Problem 1.1. Find an opponent and play a number of games of 2-spot and 3-spot sprouts. Find out the best strategy that you can for 2-spot sprouts, and write a short essay describing your finding. This essay should fit on one side of a sheet of paper.
Due Friday, January 16, 2009.

Problem 1.2. Prove that every game of sprouts starting with any finite number of spots (say with N spots) must eventually end, no matter how the players play. (Of course they must obey the rules.) If the game starts with N spots, give an upper bound on the number of moves in any game.
Due Wednesday, January 21, 2009.

Problem 1.3. Read pages 39 - 51 in our text "An Introduction to Mathematical Reasoning" and do problems 5.1 to 5.4 on page 51.
Due Monday, January 26, 2009.

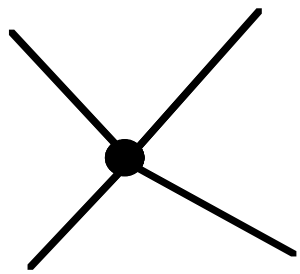
Problem 1.4. Prove by induction that
$(1/1x3) + 1/(3 \times 5) + 1/(5 \times 7) + \ldots + 1/((2n-1) \times (2n + 1)) = n/(2n + 1)$.
Due Monday, January 26, 2009.

Problem 1.5. The Fibonacci Numbers are the numbers in the sequence 1,1,2,3,5,8,13,21,34,55,144,189,....
We denote these by $f_1 = 1$, $f_2 = 1$, $f_3 = 3$, $f_4 = 5$ and
$f_{n+1} = f_n + f_{n-1}$.
Each Fibonacci number is the sum of the previous two Fibonacci numbers. There are many patterns in the Fibonacci series.
Notice that
$1^2 = 1 = 1 \times 1$
$1^2 + 1^2 = 2 = 1 \times 2$
$1^2 + 1^2 + 2^2 = 6 = 2 \times 3$
$1^2 + 1^2 + 2^2 + 3^2 = 15 = 3 \times 5$.
Prove by induction that
$f_1^2 + f_2^2 + \ldots + f_n^2 = f_n \times f_{n+1}$ for all n=1,2,3,....
Due Friday, January 30,2009.

Problem 1.6. (Returning to Sprouts) Prove that for every natural numbrer n (n = 1, 2, 3, 4, ...) there is a sprouts game, starting with n sprouts, that ends in exaactly 3n -1 moves. You can use mathematical induction in your proof.

Due Monday, February 2, 2009.

Problem 1.7. In *4-sprouts* we use the same rules as in ordinary sprouts, but we allow 4 lines to touch a spot
as in the diagram below.



A move has the same form as in regular spots, but notice that the new spot, having two lines going into it, has two freedoms in the game of 4-spots. Show that some games of 4-spots can go on forever. Give specific examples. Think about the question of how to modify the rules of 4-spots so that it will become a game that always ends in a finite number of moves.
Due Wednesday, February 4, 2009.

Problem 1.8. The Fibonacci Numbers are the numbers in the sequence 1,1,2,3,5,8,13,21,34,55,144,189,....
We denote these by $f_1 = 1$, $f_2 = 1$, $f_3 = 3$, $f_4 = 5$ and
$f_{n+1} = f_n + f_{n-1}$. We also take $f_0 = 0$ by convention.
Prove the following formula by induction.

$$f_{n-1} \text{ x } f_{n+1} = f_n^2 + (-1)^n \text{ for } n = 1,2,\ldots$$

For example $8 \text{ x } 21 = 168 = 169 - 1 = 13^2 - 1$.
Due Friday, February 6,2009.

Problem 1.9. Problem 20. page 56 of our textbook.
Due, Monday, February 9,2009.

Problem 1.9. Problem 20. page 56 of our textbook.
Due, Monday, February 9,2009.

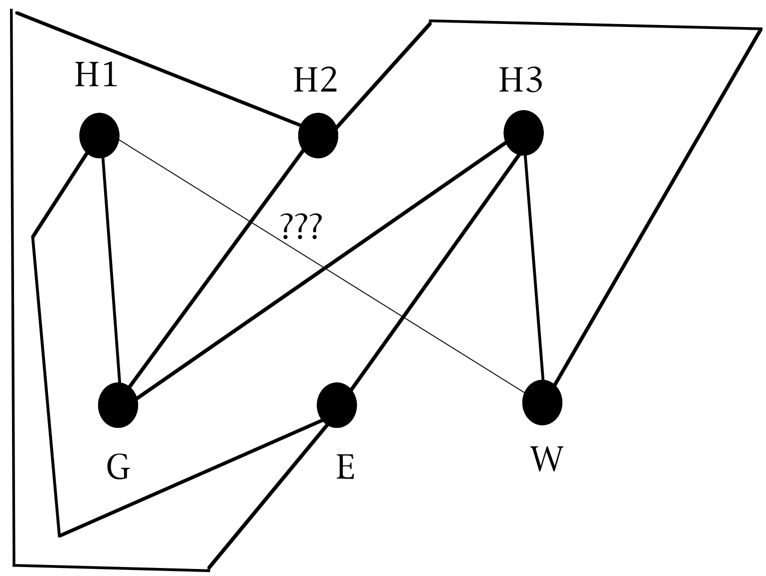
Problem 1.10. Prove by induction that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$$

for $n = 1,2,3,\ldots$.
Due, Monday, February 9,2009.

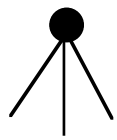
Problem 1.11. Suppose that you play a sprouts game with n starting spots and that the game lasts for $3n - 1$ moves. How many Pharisees will there be at the end of the game? Illustrate your result with a specific example.
Due, Wednesday, February 11,2009.

Problem 1.12. (The Gas-Electricity-Water Problem)
Three companies, the gas company, the electricity compay and the water company want to make connections from the gas main (G), the electrical source (E) and the water main (W) to three houses (H1, H2 and H3). They wish to lay their lines so that no two lines meet except at the sources (G,E and W) and at the houses (H1, H2 and H3). Can you find a solution to this design problem? If not, then why not?



In the illustration above the city planners have drawn a graph to help them design the connections but they have run into a difficulty with making a water line from W to H1. Everything went fine with the design up to that point, but then there does not seem to be any way to conncet from W to H1 without crossing previously created lines. It will cost the city a great deal to dig tunnels to make lines cross over one another. So these designers really need to know whether the job can be done with no crossovers, and if it cannot be done that way, then they want to know the least number of crossovers that are needed to do the job.
Due, Wednesday, February 11,2009.

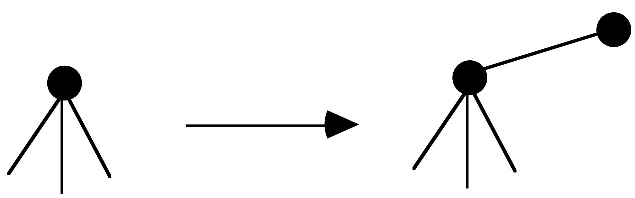
Problem 1.13. Construct a proof of Euler's formula by induction on the total number of edges and vertices in the graph G. You should consider how the graph can be built up from simpler graphs by adding edges to them. In fact, any connected graph can be built from a single vertex graph by adding new edges in two ways that I will now explain, but first we introduce an abbreviation: The diagram below stands for s*ome vertex in a larger graph*.



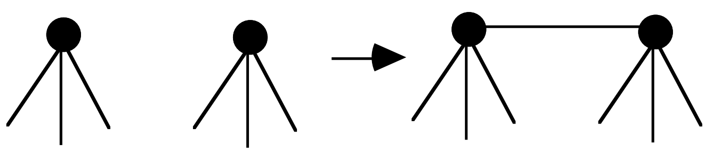
You can tell when I am using this abbreviation because the edges that go out of this vertex are not meeting any other vertices in the picuture. The picture is a shorthand for a possibly larger and more complete picture. In the abbreviation we show three edges touching the vertex. In a real situation some edges touch the vertex, but the number is not necessarily equal to three. Ok?

Now lets use this and illustrate two ways to make a larger graph.

I.

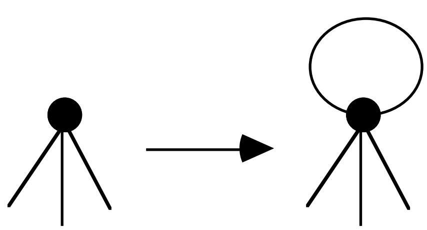


II.



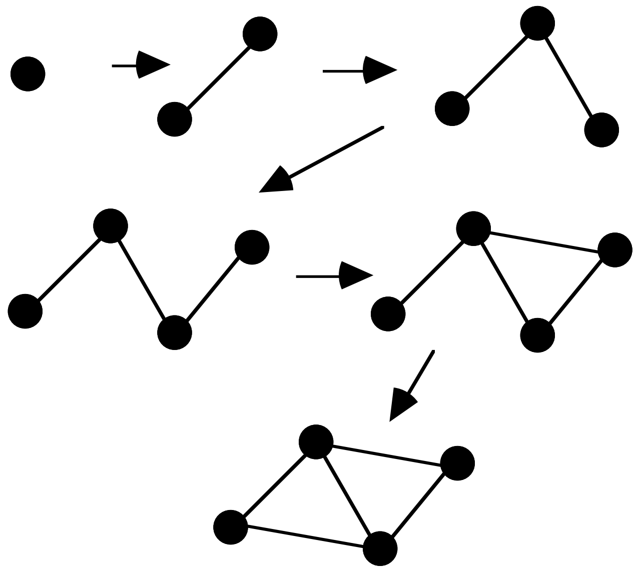
In method number I we add a new edge and a new vertex by attaching the new edge to an already existing vertex. In method number II we connect two vertices with a new edge.

Remark. We regard the move



as a special case of II.

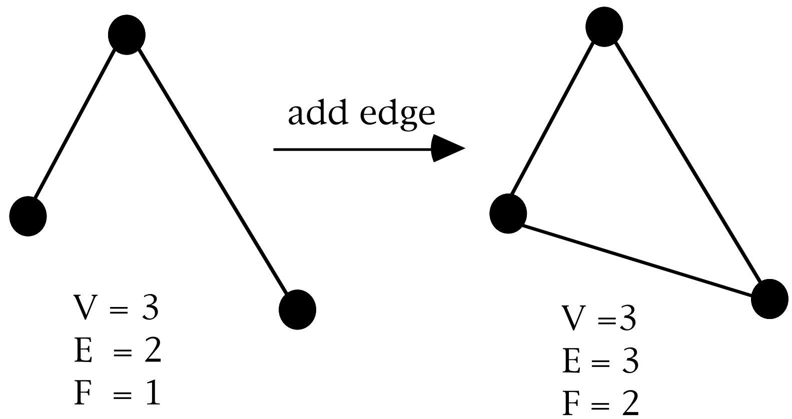
I claim that *any connected graph can be built up by performing a sequence of operations of these two types*. Here is an example.



You can use this claim in your proof, and if you want, you can also make a proof of the claim. We will discuss why the claim is true in class.

Now, to prove the Euler Theorem, you can proceed by induction, showing that V - E + F does not change its value when you perform a move of type I or type II. You will find that it is very easy to see this for type I, and that in order to see it for type II you need to start with a connected graph. If the graph is connected, then a move of type II will create a new region in the graph. Look at the example above and see how this works. You can use this fact also in your proof (that a move of type II will create a new region). You should then be able to construct an inductive proof of the Euler formula.

Here is an example:
We create a triangle graph by adding an edge to a tree.



Note that adding the edge creates a new region, and V-E + F does not change from before to after the addition of the new edge.

(c) Discuss your proof of the Euler formula with another student in the class. Do you both feel that the proof is complete? What might be missing? In this problem, it worth having the discussion. We will discuss some of the issues related to the problem in class before the problem is due.

Due, Friday, February 20,2009.

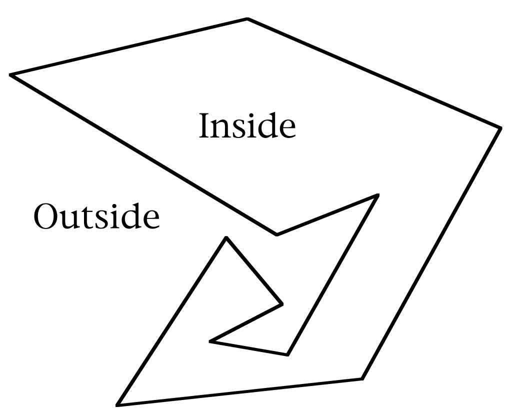
Supplement to Problem 1.13.

There is a fact about curves in the plane that you can use in thinking about regions that are created when graphs are drawn in the plane.

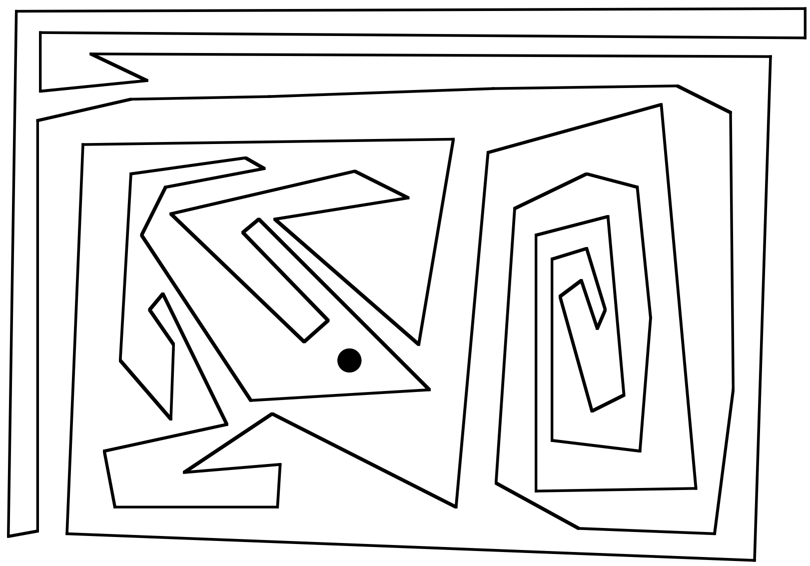
This fact is called the

Jordan Curve Theorem: *A closed curve in the plane without any self-intersections divides the plane into exactly two regions.*

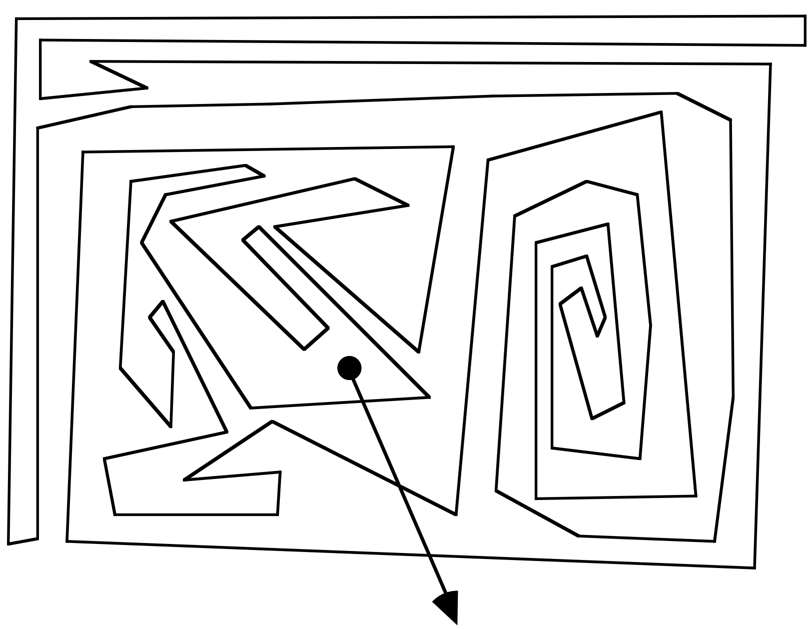
Here is an example:



You are not required to prove this result, but you can use it and it is interesting to see how complex examples can look!



Is the black dot inside or outside this curve?
Of course you can solve this like solving a maze, but look!



An arrow from the dot intesects the curve in an ODD number of points. I claim that this tells you that the point must be INSIDE.

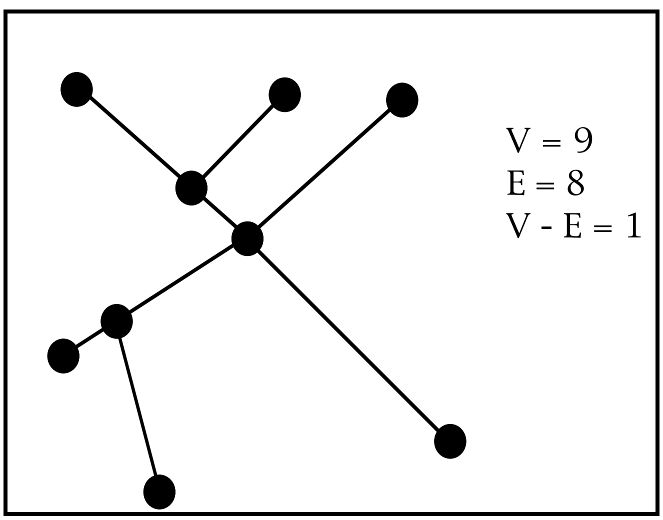
If the intersection number were EVEN, then the point would be outside. Can you explain why this works? (I say explain, and of course I am hoping that your explanation will turn into a mathematical proof. But lets explore.)

We will discuss in class why and how the Jordan Curve Theorem is relevant to proving Euler's Formula.

Remark. Another approach to the Euler formula uses the concept of a tree: A graph is said to be a *tree* if it does *not* contain any cycles (a cycle is a sequence of distinct edges such that the each edge shares its endpoints with the edges before and after it in the sequence. For example in the graph above, bce is a cycle and abcd is a cycle. When a plane graph has no cycles then the only region it can delineate is the rest of the plane other than itself, and so a tree has $F = 1$.

Show that for a connected tree, $V - E = 1$.

From this is follows that for connected plane trees $V - E + F = 2$, and so we know the Euler formula already for trees.



The picture above illustrates this result for trees. You can prove that V - E = 1 for a connected tree by induction on the number of edges in the tree.

You can then prove the Euler formula for an arbitrary connected plane graph by just making that graph by adding edges by our type II move to a tree. Think about this and try some examples.

##

Logic Problems and Other Problems From Part I.

Problem 2.1.
Proof by Contradiction.
Consider the following proof that the square root of 2 is not a rational number. At those places where we state an exercise, do the exercise.

Proof. Suppose that there is a rational number p/q whose square is equal to 2. We can assume that p/q is in reduced fractional form and that it is positive. Thus we assume that p and q are positive integers and that they do not have a common divisor other than 1.
So $(p/q)^2 = 2$.
Whence $p^2/q^2 = 2$.
Whence $p^2 = 2q^2$.
Thus p^2 is even.
This implies that p is even.
[Exercise: Prove that, when p is a natural number, if p^2 is even ,then p is even.]
Therefore p = 2r for some natural number r.
Hence $(2r)^2 = 2q^2$ by substitution into $p^2 = 2q^2$.
This is the same as saying $4r^2 = 2q^2$.
Dividing this equation by 2, we get
$2r^2 = q^2$. Thus q^2 is even.

But this means that q is even, by the exercise.
Therefore both p and q are even.
This means that p and q have the common factor 2.
But this contradicts our assumption that p and q do not have a common factor.
We conclude that the assumption that there is rational number whose square is 2, leads to a contradiction. Since we assume that mathematics is consistent, this is a proof by contradiction that the square root of 2 is irrational. Q.E.D.

This proof is a very good example of a proof by contradiction. We wish to prove P and we start by assuming ~P and find that this assumption leads to an absurdity.
We conclude from this that ~P cannot be true. But if ~P is false, then P is true!

The result that the square root of 2 is not rational was a scandal for the ancient Greek mathematicians, and even today there is a mystery about this. It means that the decimal expansion of the square root of two does not have a periodic pattern, and to know the square root of two fully as a decimal you would have to know infinitely many terms of its expansion.

[Exercise: Prove that there is no rational number whose square is equal to 3.]

Remark on the above exercise. You willl need to prove that 3 divides p^2 implies that 3 divides p. In order to do this, note that the natural numbers fall into three disjoint classes:

$$A : p = 3q,$$
$$B: p = 3q + 1,$$
$$C : p = 3q + 2.$$

A is the class of all numbers that are divisible by 3.
B is the class of all numbers that leave a remainder of 1 upon division by 3.
C is the class of all numbers that leave a remainder of 2 upon division by 3.

You can show that p^2 is of type A only if p is of type A. From this it follows that if p^2 is divisible by 3, then p is divisible by 3.

Due Monday, March 2, 2009.

Problem 2.2.
From the book.
p. 9: 1.3, 1.4.
p. 19: 2.1,2.3,2.4,2.5.
p. 53: 1,2.

Due Monday, March 2, 2009.

Problem 2.3.
p. 53: 3.

p. 54: 6.

Due Wednesday, March 4, 2009.

Remark on Problem 6 of page 54:
Here you are asked to use only the axioms given on pages 18 and 19. This is a very stringent exercise! Here is an example of how one can proceed.
We want to prove that a x 0 = 0. Here is the proof.

Proof that a x 0 = 0.
0 = 0 + 0 by (iv)
Hence a x 0 = a x (0 + 0) and
a x (0 + 0) = a x 0 + a x 0 by the distributive law (iii).
Thus a x 0 = a x 0 + a x 0.
Thus
a x 0 + (- (a x 0)) = (a x 0 + a x 0) + (- (a x 0))
and
(a x 0 + a x 0) + (- (a x 0)) = a x 0 + (a x 0 + (- (a x 0))
by the associative law (ii).
Thus
a x 0 + (- (a x 0)) = a x 0 + (a x 0 + (- (a x 0))
We know that a x 0 + (- (a x 0)) = 0 by (vi).
Thus
0 = a x 0 + 0.
Since a x 0 + 0 = a x 0 by (iv), we conclude that
0 = a x 0.
QED.

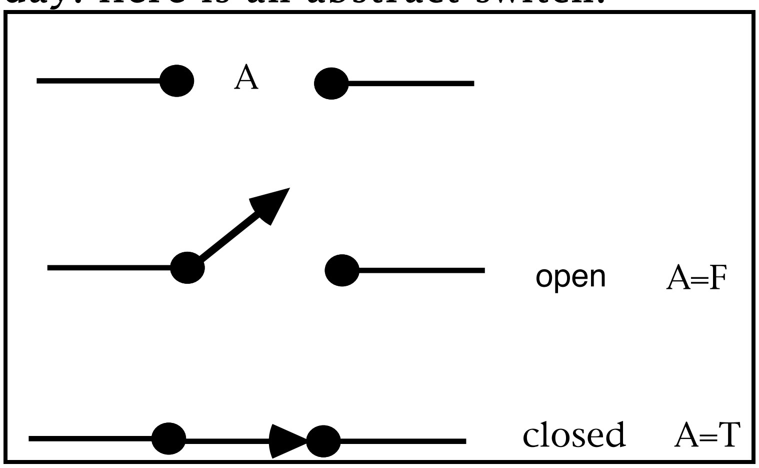
Note that in applying an axiom like "x + 0 = x for any x"
we can put whatever we like in place of x.
Elephants + 0 = Elephants?
Well, not quite anything we like. It has to be about numbers for these axioms. So we can say
27 + 0 = 27
and we can say
(xy + 2z) + 0 = (xy + 2z).
In justifying our steps in the proof, we have not explicitly indicated such substitutions.

###
Problem 2.4. p. 9 - 1.3, 1.4, 1.5

p. 29, 3.1, 3.2, 3.7

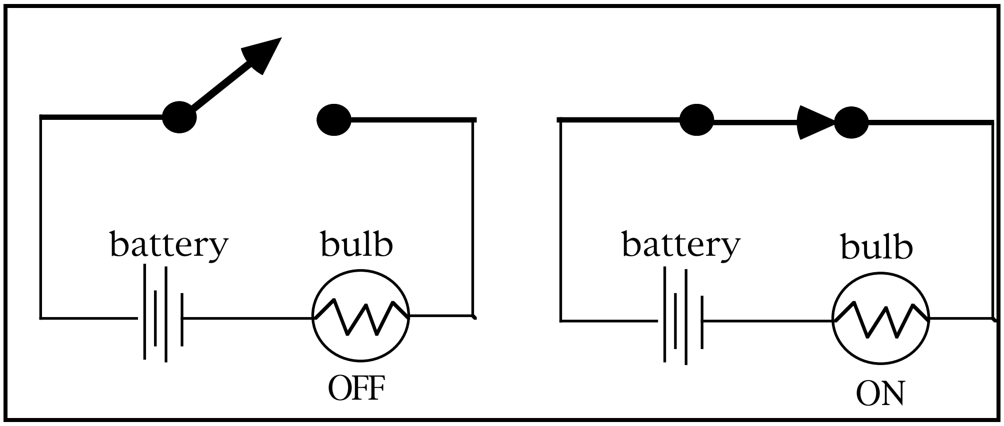
Due Monday, March 10, 2009.

Problem 2.5. This problem introduces an interpretation of logic in terms of switching circuits. This application of logic to switching circuits is the discovery of Claude Shannon
The Journal of Symbolic Logic, Vol. 18, No. 4 (Dec., 1953), p. 347
and forms the basis for the design of computers to this day. Here is an abstract switch:

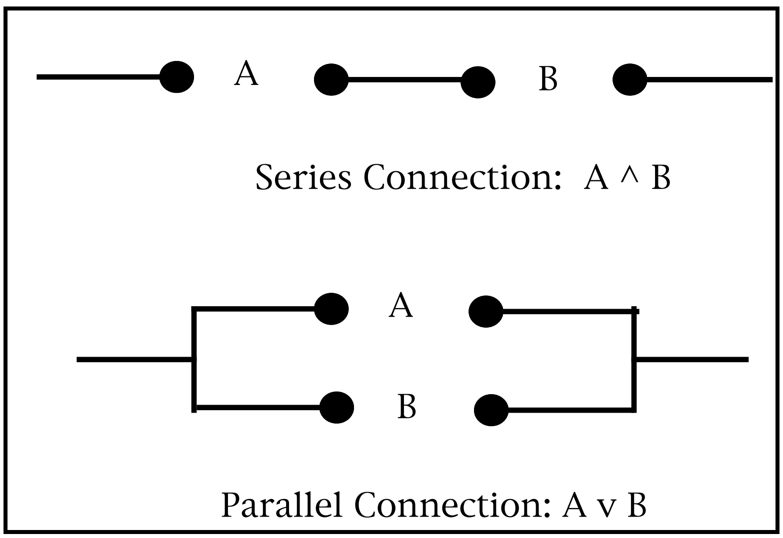


A signal can go from left to right through the switch when it is closed, and no signal can go through when the switch is open. We choose to designate a switch by a label such as A above and we let A = T correspond to the closed switch position (T for "transmit" if you like!) and we let A = F when the switch is in the open position (F = ~ T = not transmit).

The position of a switch can control a device such as a lamp.



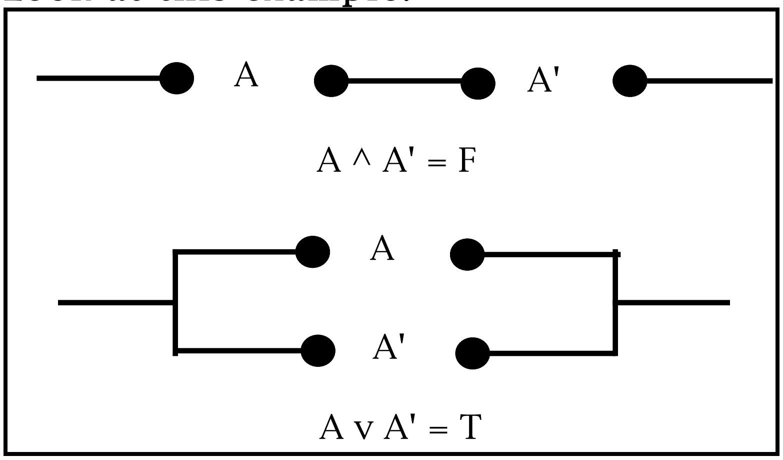
Two basic ways to put switches together are Series and Parallel Connection:



As you can see, the only way for a signal to get from left to right in the series connection of A and B is if both A and B are T (closed). Thus the series connection corresponds to A and B which we write in logic notation as A ^ B. Similarly, the only way for a signal to get from left to right in a parallel connection is if one of A or B is closed. Hence this corresponds to A or B, which we write as A v B. Thus our two basic logical operations are mirrored in the behaviour of networks that carry signals.

What about negation? An example will show you how we handle this. What we do is, we allow multiple appearances of a given label or its negation. We will write either ~A or A' for the negation of A. Here we will use A' ok?

Then in the mutiple appearances, all A's will be either closed or open and if you have A' and A is closed, then A' will be open. If A is open then A' will be closed.
Look at this example:



Each of these circuits has an A and an A'. In the series connection, this means that one switch is always open and so the value of the whole circuit is the same as a simple open circuit. On the other hand in the parallel connection of A with A' either one line will transmit, or the other line will transmit. So the circuit as a whole behaves like a single switch that is closed. Thus we see that the series connecction corresponds to the identity A ^ A' = F, while the parallel connection corresponds to the identity
A v A' = T.

Terminology: Since a label A in one of our circuits can appear in many places we will still refer to A as a 'switch' even though it may be composed of a number of elementary switches. In such a multiple switch, if you make one, there has to be a mechanical way to make sure that all the different parts of the switch work together. Also these switching patterns may, in practice, happen in elecronic circuitry that syncronizes actions at separate locations. For our purposes, we shall imagine simple mechanical switching devices.

Problem: Illustrate the two identities (distribution laws in logic) with the correspoinding switching circuits.

(i) a ^ (b v c) = (a ^ b) v (a ^ c)
(ii) a v (b ^ c) = (a v b) ^ (a v c).

In each case, draw the corresponding circuits and explain why they have the same signal transmission behaviour.

Due Wednesday, March 11, 2009.

##

Problem 2. 6. Design a switching circuit with three switch labels a,b,c such that each of a, b and c individually control transmission of the signal. That is if the circuit is open, then changing the state of any one of a,b,c will close the circuit and if the circuit is closed, then changing the state of any one of a,b,c will open the circuit.
(We discussed how to do this with in analogous case of two labels and how it is related to $(a \wedge b) \vee (a' \wedge b')$ in class.)

Due Monday, March 16, 2009.

Problem 2.7. p. 72, 6.1, 6.4, 6.5, 6.7.

Due Monday, March 16, 2009.

Problem 2.8. p. 86, 7.6, 7.7.

Due Wednesday, March 18, 2009.

Problem 2.9. Generalize Problem 2. 6. to an arbitrary number N of switch labels. That is, you would like to be able to control a single lamp with any one of N switches. You want to find a design that will work in principle for (say) a building with N floors, so that you can control the entrance light from any of the floors. The switch on any given floor will turn the entrance light on or off. That switch on a given floor will have two positions just as our labels have two states T or F. This is a hard problem.
The notes on Logic Circuits should make it accessible to you. See these notes on the website.
Due Monday, March 30, 2009.

Problem 2.10. p. 99, 8.1, 8.2, 8.3
Due Friday, March 20, 2009.

Problem 2.11. p. 113, 9.1,9.2,9.3,9.4,9.5,9.6.

Due Wednesday, April 1, 2009.

Problem 2.12. p. 115, 2., 3., 6., 8.
Due Friday, April 3, 2009.

Problem 2.13.
(a) page 132 #10.3.
(b) Prove that for all sets x,y,z: If {x.y} = {x,z}, then y = z.
(c) Prove that for all sets a,b,c,d:
If {{a},{a,b}} = {{c},{c,d}}, then a = c and b = d.
Due Wednesday, April 8, 2009.

Cantor's Theorem
In class we covered Cantor's proof that P(X) > X for any set X,
where P(X) is the set of subsets of X.
Here is the gist of the proof:
Suppse F:X ----> P(X) is any mapping of these sets.
Let C = {x in X| x is not in F(x)}.
Then C cannot equal F(z) for any z.
To see this, consider a set F(z).
If z is in F(z) then z is not in C and so C and F(z) are different.
If z is not in F(z) then z is in C and so C and F(z) are different.
Therefore C is not of the form F(z) and so F is not surjective.
We have shown that no mapping F:X ----> P(X) is surjective. Since it is easy to construct an injection from X to P(X) (e.g. x is sent to {x}) this proves that X < P(X).

Problem 2.14.
(a) Let N_n = {1,2,3,...,n} so that N_n has cardinality n.
We write |X| for the cardinality of a set X. And so write $|N_n|$ = n. Now, let P(X) denote the set of subseets of a given set X. Prove by induction on n that $P(N_n)$ has cardinality 2^n. That is, $|P(N_n)| = 2^n$.

(b) Let X = {a, b,c}. Make an explicit form for P(X).
That is, make an explicit list of all the subsets of X.
Choose a function F: X ----> P(X) (make this quite explicit).
Then apply Cantor's process to F and form the
set C = {x in X | x is not in F(x)}. You should find that
C is not of the form F(z) for any z in X. Discuss.

(c) Consider the collection Aleph of *all possible sets whose*

members are themselves sets. Sets in Aleph can be infinite or finite. In a sense, Aleph is "the collection of all sets". Aleph begins with the empty set { }. Then you get
{ { } }. After that, come things like
{ { }, {{ }}, { { }, {{ }} }, {{{ }}} }.
And there are lots of infinite sets of sets. For example
{{}, {{}}, {{{}}}, {{{{}}}}, ...}. And you can take sets of these. It grows and grows and whenever you have a element X of Aleph, you can look inside and see that all the members of X are themselves sets, and hence members of Aleph. So any member of Aleph is itself a subset of Aleph.
Any subset of Aleph is a collection of sets and so is also a member of Aleph. Thus it seems that we have proved that Aleph = P(Aleph) in contradiction to Cantor's Theorem that says that the power set of a set is always bigger than the set. Think about this. This is not exactly a puzzle. It is really a philosophical and mathematical problem about the use of the word "all". We will discuss this further in class.
Due Friday, April 10, 2009.

Problem 2.15.
Let X be a finite set and Y another finite set.
Let X^Y denote the intersection of X and Y and let
XvY denote the union of X and Y. Let |X| denote the cardinality of X. Prove, in your own words, the formula
|XvY| = |X| + |Y| - |X^Y|.
Due Monday, April 13,2009.

Problem 2.16. page 155. problems 12.1, 12.2, 12.5
Due Monday, April 20, 2009.

Problem 2.17. page 182. problem 4.
page 185. problem 17.
Due Monday, April 27, 2009.