Sample Second Exam - Math 215 - Fall 2010

Write all your proofs with care, using full sentences and correct reasoning.

1. Prove $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n}=2-\frac{n+2}{2^n}$ for all $n=1,2,3,\cdots$.

Solution. This is a straightforward induction argument. Solution omitted.

2. Prove that, for a positive integer n, a $2^n \times 2^n$ square grid with any one square removed can be covered using L-shaped non-overlapping tiles. Each tile consists in three adjacent grid squares in an L-shaped pattern.

Solution. We have discussed this in class.

3. Prove that the following two statements are equivalent:

$$A \Rightarrow (B \Rightarrow C)$$

and

$$(A \wedge B) \Rightarrow C.$$

In your proof, do *not* use truth tables. Use the fact that

$$A \Rightarrow B \ = (\sim A) \vee B,$$

and give a completely algebraic proof.

Solution.

$$A \Rightarrow (B \Rightarrow C) = (\sim A) \vee ((\sim B) \vee C)$$
$$= ((\sim A) \vee (\sim B)) \vee C) = (\sim (A \wedge B)) \vee C = (A \wedge B) \Rightarrow C.$$

4. Define the composition of the function $f : X \longrightarrow Y$ and the function $g : Y \longrightarrow Z$ to be the function $g \circ f : X \longrightarrow Z$ with $g \circ f(x) = g(f(x))$ for all $x \in X$. Prove that if f is surjective and g is surjective, then $g \circ f$ is surjective.

Solution. Let $z \in Z$. Then $z = g(y)$ for some $y \in Y$ since g is surjective. And then $y = f(x)$ for some $x \in X$ since f is surjective. Therefore $g \circ f(x) = g(f(x)) = g(y) = z$. Hence $g \circ f$ is surjective.

5. Given sets A and B, consider the following two statements about a function $f : A \longrightarrow B$.
(i) $\exists b \in B$ such that $\forall a \in A, f(a) = b$.
(ii) $\forall b \in B, \exists a \in A$ such that $f(a) = b$.

One of these statements is the definition for f to be a surjective mapping from A to B. Which one is it? For the other statement, please explain what it says and give an example of a function from $A = \{1, 2, 3\}$ to $B = \{1, 2\}$ that has this property.

Solution. The correct choice is **(ii)**. The rest of the solution is omitted.

6. (a) Let $N = \{1, 2, 3, \cdots\}$ be the set of natural numbers. Let $P(N)$ denote the set of subsets of N. Let $F : N \longrightarrow P(N)$ be any well-defined mapping from N to its power set $P(N)$. Show that F is not surjective. Your proof should not depend upon any particular choice of F.

Solution. Let $C = \{n \in N | n \notin F(n)\}$. Then it follows at once that C is not of the form $F(n)$ for any $n \in N$. For if $C = F(m)$ for some m, then $m \in C$ iff $m \notin F(m)$. But this means $m \in C$ iff $m \notin C$. This is a contradiction, and we conclude that C is not equal to $F(m)$.

(b) Prove that there is a $1 - 1$ correspondence between the set

$$O = \{1, 3, 5, 7, 9, 11, \cdots\}$$

of odd natural numbers and the set N of all natural numbers.

Solution. Map $n \in N$ to $2n - 1 \in O$.

(c) Make the special assumption that if x is any set, then it is not the case that x is a member of x. On the basis of this special assumption prove that there does not exist a set U such that for all sets $y, y \in U$.

Solution. If U is a set, then U would have to be a member of itself. We have said that no set is a member of itself. Therefore U is not a set.

7. (a) Prove that $\sqrt{5}$ is irrational.

Solution. Omitted.

(b) Prove that there exist irrational numbers a and b such that a^b is rational.

Solution. Consider $\sqrt{2}$. If $\sqrt{2}^{\sqrt{2}}$ is rational, we can take $a = b = \sqrt{2}$. If $\sqrt{2}^{\sqrt{2}}$ is irrational, then we take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. Then $a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$, and so it is rational. This proves the result without deciding whether $\sqrt{2}^{\sqrt{2}}$ is irrational or rational. In fact, $\sqrt{2}^{\sqrt{2}}$ is irrational, but this is much harder to prove.

8. Using discrete calculus, find a formula for

$$S(n) = 1^3 + 2^3 + 3^3 + \cdots + n^3.$$

Solution. Hint: Use the fact that $\Delta S(n) = (n+1)^3$. Lets start at the bottom and do the needed work. Recall that

$$\Delta(F(n)) = F(n+1) - F(n).$$

We are using the fact that

$$\Delta(n^{(r)}) = rn^{(r-1)}$$

where

$$n^{(r)} = n(n-1)(n-2)\cdots(n-r+1).$$

You have to discriminate between n^r and $n^{(r)}$. Let $Int(f(n))$ denote any function such that $\Delta(Int(f(n)) = f(n)$.

1. $n = n^1 = n^{(1)}$ so nothing new here. Thus $Int(n) = (1/2)n^{(2)} + k = (1/2)n(n-1) + k$.

2. $n^2 = n(n-1) + n = n^{(2)} + n^{(1)}$. Therefore

$$Int(n^2) = (1/3)n^{(3)} + (1/2)n^{(2)} + k.$$

3. Start with $n^{(3)} = n(n-1)(n-2) = n^3 - 3n^2 + 2n$ and rewrite as

$$n^3 = n^{(3)} + 3n^2 - 2n.$$

Then substitute the formula you have for n^2 and get

$$n^3 = n^{(3)} + 3n^{(2)} + n^{(1)}.$$

(calculations omitted, but you can check this.) Now you can find $Int(n^3)$ and also $Int((n+1)^3)$, which is what is needed to solve the present problem.

4. By multiplying out $n^{(4)} = n(n-1)(n-2)(n-3)$ and using what we already have, you should be able to show that

$$n^4 = n^{(4)} + 6n^{(3)} + 7n^{(2)} + n^{(1)}$$

and from this you can find $Int(n^4)$. And you can use this to solve the homework problem about the sum of fourth powers!

9. The following problem is due to the Reverend Charles Lutwidge Dodgson (27 January 1832 to 14 January 1898), also known as Lewis Caroll. He is the author of books on Symbolic Logic and also the books "Alice's Adventures in Wonderland" and "Through the Looking-Glass."

From the following three assertions we are to make whatever deductions are possible.

(i) Nobody who really appreciates Beethoven fails to keep silence while the Moonlight Sonata is being played.

(ii) Guinea-pigs are hopelessly ignorant of music.

(iii) No one who is hopelessly ignorant of music ever keeps silence while the Moonlight Sonata is being played.

These can be interpreted as statements about various sets. Let
G = the set of guinea-pigs.
H = the set of creatures that are hopelessly ignorant of music.
K = the set of creatures who keep silence while the Moonlight Sonata is being played.
R = the set of creatures that really appreciate Beethoven.

Rewrite each of **(i), (ii), (iii)** as a statement about sets in set theoretic notation. For example, statement (i) says that $R \subseteq K$. Use this rewrite to deduce that "Guinea pigs do not really appreciate Beethoven."

Solution.

(i) $R \subseteq K$.

(ii) $G \subseteq H$.

(iii) $H \subseteq K^c$.

Conclusion: $G \subseteq K^c$, hence, since $R \subseteq K$, $G \subseteq R^c$. That is, "Guinea pigs do not really appreciate Beethoven.".