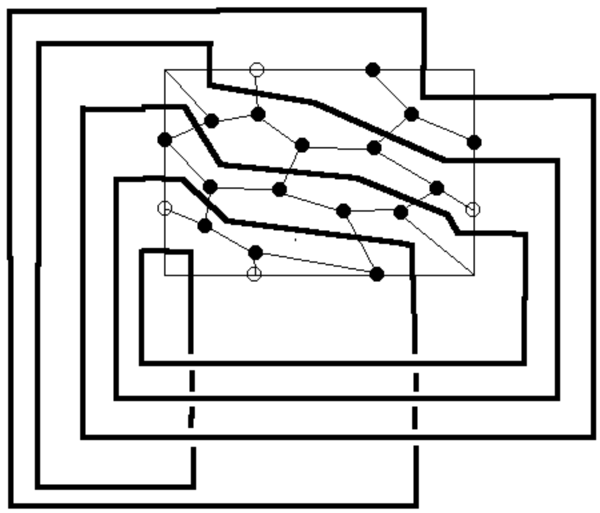
Knots in the Seven Color Map

by Louis H. Kauffman

Introduction



In the figure above you see a knot traced by the bold lines. This knot is topologically equivalent to a simple trefoil knot. See the deformation in section 1 if you do not believe this!

Underneath the knot there is a rectangular bit of hexagonal paving. The paving represents a hexagonal tiling of the surface of a torus. The torus is obtained by identifying the top edge of the rectangle with the bottom edge, and the left edge with the right edge. See section 2 for more about how the torus is obtained via identifications.

These identifications can actually be performed so that the topology of the knot is preserved and the knot appears on the surface of the torus with the torus embedded in three dimensional space in the "usual" way. The picture above can be interpreted as instructions for drawing a curve (with no self crossings) on the surface of a torus so that the following two conditions are met:

1. The curve winds two times around the torus in one direction and three times around it in the other direction, forming a (2,3) torus knot in three space (via the standard embedding of the torus in 3-space).
2. The curve goes through each of the seven hexagaonal tiles of the tiling of the torus exactly once, meeting the boundaries of the tiles transversely.

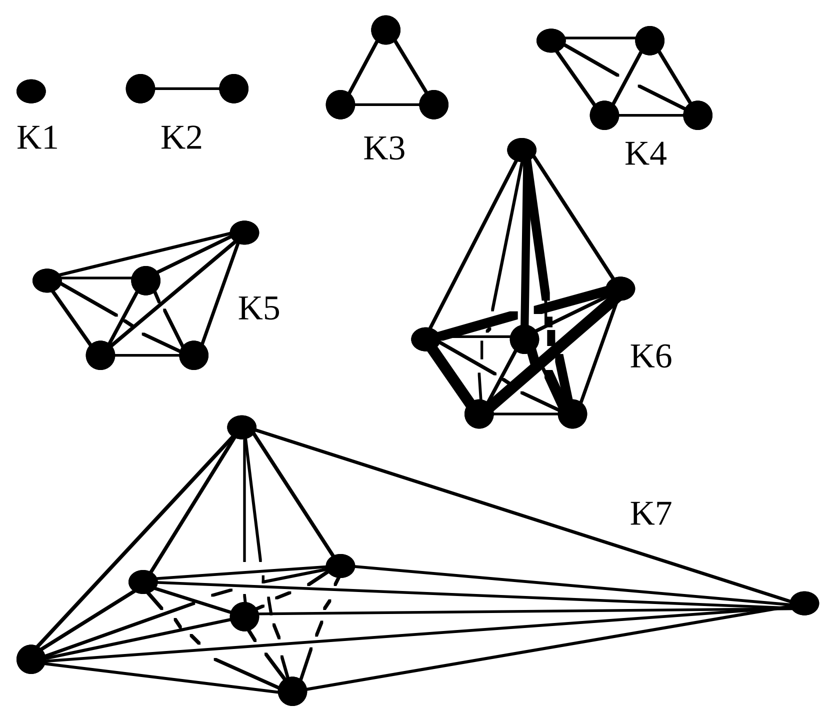
The hexagonal tiling of the torus shown above is often called the seven color map on the torus. That is, if we desire to color the regions of the tiling so that two adjacent regions have distinct colors then seven colors are needed (since each hexagon touches six neighbors).

By other means (the Euler formula) one can show that every map on the torus can be colored with no more than seven colors. The seven color map shows that seven colors are sometimes needed. This is the final fact that is needed for the

Seven Color Theorem: *Every map on the torus can be colored with no more than seven colors, and seven is the least number for which this can be stated.*

It is a strange and beautiful fact that the seven color map on the torus winds upon the surface of the torus forming a kind of vortex when you model it in three dimensions. There is a topological reason for this vortex, and it is related to the knot that we have drawn!

A Theorem due to Conway and Gordon that says that *the complete graph on seven nodes is intrinsically knotted*. A complete graph on a set of nodes is a graph in which there is exactly one edge between each pair of possible nodes. See the figure below for a depiction of the complete graphs K1 to K7.

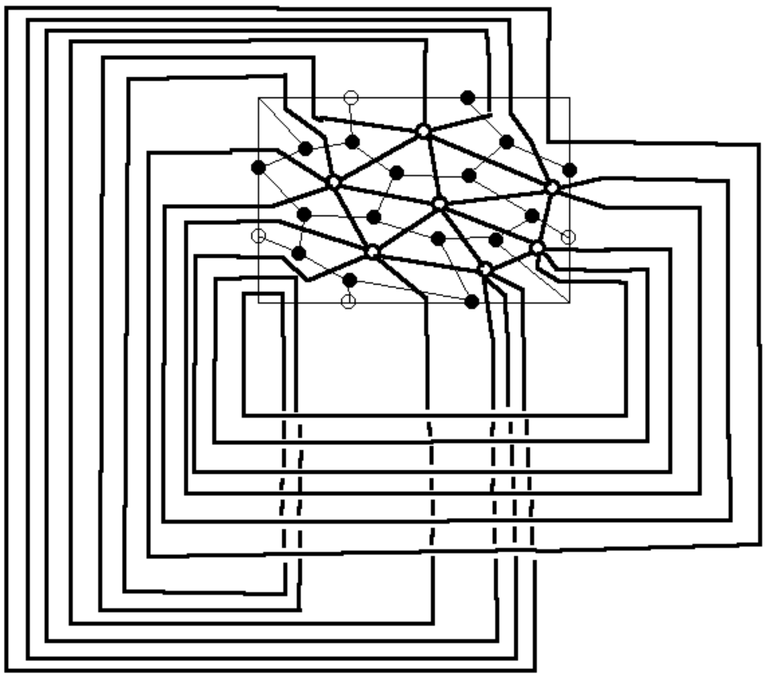


Complete Graphs

In the Figure above you will see a specific embedding of K7 in 3-space. We leave it to you to find the knot here! The knot will be in the form of a walk along the edges of K7 (not all of them!) that goes through each node once. (If you could find such a walk that used less than all the nodes, that would be fine so long as it is knotted in 3-space. In this example there is no such walk.) In K6 you will be able to find two curves that are linked with one another. This is a related Theorem: *K6 is intrinsically linked*. It is much easier to find the link in K6 as shown above and in fact we have illustrated it above with bold lines for the walk.

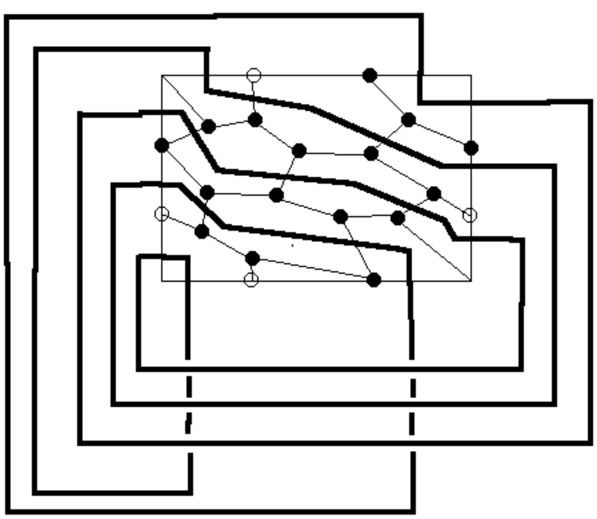
The key point about these theorems about the intrinsic linkedness of K6 and the intrinsic knottedness of K7 is that *the linking or knotting will occur no matter how the graph is embedded in three space*. Thus you can make a drawing yourself of K6 or K7 and it is guaranteed that no matter how you draw it, how you set the self-crossings, how you make the connections in 3-space, there will be a link in K6 and there will be a knot in K7!

Now seven nodes connected each to all the others is very like seven hexagons such that each hexagon shares boundary with each of the others. And indeed the fact that there is a paving of the torus with seven hexagons tells us that there is a drawing of the graph K7 in the surface of the torus with one node for each hexagon. Here is a drawing of that graph:



K7 on the Torus

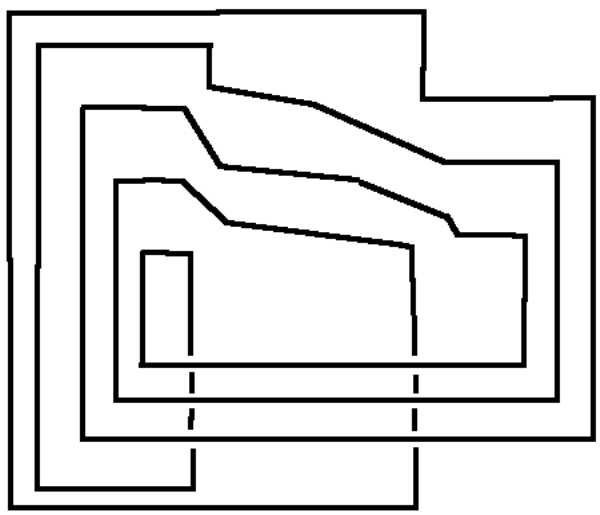
In this drawing we show the nodes of the K7 as little white circles, (please do not confuse these with the less bold white circles on the original seven color map) one node for each hexagonal tile on the torus. Even in the almost planar embedding of the K7 of this drawing, Conway-Gordon Theorem tells us there must be a knot and in fact, that is the knot that was the first diagram in this article. We reproduce the diagram below so that you can make the comparison.



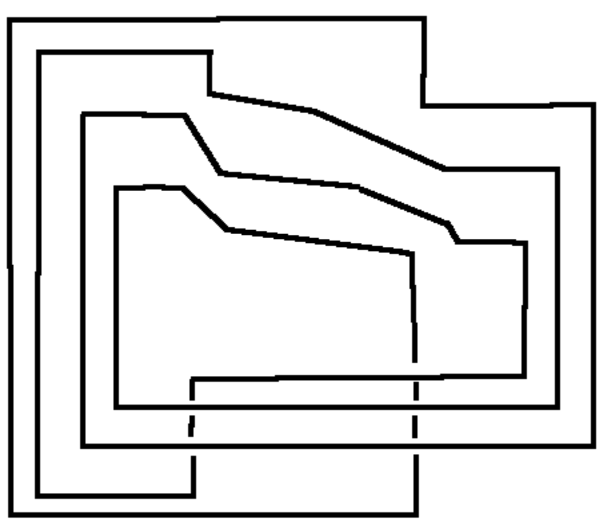
So by the Conway-Gordon theorem there has to be a knot inside the K7 that is embedded in the torus (in three space) that is determined by the seven - color map on the torus. A knot that is induced from a curve on a torus must wind around the torus. The simplese such knot winds around the torus three times in the meridian direction on the torus and twice in the longitude direction. This means that it is not an accident that the hexagonal paving on the torus winds around it. The winding is needed for the Conway-Gordon Theorem to be true! This is the topology of the seven color map.

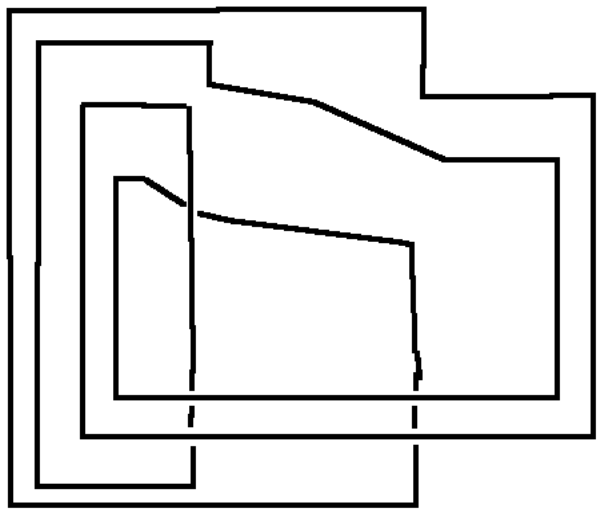
I. Simplifying the Knot

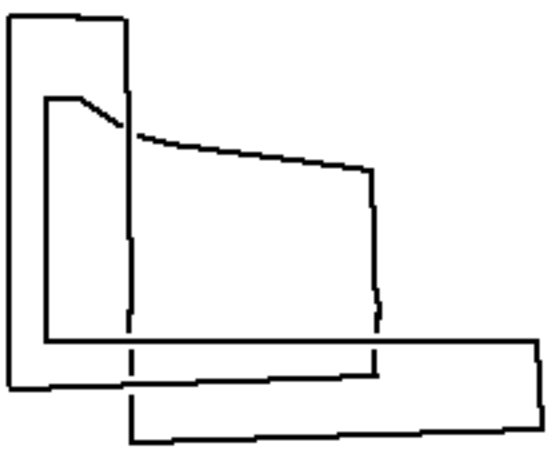
Here is our knot as drawn at the beginning.

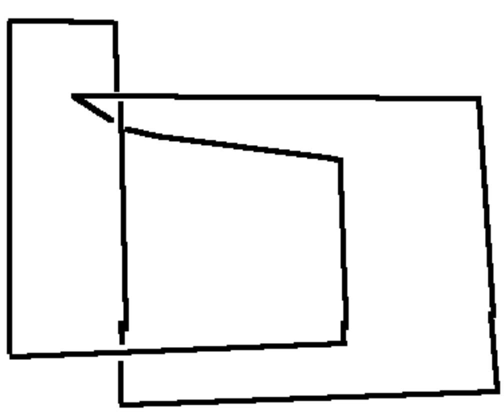


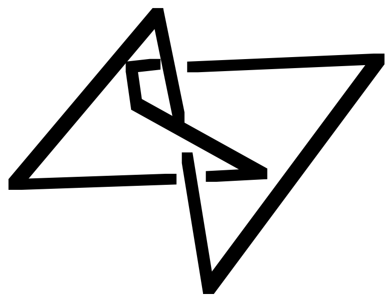
The next sequence of pictures simplify it to a trefoil.





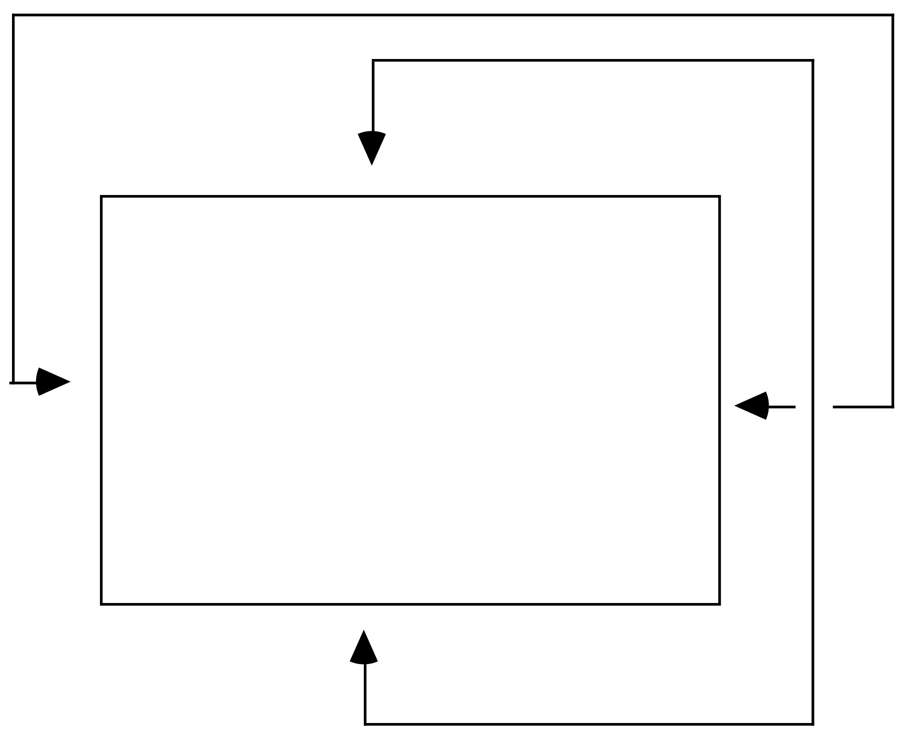




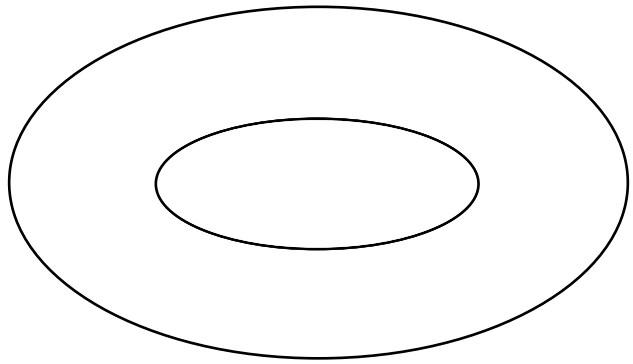


II. The Torus

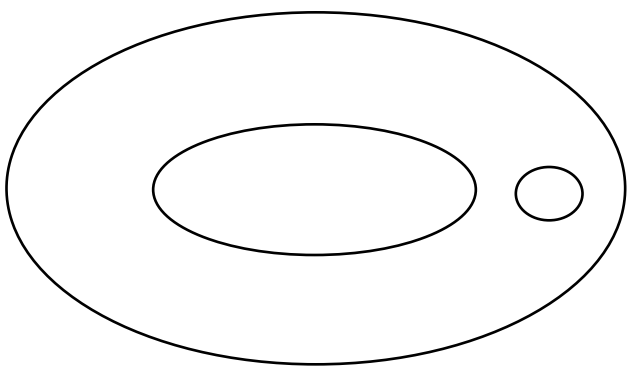
We take a rectangle and identify the opposite sides.



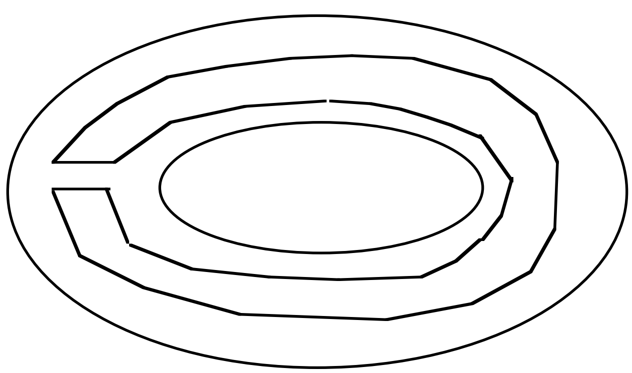
This gives a torus.



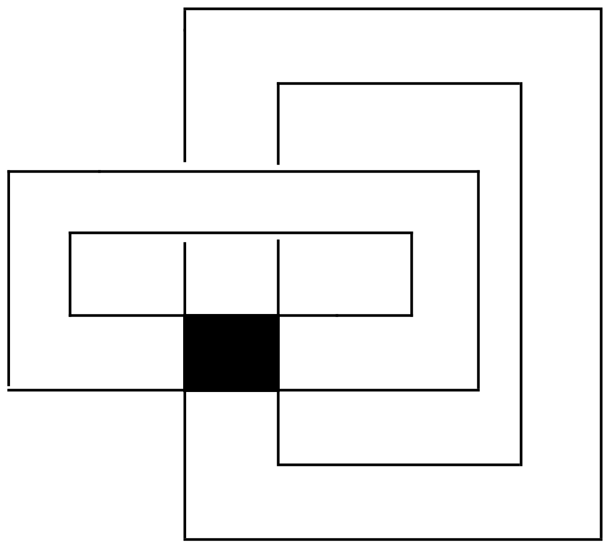
Suppose you cut a hole in the surface of the torus.



Now enlarge the hole.

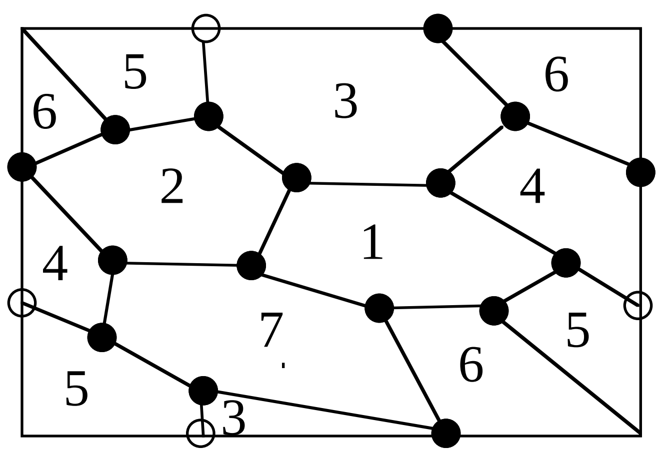


Flatten the remaining material and you find that you have two annuli attached to one another. This can be described by saying that you have a (black) rectangle with the top and bottom attached by a strip and the left and right edges attached by another strip as shown below. The picture below is therefore topologically a torus with a hole removed!

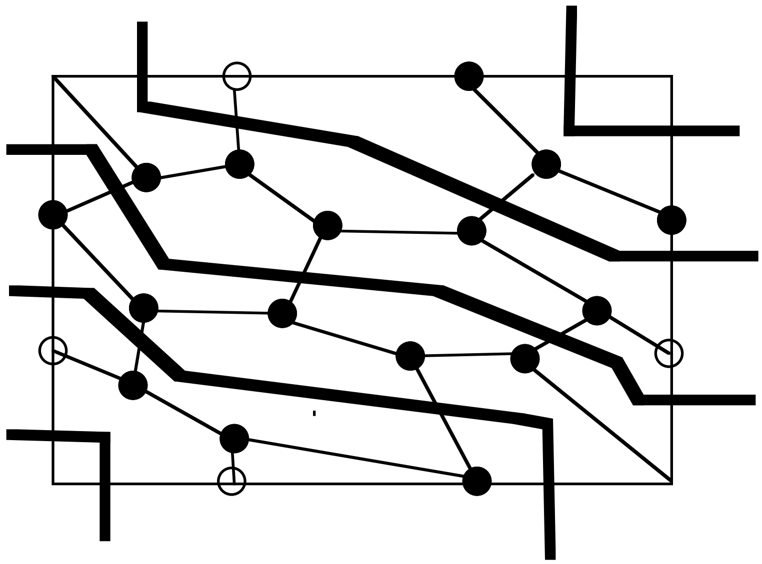


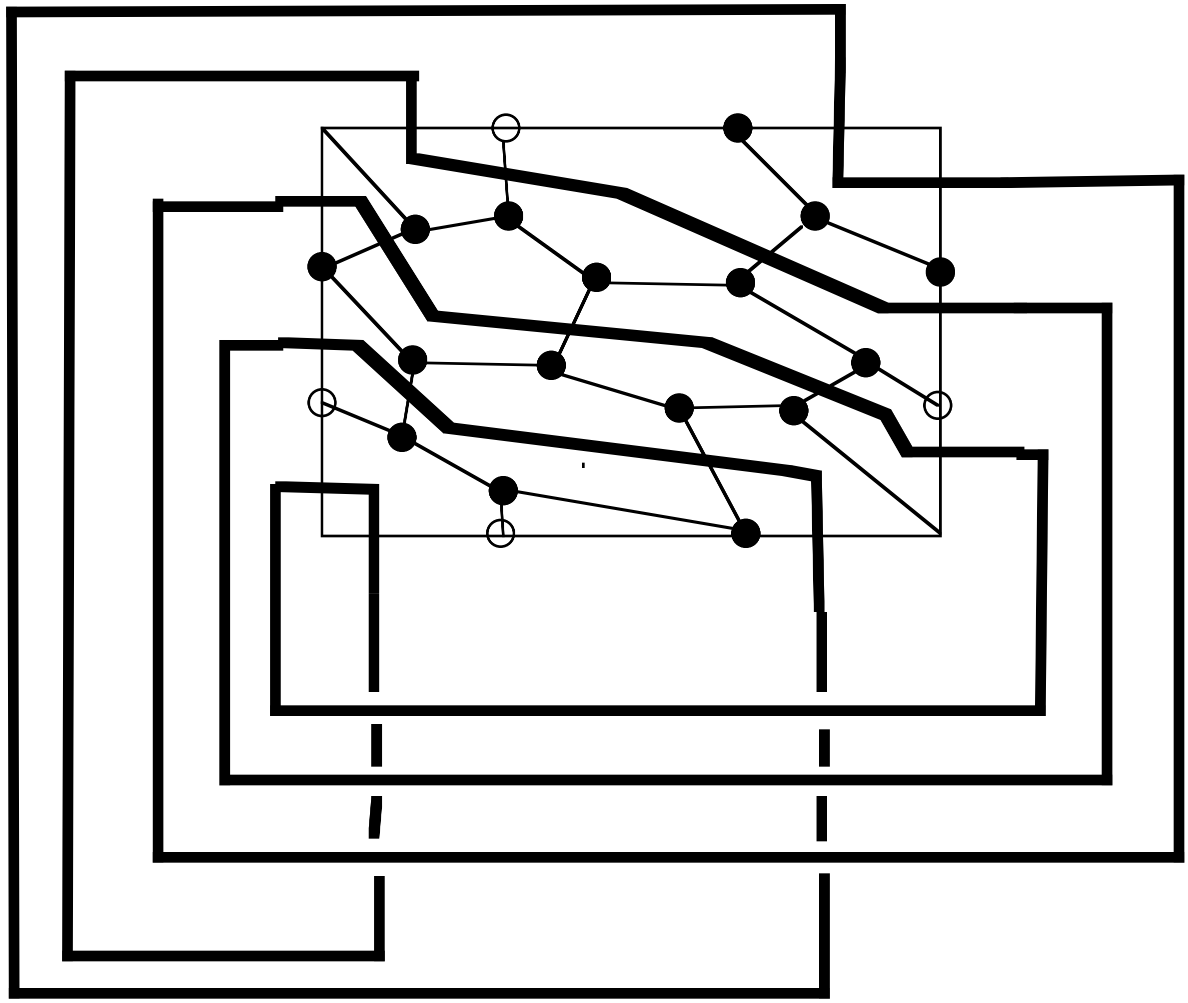
A Punctured Torus

Here is the seven color map on the torus, depicted by thinking of the torus as a rectangle with opposite edges identified.

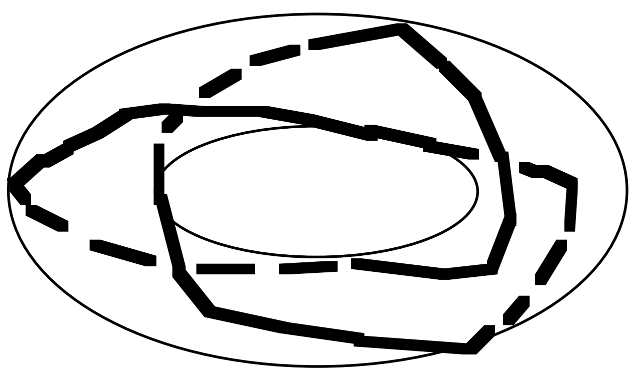


Here is the drawing of the curve on the torus that becomes our trefoil knot.

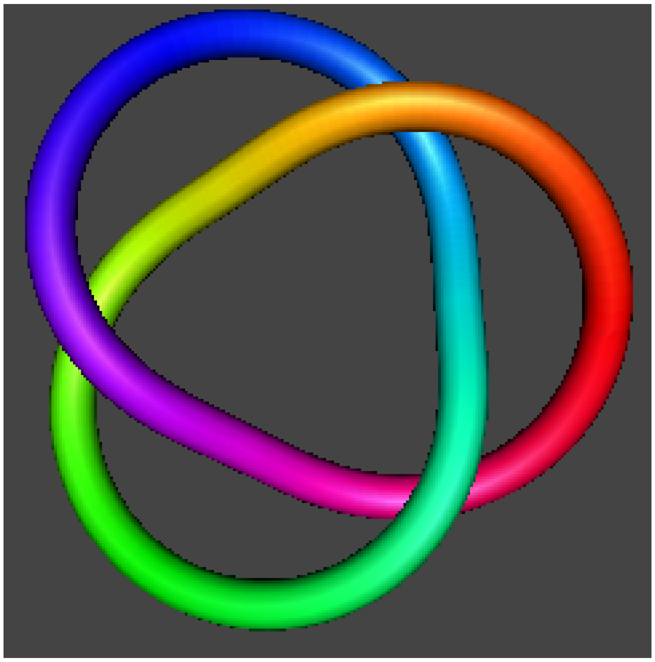




Finally, here is the trefoil knot as it appears on the torus itself.



Or maybe you would prefer the depiction below (which along with being colored and rendered is the mirror image of the trefoil above).



FINIS FOR NOW.