - 3 \quad = 6.$$

This is obviously the correct answer, but it also illustrates the fact that we must expect a considerable amount of cancellation in the sum. The formula is clearly not very practical for explicit calculation, but also it is unlikely to be of great direct theoretical use in attempts to establish $K > 0$, owing to this cancellation.

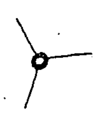
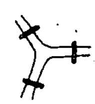
At this point we turn our attention to a "negative dimensional" ATS. It is a remarkable fact that the above formula for K has a curious analog in which 3 is replaced by -2:

$$K = (-\tfrac{1}{4})^{\frac{1}{2}V} \sum (-1)^X (-2)^C$$

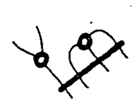
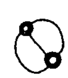
where $V = \frac{2}{3}E$ is the number of vertices of the graph. Note that the value of the summation must now be much larger, even though the individual terms have smaller absolute value than before. Thus the amount of cancellation has been considerably reduced. The above worked example gives:

$$\begin{aligned} K = -\tfrac{1}{4}\{(-2)^3 - (-2)^2 - (-2)^2 - (-2)^2 \\ + (-2) + (-2) + (-2) - (-2)\} \\ = \tfrac{1}{4}(8 + 4 + 4 + 4 + 2 + 2 + 2 - 2) = 6. \end{aligned}$$

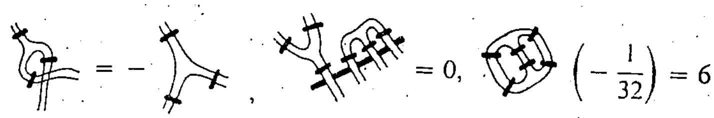
The basis of the above formula stems from consideration of the system of binors (i.e. the regular system $\{\overset{c}{\mathscr{I}}\}$ with $\nu = -2$). We consider a correspondence

 $\leftrightarrow \dfrac{i}{4\sqrt{2}}$ 

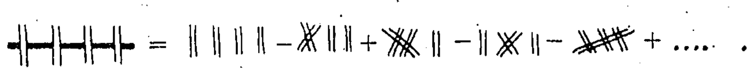
where the arms on the left-hand figure are ordinary Cartesian 3-tensor lines (as before, i.e. $\nu = 3$), but where the arms on the right-hand figure are binor lines ($\nu = -2$). The point about this correspondence is that the algebraic properties of the $i\varepsilon_{abc}$ object on the left and of the binor object on the right—indices suitably clumped in pairs as indicated—are formally identical. In other words, the correspondence establishes an embedding of the $(\delta_{ab}, \varepsilon_{abc};\ \nu = 3)$ system, considered earlier, in the binor system $(\delta_{\alpha\beta};\ \nu = -2)$ for which indices are clumped in pairs (e.g. $a = (\alpha, \alpha_0)$, $b = (\beta, \beta_0)$, ...). I shall not go into the matter in full detail here. Essentially, what has to be verified is that the relations

 $= -$     $= 0.$  $= 6$

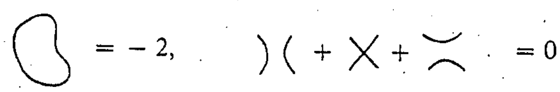
are also satisfied by the proposed binor object. That is,



where

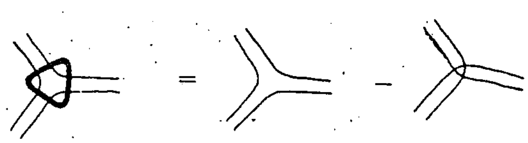


The verification of these relations requires the properties

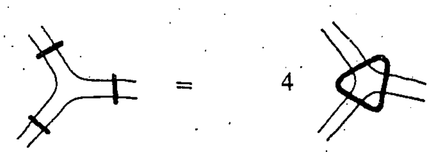


which characterize the binor system. It is easiest not to carry out the full verification of the middle relation directly, but to use some general results. These will not be entered into here. The third relation is really just the example considered earlier.

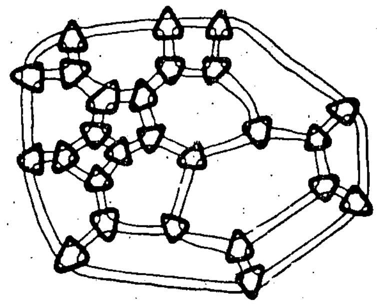
By using binors, we can also establish some other formulae for K. Set



Then we have the binor relation



This may be verified directly using the basic binor identical relation. This leads to yet another formula for K:

$$K = \left(-\frac{1}{2}\right)^{\frac{1}{2}V}$$ 

If we assign an orientation "*counter-clockwise*" to each face, we may describe this formula in terms of *directed* circuits (i.e. this falls within the framework of the non-Cartesian system $\{\mathscr{S}\}$ with $\nu = -2$). (We could not do this previously because the orientations would not have been consistent with crossed edges.) Then our formula can be described as follows:

$$K = (-\tfrac{1}{2})^{\frac{1}{2}V} \sum (-1)^Y (-2)^{C'}$$

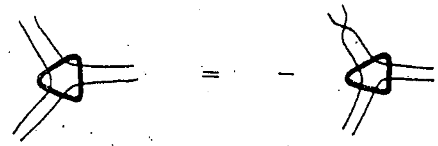
where there are 2^V terms in the sum, one term corresponding to each assignment of a "+" or of a "−" to every vertex. Here Y is the number of "−" vertices; C' is the number of directed circuits required, where we cover every edge of the graph exactly twice and where we turn to the *left* at every "+" vertex and to the *right* at every "−" vertex, as we follow the circuits. In fact, it can be seen fairly easily that the number of circuits has the same parity in each term. Hence

$$K = (\tfrac{1}{2})^{\frac{1}{2}V} \sum (-1)^Y 2^{C'}$$

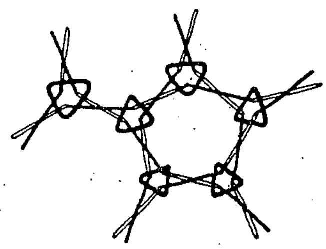
(So apart from an overall sign $(-1)^{\frac{1}{2}V}$, we may interpret our last diagram for K as applying within the system $\{\mathscr{S}\}$ or $\overset{c}{\{\mathscr{S}\}}$ with $\nu = 2$.) Our example now gives:

 $= K = \frac{1}{2}\{8 - 2 - 2 + 8\} = 6.$

Let us consider one final expression for K. We observe that our binor vertex possesses the skew-symmetry



(since we saw that it was equal to an expression in which this skew-symmetry was manifest). Thus, at most the overall sign is altered if we replace our last diagram for K by one in which all edges become crossed edges. If we depict the diagram in the following way,



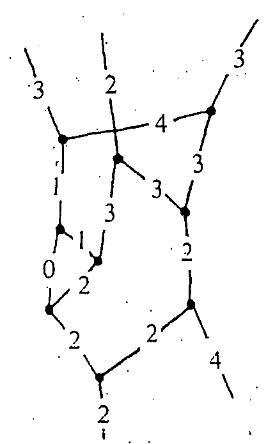
then an alternative formula for K presents itself:

$$K = (\tfrac{1}{2})^{\frac{1}{2}V} \sum (-1)^{Y} (-2)^{C''}.$$

Here, as before, there are 2^V terms in the sum; each term corresponds to an assignment of a "+" or a "−" to every vertex; Y is the number of "−" vertices. Now, C'' is the number of (undirected) closed circuits, consisting of alternately black and white line segments, which just cover every edge twice, once with a black segment and once with a white segment, where at each "+" vertex we turn *right* from white to black and *left* from black to white and *vice-versa* at each "−" vertex.

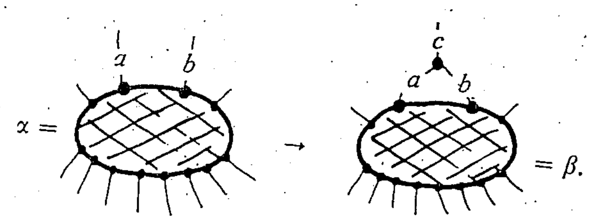
There is another application of binor algebra which is worth mentioning here. This is to the theory of quantum mechanical angular momentum. I shall not go into much detail here. A more extended account has been given elsewhere [6].

Consider an undirected graph of degree 3, but which may possess also some free ends. Each edge of the graph (including each free end) is assigned a non-negative integer. I suppose that the sum of the three integers at each vertex is *even* and that no integer at a vertex exceeds the sum of the other two. I call such a graph a *spin-network*. An example of a spin-network is the following:



Each spin-network will have a physical interpretation—or, rather, a number of different physical interpretations, depending on a choice of "time-ordering" for the graph. Let us think of the network as a kind of space–time diagram of a portion of the universe. For the spin-network depicted above, we shall think of time as progressing from the bottom of the diagram to the top. Each line segment is to be pictured as describing a part of the universe—called a *unit*—which is effectively stationary and isolated from the remainder of the universe (e.g. a "free particle"). I suppose that a unit also has a well-defined total angular momentum. According to quantum mechanics, the value of this total angular momentum must be a (non-negative) *integral* multiple of $\frac{1}{2}\hbar$. If the integer depicted on the line segment is n, then the total angular momentum of the unit is to be $\frac{1}{2}n\hbar$ and the unit is called an *n-unit*.

We may interpret the above diagram as follows: A 2-unit splits into two 2-units one of which combines with a 4-unit to make another 2-unit; the other 2-unit splits into a 0-unit and yet another 2-unit, which subsequently splits into a 1-unit and a 3-unit, the 1-unit combining with the 0-unit to make another 1-unit; ..... We are interested in the following type of question. Suppose we are given a spin-network in which an a-unit and a b-unit emerge. Suppose that the a-unit and the b-unit combine to form a c-unit:



We ask: what is the probability value for each possible value of the integer c, if the remainder of the spin-network is all that we are given? Now, quantum mechanics gives a well-defined answer to this question (assuming the effect of relative motion between the different units may be neglected). There is a standard procedure for obtaining this probability, but calculations tend to get very involved.

An alternative (but equivalent) procedure is to use binors. With each spin-network we associate a binor, where an n-unit is represented by n lines with a "skew-symmetrizer" across. The lines are then joined up at each vertex in an essentially unique way. For example, the above spin-network is represented as

APPLI

Define a norm $||\kappa||$
obtained by joining
ing contracting over
a-unit and the b-un
resulting unit is a

where

It may be finally
network norms, by
various shortcuts
number of ways of
spin-network, with
polynomial. In thi
momentum formul
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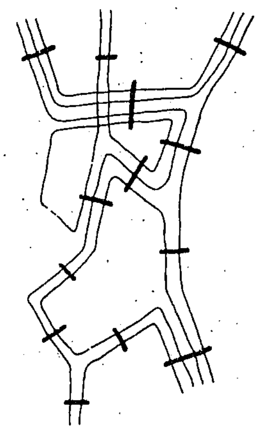
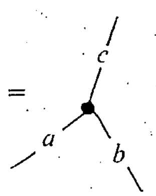
† When applied to th
having $n = 2$), this me
of covering the edges



Define a norm $||\kappa||$, for a spin-network κ, to be the modulus of the scalar obtained by joining together two identical copies of the binor for κ, connecting contracting over corresponding free arms. Then, my assertion is that when a-unit and the b-unit combine in the above process, the probability that the resulting unit is a c-unit is:

$$\text{probability of } c = \frac{\|\beta\|\,(c+1)\,(a!)^2(b!)^2}{\|\alpha\|\;\|\gamma\|}$$

where

$\gamma =$ 

It may be finally remarked that although the *direct* calculation of spin-network norms, by the above prescription, can be very impractical, there are various shortcuts possible. One such method† involves counting the total number of ways of colouring a certain series of graphs, associated with the spin-network, with n-colours and then putting $n = -2$ in the resulting polynomial. In this way it is possible to obtain some standard angular momentum formulae (e.g. Racah's expression for the $6-j$ symbol) very rapidly. All this will be described fully elsewhere.

† When applied to the particular case of a planar graph (no free ends and all spin values having $n = 2$), this method yields yet *another* expression for K. Consider each different way of covering the edges of the graph with a number of circuits, where each edge is covered

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exactly twice and where no individual circuit covers any edge more than once. For each such covering, colour the circuits with ν colours so that no two similarly coloured circuits cover a common edge. Count the number of distinct colourings for each such covering and sum over all possible such coverings. The result will be a polynomial in ν. Substituting the value $\nu = -2$ into this polynomial we get $(-4)^{V\frac{1}{4}}K$. (If we substitute $\nu = 3$ into the polynomial, the result is in fact also just K, but this is less subtle, being quite easy to see directly.)

The above polynomial in ν can also be obtained in another way. We assign a string of ν symbols, consisting solely of "zeros" and "ones", to every *face* of the graph (counting the external "sea" as a face). The strings of symbols assigned to two adjacent faces are to differ in exactly *two* places. The number of ways of making such an assignment is then just 2^ν times the required polynomial.