Quiz 1 and Answers - Math 215 - Fall 2009

1. Give a complete and careful proof by induction that

$$2^0 + 2^1 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 1$$

for $n = 0, 1, 2, \cdots$.

Answer. Let P_n for $n = 0, 1, 2, \cdots$. be the statement

$$2^0 + 2^1 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 1.$$

Note that P_0 states that $2^0 = 2^1 - 1$, and that this is the same as the statement $1 = 2 - 1$, a true statement in arithmetic. This establishes the base of the induction argument. For the induction step, suppose that P_k is true for some specific non-negative integer k. Then we calculate, using the fact that P_k is true:

$$\begin{aligned} &2^0 + 2^1 + 2^2 + 2^3 + ... + 2^k + 2^{k+1} \\ &= (2^0 + 2^1 + 2^2 + 2^3 + ... + 2^k) + 2^{k+1} \\ &= (2^{k+1} - 1) + 2^{k+1} \\ &= 2 \times 2^{k+1} - 1 \\ &= 2^{(k+1)+1} - 1. \end{aligned}$$

Our calculation shows that

$$2^0 + 2^1 + 2^2 + 2^3 + ... + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1.$$

Thus we have proved that $P_k \Rightarrow P_{k+1}$ for all $k = 0, 1, 2, \cdots$. This completes the induction step, and thus we have proved the theorem by mathematical induction. //

2. (a) The *contrapositive* of $P \Rightarrow Q$ is the statement $(\sim Q) \Rightarrow (\sim P)$. Use the following basic facts of logic

$$(A \Rightarrow B) = (\sim A) \vee B, \ \ \sim\sim A = A, \ \ A \vee B = B \vee A,$$

to show algebraically (without recourse to truth tables) that a statement and its contrapositive are logically equivalent.

Answer.

$$\begin{aligned} P \Rightarrow Q &= (\sim P) \vee Q = Q \vee (\sim P) \\ &= (\sim\sim Q) \vee (\sim P) \\ &= (\sim Q) \Rightarrow (\sim P). \end{aligned}$$

3. The following is a "proof" that $1 = 0$. What is wrong with this proof? (Note that you need to trace through the steps of the proof and find an error, but the fact that the last line of the proof says $1 = 0$ is not the error. The error occurs earlier than that!)

Theorem. $1 = 0$.

Proof. Begin with x and y non-zero and $x = y$. Then $x = y$ implies that $x^2 = xy$ and subtracting y^2 from both sides, we have $x^2 - y^2 = xy - y^2$. Now divide both sides by $x - y$ and get $x + y = y$. But since $x = y$ we then have $2y = y$ and since y is non-zero we divide by y and get $2 = 1$. Subtracting 1 from both sides, we have shown that $1 = 0$. **QED.**

Answer. Since $x = y$, we have $x - y = 0$. Therefore it is not possible to divide by $x - y$, since one cannot divide by 0.