Reflections Remarks and Review -- Math 215

I. Primes

In the last few days of the course we proved the uniqueness of prime factorization of natural numbers and we also discussed generating functions for partitions of integers. These ideas about generating functions for partitions led Euler to formulas like

$$1 + p(1)x + p(2)x^2 + p(3)x^3 + \ldots = [1/(1-x)][1/(1-x^2)][1/(1-x^3)]\ldots$$

where p(n) is the number of partitions of n and we regard all these infinite series in a strictly formal way. You are encouraged to read the Wiki article about partitions and to think about this subject!

Euler used a very similar idea to prove that there are infinitely many
prime numbers. His proof is quite different from the Euclid proof (which I will recall in a moment). Euler's proof depends on the fact that the series

$$1 + 1/2 + 1/3 + 1/4 + 1/5 + \ldots \quad \text{diverges.}$$

Can you prove this?

Given that fact, Euler observes that we have the formal identity

$$1 + 1/2 + 1/3 +1/4 +1/5+1/6 +1/7 + \ldots$$
$$= [1/(1-1/2)][1/(1-1/3)][1/(1-1/5)][1/(1-1/7)]\ldots$$

where the numbers on the right hand side (2,3,5,7,...) run through all the prime numbers. If there were only a finite number of prime numbers, then the right hand side of this formula would be a finite rational number. This would contradict that fact the the series on the left hand side diverges (goes to infinity). Thus we have a proof by contradiction that there must be infinitely many prime numbers!

This observation of Euler about proving the infinity of the primes was the beginning of many new aspects of the theory of numbers and the beginning of our understanding of questions about the distribution of the prime numbers. One of the first things that Euler did was to take the logarithm of both sides of the above formula and analyze the results. He was then able to prove (and you can too!) that the sum of the reciprocals of all the prime numbers diverges.

That is

$$1 + 1/2 + 1/3 + 1/5 + 1/7 + 1/11 + 1/13 + 1/17 + 1/19 + 1/23 + \ldots$$

is a divergent series. This means that not only are there infinitely many primes, but they occur with enough frequency in the natural numbers to make the sum of their reciprocals as larger as you please if you take enough of them.

Remark. Euclid proved that there were infinitely many primes by giving a procedure to produce new primes: Let $\{p_1, p_2, \ldots, p_n\}$ be a set of n distinct prime numbers. Form the number

$$N = 1 + p_1 \times p_2 \times \ldots \times p_n$$

N is one plus the product of all those primes. Now N is not divisible by any of the primes in our set of n primes (since we'll get a remainder of 1 in each case of division). Therefore N is either a new prime number, or by prime factorization N must have prime factors that are outside the set of n primes that we started with. In either case, we have produced new primes and we now know more than n prime numbers. Hence for any n, there are more than n prime numbers. There are infinitely many prime numbers.

II. Favorite Theorems

I would like you to be prepared to prove the following Theorems on the final exam.

1. The square root of two is irrational.

2. There are infinitely many prime numbers.

3. The set of rational numbers is countable.

4. If S is the set of all infinite sequences of 0's and 1's, then S is uncountable. (Cantor diagonal argument).

5. If X is any set, then $|X| < |P(X)|$. This is Cantor's general diagonal argument.

6. The inclusion-exclusion principle.

7. The binomial theorem.

Of course there are many things that we could ask you to prove, some that you have never seen before, but will be able to understand from the problem statement. The reason for asking you to know the proofs of the above results is that knowing these proofs is a major step in understanding both the course and the meaning of proof in mathematics.

III. General Review

In this course we have studied symbolic logic with truth tables and algebraically, logic with quantifiers (for all, there exists), the logic of sets and Venn diagrams, mappings of sets and constructions of new sets from given sets (such as the cartesian product and the power set). We have worked with the concepts of injection, surjection and bijection and have studied the notion of counting both finite and infinite sets using bijection (1-1 correspondence) as the definition of same cardinality. We have studied combinatorial counting arguments such as the principle of inclusion/exclusion and we have studied arguments about the "sizes" of infinite sets such as Cantor's diagonal argument. We have begun a bit of number theory with the proofs that natural numbers have a unique prime factorization and the proofs that there are infinitely many primes.

In the course of this we have worked with a number of types of proof including direct deduction, proof by contradiction, use of the contrapositive and proof by mathematical induction.

What has been done in the course forms a basis for studying many areas of mathematics and everything we have done can itself be studied more deeply. There are questions and investigations to make at every turn. Good luck in all your further studies!