Notes on Signature Theorems

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I) Characteristic Classes

a) Complex line bundles

A complex line bundle is a vector bundle with fiber $\cong \mathbb{C}$. Let $\mathcal{L}(X)$ = set of isom classes of complex line bundles over X. Let $\Lambda \longrightarrow \mathbb{CP}^\infty$ be the standard (univ) line bundle over $\mathbb{CP}^\infty$ and $z \in H^2(\mathbb{CP}^\infty)$ the generator. Note that $\mathbb{CP}^\infty \cong K(\mathbb{Z},2)$ and that standard bundle theory $\Rightarrow$

$$\mathcal{L}(X) \longleftrightarrow [X, \mathbb{CP}^\infty]$$

is a 1-1 correspondence. Here $[\,,\,]$ = homotopy classes of maps and given $f: X \longrightarrow \mathbb{CP}^\infty$ we have

$$\begin{array}{ccc} f^*\Lambda & \longrightarrow & \Lambda \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & \mathbb{CP}^\infty \end{array}$$

the corres pull-back bundle.

Define $c_1(f^*\Lambda) = f^*(z) \in H^2(X,\mathbb{Z})$. This is called the 1st Chern class.

Now $\mathcal{L}(X) \cong [X, \mathbb{CP}^\infty] \xrightarrow{\cong} H^2(X,\mathbb{Z})$

$$f \longmapsto f^*(z)$$

Thus line bundles are classified by 1st Chern class.

Example: $X = S^2 \Rightarrow \mathcal{L}(X) \cong H^2(S^2) \cong \mathbb{Z}$ & $c_1(E) \in \mathbb{Z}$ for $E \in \mathcal{L}(X)$.

There are other ways to get this integer:

i) $S^2 \subset E$ as zero section, let $\tilde{S}^2 \pitchfork S^2$ in E then $c_1(E) = S^2 \cdot \tilde{S}^2$ (inter # in E).

ii) trivialize E over $D^2_\pm$, $S^2 = D^2_+ \cup D^2_-$ then get $\alpha: S^1 \longrightarrow Autom(\mathbb{C}) \sim SO(2) = S^1$ & $\therefore$ an elt $[\alpha] \in \pi_1(S^1) = \mathbb{Z}$.

b) Constructions:

Given vector bundles E, E' over B can form $E \oplus E'$ = Whitney sum (fibers of $E \oplus E'$ are direct sums of fibers of E, E' resp) and tensor product $E \otimes E'$. Note:

$$\mathcal{L}(B) \times \mathcal{L}(B) \longrightarrow \mathcal{L}(B)$$
$$E, E' \longmapsto E \otimes E'$$

Claim: If $E \in \mathcal{L}(B)$ & E^* = complex conjugate bundle (switch cx str in each fiber) then

then $\boxed{E\otimes E^* \simeq \mathcal{E} = \text{trivial bundle over } B}$.

To see this we digress for a moment about transition fcns. If $E \xrightarrow{\pi} B$ is a line bndle then it may be described via an open cover $\{U_\alpha\}$ on which E_{U_α} is trivial for each α, and transition fcns $g_{\alpha\beta}: U_\alpha \cap U_\beta \longrightarrow S^1 = SO(2)$ which provide the pasting data. There is a compatibility condition:
$g_{\alpha\gamma} = g_{\alpha\beta}g_{\beta\gamma}$ when restricted to $U_\alpha\cap U_\beta \cap U_\gamma$.
One way to classify bundles is via equivalence classes of such sets of transition fcns. Now let E be specified by $\{U_\alpha \mid g_{\alpha\beta}\}$. Then E^* is (by definition) specified by $\{U_\alpha \mid \overline{g}_{\alpha\beta}\}$ where $\overline{g}_\alpha(b) = \overline{g_\alpha(b)}$; the second $\overline{}$ denotes complex conjugation. Furthermore, if $E \leftrightarrow \{U_\alpha \mid g_{\alpha\beta}\}$, $E' \leftrightarrow \{U_\alpha \mid g'_{\alpha\beta}\}$ (same covering — can do via common refinement) then
$E\otimes E' \leftrightarrow \{U_\alpha \mid g_{\alpha\beta}g'_{\alpha\beta}\}$. Hence $E\otimes E^*$ obviously trivial. Exercise: Verify this transition fcn descrip of $E\otimes E'$ from usual defn of $E\otimes E'$ as a quotient of $E\oplus E'$ by tensor relations $(a+b, c) \sim (a,c)+(b,c)$ ... etc.

Thus $\mathcal{L}(B)$ is a group with mult corres to $\otimes$. We claim that the correspondence $\mathcal{L}(B) \rightsquigarrow H^2(B,\mathbb{Z})$ is an isomorphism of groups.

There are many ways to do this. Here's one. First we give a pull-back description of tensor product of bundles. Any bundle of fiber dim k over base space B can be obtained via a map $f: B \longrightarrow G_k(\mathbb{C}^n)$ where $G_k(\mathbb{C}^n) = k$-planes in $\mathbb{C}^n$ (see page 5). There is a map $\mu: G_k(\mathbb{C}^n)\times G_\ell(\mathbb{C}^m) \longrightarrow G_{k\ell}(\mathbb{C}^{nm})$ given by taking tensor product of the corresponding subspaces. Given $f: B \to G_k(\mathbb{C}^n)$, $g: B \longrightarrow G_\ell(\mathbb{C}^m)$ then $\mu\circ(f\times g): B \longrightarrow G_{k\ell}(\mathbb{C}^{nm})$ pulls-back the canonical bundle to the tensor product of the bundles corresponding to f and g. In the case of line bundles, we get an H-space structure $\mu: \mathbb{CP}^\infty \times \mathbb{CP}^\infty \longrightarrow \mathbb{CP}^\infty$. It then suffices to show that if $[f],[g] \in [B,\mathbb{CP}^\infty]$ then $[f]+[g]$ in $[B,\mathbb{CP}^\infty] \simeq H^2(B,\mathbb{Z})$ corresponds to $[\mu\circ(f\times g)]$. To see this, let $H = \mathbb{CP}^\infty$ and note that $[\ ,H\times H] \simeq [\ ,H]\times[\ ,H]$ with group str on left corresponding to direct sum of grp strs on right. Thus the diagram on left induces homomorphisms of groups:

$$\left.\begin{tikzcd} H \arrow[d, "i"'] \arrow[dr, "1"] & \\ H\times H \arrow[r, "\mu"] & H \\ H \arrow[u, "j"] \arrow[ur, "1"'] & \end{tikzcd}\quad \begin{matrix} i(x) = (x,*) \\ j(x) = (*,x) \end{matrix}\right\} \qquad \left.\begin{tikzcd} {[B,H]} \arrow[d, "i_*"'] \arrow[dr, "1"] & \\ {[B,H]\times[B,H]} \arrow[r, "\mu_*"] & {[B,H]} \\ {[B,H]} \arrow[u, "j_*"] \arrow[ur, "1"'] & \end{tikzcd}\right\}\quad \begin{matrix} a+b \\ \| \\ \mu_* i_* a + \mu_* j_* b \\ \| \\ \mu_*(i_* a + j_* b) \\ \| \\ \mu_*((a,1)\cdot(1,b)) \\ \| \\ \mu_*((a,b)) \end{matrix}$$

Q.E.D.

c) General Chern Classes

Given a complex vector bundle $E \xrightarrow{p} B$, we wish to show that there are cohomology classes $C_i(E) \in H^{2i}(B, \mathbb{Z})$ (Chern classes) enjoying the following properties:

0) $C_i(E) \in H^{2i}(B)$ and $C_i(E) = 0$ for $i > n$ where $n =$ complex fiber dimension of E.
We write $C(E) = 1 + C_1(E) + C_2(E) + \cdots + C_n(E)$
Here $C(E) \in \bigoplus_i H^{2i}(B)$.

1) If E and $\overline{E}$ are isomorphic over B, then $c(E) = c(\overline{E})$. If $f : \overline{B} \to B$ is a map, then $f^* c(E) = c(f^* E)$.

2) Given $E, \overline{E}$ vector bundles $/B$, then $c(E \oplus \overline{E}) = c(E)\, c(\overline{E})$ (cup multiplication).

Note: The canonical line bundles are the conjugates of those induced by inclusion.

3) For the canonical line bundle $\lambda \to S^2 = \mathbb{CP}^1$, $c_1(\lambda) = g =$ generator of $H^2(S^2)$.

4) For the canonical line bundle $\Lambda / \mathbb{CP}^\infty$, $c_1(\Lambda) = \hat{\imath} =$ gen of $H^2(\mathbb{CP}^\infty)$.

Thus $C_1 = c_1$ for line bundles as before. We shall see that the Chern classes are uniquely characterized by these properties. The key to this is the splitting principle (see below) which reduces all calculations to computing Chern classes of sums of line bundles. In order to prove the splitting principle we first prove the Leray-Hirsch Theorem which gives useful information on bundle cohomology.

Theorem (Leray-Hirsch): Let $E \xrightarrow{p} B$ be any fiber bundle of finite type, and $E_0 \subset E$ an open subspace. Let (F, F_0) be a pair such that $\forall b \in B$ $\exists\, j_b : (F, F_0) \xrightarrow{\simeq} (p^{-1}(b), p^{-1}(b) \cap E_0) \subset (E, E_0)$.
Let $a_1, a_2, \ldots, a_r \in H^*(E, E_0)$ be homogeneous elements such that $\forall b \in B$, $j_b^*(a_1), \ldots, j_b^*(a_r)$ is a basis for $H^*(F, F_0)$.

Then $H^*(E, E_0)$ is a free $H^*(B)$ module with basis $a_1, \ldots, a_r$ under the action described by $p^*: H^*(B) \longrightarrow H^*(E, E_0)$.

Proof. Let $\mathcal{U} \subset B$ be an open set s.t. have bundle homeom $(\mathcal{U}\times F, \mathcal{U}\times F_0) \longrightarrow (E_{\mathcal{U}}, E_{\mathcal{U}} \cap E_0)$. Clearly theorem true for the bundle $E_{\mathcal{U}} \longrightarrow \mathcal{U}$. Thus it suffices to show that if theorem holds for $\mathcal{U}, V$ and $\mathcal{U} \cap V$ then it holds for $\mathcal{U} \cup V$. To this end, define functors $K^n(\mathcal{U})$, $L^n(\mathcal{U})$ on open subsets:

$$K^n(\mathcal{U}) = \sum_{i=1}^{r} H^{n-n(i)}(\mathcal{U})\, x_i$$

$$n(i) = \deg(a_i) \quad , \quad x_i = \text{alg. var.}$$

$$L^n(\mathcal{U}) = H^n(E_{\mathcal{U}}, E_{\mathcal{U}} \cap E_0).$$

$$\Theta_{\mathcal{U}} : K^n(\mathcal{U}) \longrightarrow L^n(\mathcal{U})$$

$$\Theta_{\mathcal{U}}\left(\textstyle\sum_i c_i x_i\right) = \textstyle\sum_i p^*(c_i)\, a_i \, .$$

Thus Thm true for $E_{\mathcal{U}} \Leftrightarrow \Theta_{\mathcal{U}}$ isom. Now apply Mayer-Vietoris sequences for functors L^n, K^n and watch theorem appear from behind the 5-lemma. //

The idea behind the splitting principle is to give a map $f: \overline{B} \longrightarrow B$ such that $f^*E \cong L \oplus E'$ where L is a line bundle and $f^*: H^*(B) \longrightarrow H^*(\overline{B})$ is monic. To do this, we define $P(E) \xrightarrow{q} B$ corresponding to a vector bundle $E \xrightarrow{p} B$. The fibers of $P(E)$ are the projective spaces of the fibers of E:

$E_0 = E - (\text{zero section})$

$P(E) = E_0 / \mathbb{C} - \{0\}$ where $\mathbb{C} - \{0\}$ acts by complex multiplication on the fibers.

Thus we may identify

$$q^{-1}(b) = \{\text{lines } \ell \text{ in } p^{-1}(b)\}$$

Let $L \xrightarrow{p'} P(E)$ be the bundle of lines i.e. $p'^{-1}([\ell]) = \ell$. Note that $L \subset q^*(E)$.

(over)

for $q^*(E)_{[\ell]} = E_b \supset \ell$ where $q[\ell] = b$.

Claim: $q^*(E) \simeq L \oplus E'$.

Proof. Let $E \xrightarrow{p} B$ be any vector bundle of finite type and fiber dimension n. Let $G_n(\mathbb{C}^N)$ = space of linear subspaces of $\mathbb{C}^N$ of dim n. Let $E_n(\mathbb{C}^N) \subset G_n(\mathbb{C}^N) \times \mathbb{C}^N$ be $E_n(\mathbb{C}^N) = \{([V], x) \mid x \in V\}$. Then $\pi : E_n(\mathbb{C}^N) \longrightarrow G_n(\mathbb{C}^N)$ is a vector bundle. $\pi([V], x) = [V]$. Claim: $E \simeq f^* E_n(\mathbb{C}^N)$ for some large N. To see this choose covering $U_1, U_2, \ldots, U_M$ of B s.t. E_{U_i} trivial. Let $s_i : U_i \times \mathbb{C}^n \to E_{U_i}$ isoms. Use partition of unity to get $s : E \longrightarrow \sum_i \mathbb{C}^n = \mathbb{C}^N$ so that we may view fibers of E as varying family of subspaces of $\mathbb{C}^N$ ($b \neq b' \Rightarrow s(E_b) \cap s(E_{b'}) = \{0\}$). We call $s : E \to \mathbb{C}^N$ a Gauss map for E. We use it to define an inner product $\langle , \rangle : E \oplus E \longrightarrow \mathbb{C}$ $\langle x, y \rangle = \langle s(x), s(y) \rangle$ where the latter is standard Hermitian inner product on $\mathbb{C}^N$. Thus have notion of orthogonality.

Apply this to $q^*(E)$ and let $E^\perp$ = sub-bundle orthogonal to L. //

Observe: If
$$\begin{array}{ccc} j_b : \mathbb{CP}^{n-1} & \hookrightarrow & P(E) \\ \downarrow & & \downarrow q \\ \{b\} & \hookrightarrow & B \end{array}$$
is inclusion of fiber over b

then $j_b^*(L) \sim$ $\overset{\text{conjugate of}}{\text{canonical}}$ line bundle over $\mathbb{CP}^{n-1} = \Lambda_{n-1}$

Recall: $H^*(\mathbb{CP}^\infty) \simeq \mathbb{Z}[z]$, z = gen of $H^2(\mathbb{CP}^\infty)$

$$H^*(\mathbb{CP}^n) \simeq \mathbb{Z}[z_n]/(z_n^{n+1}) \qquad z_n = \text{gen of } H^2(\mathbb{CP}^n)$$

$$S^2 = \mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \cdots \subset \mathbb{CP}^n \subset \cdots \subset \mathbb{CP}^\infty$$

$$\begin{array}{ccccccccc} H^*(S^2) = H^*(\mathbb{CP}^1) & \leftarrow & H^*(\mathbb{CP}^2) & \leftarrow \cdots \leftarrow & H^*(\mathbb{CP}^n) & \leftarrow \cdots \leftarrow & H^*(\mathbb{CP}^\infty) \\ g = z_1 & \leftarrow\!\shortmid & z_2 & \leftarrow\!\shortmid \cdots \leftarrow\!\shortmid & z_n & \leftarrow\!\shortmid \cdots \leftarrow\!\shortmid & z \end{array}$$

We have $L \longrightarrow P(E)$. Let $f: P(E) \longrightarrow \mathbb{CP}^\infty$ s.t. $L^* = f^*(\Lambda)$. Let $a = f^*(i) \in H^2(P(E))$.

Theorem. The classes $1, a, a^2, \ldots, a^{n-1}$ form a basis for the $H^*(B)$ module $H^*(P(E))$. Moreover the map $g^*: H^*(B) \longrightarrow H^*(P(E))$ is a monomorphism.

Proof. Note that $j_b^*(f^*\Lambda) =$ conjugate of canonical line budle / $\mathbb{CP}^{n-1}$.
$\therefore$ up to automorphism of $\mathbb{CP}^{n-1}$, $f \circ j_b \simeq$ the inclusion $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^\infty$. Therefore $j_b^*(1), j_b^*(a), \ldots, j_b^*(a^{n-1})$ is a $\mathbb{Z}$ basis for $H^*(\mathbb{CP}^{n-1})$. Hence this theorem follows from the Leray-Hirsch Theorem. //

Definition. Let $p: E \to B$ be a vector budle. A splitting map of E is a map $f: \overline{B} \longrightarrow B$ such that $f^*(E) \simeq$ direct sum of line budles, and $f^*: H^*(B) \longrightarrow H^*(\overline{B})$ is monic.

Proposition. Splitting maps exist.

Proof. Induct on dimension, using theorem above. //

Now suppose we have Chern classes and let $E \xrightarrow{P} B$ be a vector budle & $f: \overline{B} \longrightarrow B$ a splitting map, so that $f^*E \simeq L_1 \oplus L_2 \oplus \cdots \oplus L_n$.
Then
$$f^*c(E) = cf^*E = c(L_1 \oplus \cdots \oplus L_n)$$
$$= c(L_1)c(L_2)\cdots c(L_n)$$
$$= (1+c_1(L_1))(1+c_1(L_2))\cdots(1+c_1(L_n)).$$

Let $\sigma_i(x_1, \ldots, x_n) = i^{th}$ elem symm fcn in $x_1, \ldots, x_n$

$$\sigma_1(x_1, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$$
$$\sigma_2(x_1, \ldots, x_n) = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n$$
$$\vdots$$
$$\sigma_n(x_1, \ldots, x_n) = x_1x_2\cdots x_n$$

Let $\alpha_i = f^{*-1}c_1(L_i)$. Then we have

$$c(E) = (1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_n)$$
$$= 1 + \sigma_1(\alpha_1,\ldots,\alpha_n) + \sigma_2(\alpha_1,\ldots,\alpha_n) + \cdots + \sigma_n(\alpha_1,\ldots,\alpha_n).$$

Hence $c_i(E) = \sigma_i(\alpha_1,\ldots,\alpha_n)$.

Conclusion: The Chern classes are uniquely determined by the axioms.

In fact we are in a position now to give a definition of the Chern classes.

Definition. Let $E \xrightarrow{p} B$, $q: P(E) \longrightarrow B$, $f: P(E) \longrightarrow \mathbb{CP}^\infty$ with $f^*(z) = a$ be as in the theorem on previous page. Then we have $H^*(P(E)) =$ $H^*(B)$-module with basis $1, a, a^2, \ldots, a^{n-1}$. Hence there exist unique $c_i \in H^{2i}(B)$ such that

$$a^n = -\sum_{1\le i\le n} c_i a^{n-i}$$

Let $c = 1 + c_1 + \cdots + c_n$ and

define $c(E) = c \in H^*(B)$. //

Proposition: If $E = L_1 \oplus \cdots \oplus L_n$, a direct sum of line bundles, then
$$c(E) = (1 + c_1(L_1)) \cdots (1 + c_1(L_n))$$
where $c(E)$ = above defn and $c_1(L_i)$ = 1st Chern class as defined previously.

Proof: I) Suppose E = a line bundle. Then $P(E) = B$ and $q = id$ so that $L = q^*E = E$. $a = f^*(z)$ where $f: B \longrightarrow \mathbb{CP}^\infty$ classifies E^*. The definition above gives $a = -c_1$. This is in exact agreement with our old defn.

II) $E = L_1 \oplus \cdots \oplus L_n$.

~~Then $q^*E = L \oplus E'$ so we have~~

an exact sequence $0 \to L \to q^*L_1 \oplus \cdots \oplus q^*L_n \to E' \to 0$

$\therefore \quad 0 \longrightarrow \varepsilon = L^* \otimes L \to (L^* \otimes q^* L_1) \oplus \cdots \oplus (L^* \otimes q^* L_n) \to L^* \otimes E' \to 0.$

trivial budle.

Thus $L^* \otimes q^* E$ has a nontrivial cross-sections s that projects to cross section S_i in $L^* \otimes q^* L_i$.
Let V_i = open subset over which $S_i \neq 0$.
The image of $c_1(L^* \otimes q^* L_i)$ is zero in $H^2(V_i)$
$\therefore$ it pulls back to $H^2(B, V_i)$.
But $c_1(L^* \otimes q^* L_1) \cdots c_1(L^* \otimes q^* L_n) \in H^*(B, \bigcup_{i=1}^{n} V_i) = 0.$

$$\therefore \quad \prod_{i=1}^{n} (c_1(L^*) + c_1 q^* L_i) = 0$$

$$\text{But } a = c_1(L^*)$$

$$\text{Hence } \prod_{i=1}^{n} (a + c_1 q^* L_i) = 0$$

$\therefore$ This is the equation $a^n = -\sum_{i=1}^{n} c_i a^{n-i}$

and this proves the proposition.

Corollary. Our definition of $c(E)$ satisfies the axioms. Hence Chern classes exist and are uniquely determined by these axioms.

Proof. 0) ✓
1) Given $f: \bar{B} \to B$ get f morphism $u: P(f^*E) \longrightarrow P(E)$.
$\therefore u^*(L_E)$ and L_{u^*E} isom over $\bar{B}$.
$\therefore$ In $H^*(Pf^*E)$ have $u^*(a) = a'$ where $a = a_E$, $a' = a_{f^*E}$. ✓
2) Use splitting principle and above result on line budles.
3) ✓ //

<u>Example</u>: Let $E = \tau\,\mathbb{C}P^n$, the tangent bundle to $\mathbb{C}P^n$.

$$\mathbb{C}P^n = S^{2n+1}/S^1 \qquad \vec{u} \sim \lambda\vec{u}, \quad \lambda \in \mathbb{C} \quad |\lambda| = 1.$$

$$E = \{[\vec{u},\vec{v}] \mid \|\vec{u}\| = 1,\ \vec{u}\cdot\vec{v} = 0,\ (\vec{u},\vec{v}) \sim (\lambda\vec{u},\lambda\vec{v})\}$$

$\Lambda_n \to \mathbb{C}P^n$ standard line bundle.

$\Lambda_n = \{[\vec{u},\rho] \mid \vec{u} \in S^{2n+1},\ \rho \in \mathbb{C},\ (\vec{u},\rho) \sim (\lambda\vec{u},\lambda\rho)\}$.

Let $E' = \Lambda_n \oplus \Lambda_n \oplus \cdots \oplus \Lambda_n$ ($n+1$ copies)

Then E' : $(\vec{u},\vec{v}) \in S^{2n+1} \times \mathbb{C}^{n+1}$, $(\vec{u},\vec{v}) \sim (\lambda\vec{u},\lambda\vec{v})$

$\therefore E' \supset E$.

But E' has cross-section $u \in S^{2n+1} \longmapsto (\vec{u},\vec{u})$.

$\therefore E' \simeq \tau\,\mathbb{C}P^n \oplus \varepsilon$

$\therefore c(\tau\,\mathbb{C}P^n) = c(E') = (1 + c_1(\Lambda_n))^{n+1}$

Letting $c_1(\Lambda_n) = \alpha_n$ we have

$c(\tau(\mathbb{C}P^n)) = (1+\alpha_n)^{n+1}$. ($\alpha_n$ gen $H^2(\mathbb{C}P^n)$)

d) Pontryagin Classes

Let $E \xrightarrow{p} B$ be a real vector bundle.

Get complex v-bundle $\hat{E} = E \otimes_{\mathbb{R}} \mathbb{C}$. (replace each fiber V by $V \otimes_{\mathbb{R}} \mathbb{C} = F \oplus iF$. Note that $\hat{E}^* \simeq \hat{E}$ via $x+iy \longmapsto x - iy$.

<u>Note also</u>: If E' is a complex v-bundle, then $c_i(E'^*) = (-1)^i c_i(E')$ (suffices to check for line bundles & sums of line bundles).

$\therefore$ $\boxed{2c_{2i+1}(\hat{E}) = 0}$ (odd chern classes have order 2)

<u>Defn</u>. The ith Pontryagin class $P_i(E) \in H^{4i}(B)$ for $E \to B$ a real v-bundle is

$$P_i(E) = (-1)^i c_{2i}(E \otimes_{\mathbb{R}} \mathbb{C}).$$

$$P(E) = 1 + P_1(E) + \cdots + P_{[n/2]}(E)$$

where E has real fiber dim n and $[n/2]$ = greatest integer $\leq n/2$.

Theorem. $2\,(P(E\oplus E') - P(E)P(E')) = 0$ for real vector bundles E, E'.

Proof. $(E\oplus E')\otimes\mathbb{C} \simeq (E\otimes\mathbb{C})\oplus(E'\otimes\mathbb{C})$.

$\therefore\ c_k((E\oplus E')\otimes\mathbb{C}) = \sum_{i+j=k} c_i(E\otimes\mathbb{C})\,c_j(E'\otimes\mathbb{C})$

$$\therefore\ c_{2k}((E\oplus E')\otimes\mathbb{C}) \equiv \sum_{i+j=k} c_{2i}(E\otimes\mathbb{C})\,c_{2j}(E'\otimes\mathbb{C}) \pmod 2 \quad //$$

Lemma. For any complex v-bundle ω, the complexification $\omega_{\mathbb{R}}\otimes\mathbb{C}$ of the underlying real v-bundle is $\simeq \omega\oplus\omega^*$.

Pf: $V\otimes\mathbb{C} \simeq V\oplus V$ cx str $J(x,y) = (-y, x)$.
If $V = F_{\mathbb{R}}$ where F = fiber of cx v-space.
Then $g: F \to V\oplus V$ $x \mapsto (x, -ix)$ is cx-linear i.e. $g(ix) = J(g(x))$. & $h: F \to V\oplus V$ $x \mapsto (x, ix)$ is conj-linear. Decompose $(x,y)\in V\oplus V$ via
$(x,y) = g\left(\frac{x+iy}{2}\right) + h\left(\frac{x-iy}{2}\right)$. //

Cor. For a complex n-plane bundle ω, $c_i(\omega)$ determine $p_k(\omega_{\mathbb{R}})$ via $1 - p_1 + p_2 - \cdots \pm p_n = (1 - c_1 + c_2 - \cdots \pm c_n)(1 + c_1 + \cdots + c_n)$.

Ex: $\tau = \tau\,\mathbb{CP}^n$, $c(\tau) = (1+a)^{n+1}$

$\therefore$ if $p_k = p_k(\tau_{\mathbb{R}})$ then

$$\begin{aligned} 1 - p_1 + \cdots \pm p_n &= (1 - c_1 + \cdots \pm c_n)(1 + c_1 + \cdots + c_n) \\ &= (1-a)^{n+1}(1+a)^{n+1} \\ &= (1-a^2)^{n+1} \end{aligned}$$

$$\therefore\ 1 + p_1 + \cdots + p_n = (1+a^2)^{n+1}$$

$$\therefore\ p_k(\mathbb{CP}^n) = \binom{n+1}{k} a^{2k}, \quad 1 \le k \le n/2.$$

Pontryagin Numbers

A partition of $k \ge 0$ is unordered seq $I = i_1, \ldots, i_r$ of pos integers with sum k.

$I = i_1, \ldots, i_r \qquad J = j_1, \ldots, j_s$

$IJ = i_1, \ldots, i_r, j_1, \ldots, j_s$.

$\underbrace{I = i_1 \ldots i_r}_{\text{partition of } n}$, M^{4n} smooth compact oriented manif.

$$\Rightarrow\ p_I[M^{4n}] = \langle p_{i_1}(\tau^{4n}) \cdots p_{i_r}(\tau^{4n}), \mu_{4n} \rangle$$

where μ_{4n} = fund class in $H_{4n}(M)$

τ^{4n} = tangent bdle of M.

Here $P_{i_1} \cdots P_{i_r}[\mathbb{CP}^{2n}] = \binom{2n+1}{i_1} \cdots \binom{2n+1}{i_r}$ if $i_1 + \cdots + i_r = n$.

N.B. $P_I(M^{4n}) = -P_I(-M^{4n})$ where $-M^{4n} = M$ with reversed orientation.

Theorem. If $M^{4n} = \partial B^{4n+1}$, B smooth compact ori. manif $\Rightarrow$ all Pontryagin numbers $P_I(M^{4n})$ vanish.

Pf: $\mu_B \in H_{4n+1}(B, M) \xrightarrow{\partial} H_{4n}(M) \ni \mu_M$

$$\partial \mu_B = \mu_M.$$

$$v \in H^{4n}(M) \Rightarrow \langle v, \partial \mu_B \rangle = \langle \delta v, \mu_B \rangle$$

where $\delta : H^{4n}(M) \to H^{4n+1}(B, M)$.

$$\tau_B | M = \tau_M \oplus \varepsilon$$

$$\therefore P_i(\tau_B|M) = P_i(\tau_M).$$

$$H^{4n}(B) \xrightarrow{i^*} H^{4n}(M) \xrightarrow{\delta} H^{4n+1}(B, M)$$

$$\Rightarrow \delta(P_I) = 0$$

$$\therefore \langle P_I, \partial \mu_B \rangle = \langle \delta P_I, \mu_B \rangle = 0 \quad //$$

Hence: $\mathbb{CP}^{2n}$ not oriented bndry nor is any sum of these.

Let Ω_n = oriented cobordism group of n-manifolds.

Then one can show:

Cobordism Facts

- Ω_n finite $n \not\equiv 0$ (4)
- $\Omega_{4k} \otimes \mathbb{Q}$ has basis $\{ \mathbb{CP}^{2i_1} \times \cdots \mathbb{CP}^{2i_r} \mid I = i_1 \ldots i_r$ partition of $4k \}$
- N.B. Ω_* = ring : $\left\{ \begin{array}{l} + = \# \text{ conn. sums} \\ \times = \text{cartesian product} \end{array} \right.$

This result of Thom is actually proved in Milnor's book on char classes. (for $\Omega_* \otimes \mathbb{Q}$)

II) Hirzebruch Signature Theorem

The idea is to produce combinations of Pontryagin numbers that behave formally like the signature and then to use cobordism theory to check agreement on the relevant examples. Recall the important properties of signature: 1) $\sigma(M_1+M_2)=\sigma(M_1)+\sigma(M_2)$

2) $\sigma(\partial N^{4k+1})=0$ $+$ ∴ M_1^{4k} cobordant to $M_2^{4k}\Rightarrow\sigma(M_1)=\sigma(M_2)$.

3) $\sigma(M_1^{4k}\times M_2^{4k})=\sigma(M_1)\sigma(M_2)$

Thus $\sigma:\Omega_* \longrightarrow \mathbb{Z}$ is a homomorphism.

Pontryagin numbers already obey 1) and 2). Thus we need to cook up 3) somehow. Here's the algebra:

Let Λ be a commutative ring with unit, 1. Let $A^*=(A^0,A^1,A^2,\ldots)$ be a graded Λ-algebra. Let $A^\pi=\{a_0+a_1+a_2+\cdots \mid a_i\in A^i\}$, the associated formal power series ring. Let $K_i(x_1,x_2,\ldots,x_i)$ be a sequence of polynomials such that each K_n is homogeneous of degree n. Let $K: A^\pi\longrightarrow A^\pi$ via $K(a)=1+K_1(a_1)+K_2(a_1,a_2)+\cdots$. We say that K is <u>multiplicative</u> if $K(ab)=K(a)K(b)$ for all $a,b\in A^\pi$.

<u>Lemma</u>. Given a formal power series $f(t)=1+\lambda_1 t+\lambda_2 t^2+\cdots$ $\exists!$ multiplicative sequence $\{K_n\}$ such that
$$K(1+t)=f(t).$$

<u>Proof</u>. A) <u>uniqueness</u> Let $A^*=\Lambda[t_1,t_2,\ldots,t_n]$ and $\sigma=(1+t_1)(1+t_2)\cdots(1+t_n)$ and $\sigma_1,\ldots,\sigma_n$ be the elem symm fcns so that $\sigma=1+\sigma_1+\sigma_2+\cdots+\sigma_n$. Then $K(\sigma)=K(1+t_1)K(1+t_2)\cdots K(1+t_n)=f(t_1)f(t_2)\cdots f(t_n)$. Thus $K(\sigma_1,\sigma_2,\ldots,\sigma_n)$ is uniquely determined by $f(t)$. Since $\sigma_1,\sigma_2,\ldots,\sigma_n$ are algebraically ind this proves the uniqueness.

B) <u>existence</u> If $I=i_1,\ldots,i_r$ is a partition of k we may define $S_I(\sigma_1,\ldots,\sigma_n)=\sum t_1^{i_1}t_2^{i_2}\cdots t_n^{i_r}$ where the $\sum$ sum means that we sum over all ~~permutations~~ choices of ~~[illegible]~~ r-subsets thereby obtaining a symmetric fcn & hence a polynomial in the elementary symmetric functions $\sigma_1,\ldots,\sigma_n$. [example $k=3$ $I=1,2$ then $S_I=t_1t_2^2+t_1t_3^2+t_2t_3^2+t_2t_1^2+t_3t_1^2+t_3t_2^2$]

$\lambda_I = \lambda_{i_1} \cdots \lambda_{i_r}$

These polynomials form a basis for the symmetric homog polyns of degree K in variables $t_1, \ldots, t_n$. Thus we may write $K_n(\sigma_1, \ldots, \sigma_n) = \sum \lambda_I \, S_I(\sigma_1, \ldots, \sigma_n)$ where I ranges over all partitions of n. One then verifies that $S_I(ab) = \sum_{HJ=I} S_H(a) S_J(b)$ where HJ denotes partition obtained by juxtaposition. Whence

$$K(ab) = \sum_I \lambda_I \, S_I(ab) = \sum_I \lambda_I \sum_{HJ=I} S_H(a) S_J(b)$$

$$= \sum_{H,J} \lambda_H S_H(a) \lambda_J S_J(b) = K(a) K(b) \qquad //$$

Let $\{K_n(x_1, \ldots, x_n)\}$ be a mult seq of polyns with rational coeffs. M^m a smooth, compact, oriented m-manif.

<u>Def.</u> $K[M^m] = 0$ if $4 \nmid m$ <u>The K-Genus</u>

$$K[M^{4n}] = K_n[M^{4n}] = \langle K_n(p_1, \ldots, p_n), \mu_{4n} \rangle$$

where p_i = ith Pontryagin class of τM.

<u>Lemma.</u> If $\{K_n\}$ is any mult sequence with rational coeffs, then the correspondence $M \longmapsto K[M]$ defines a ring homom $\Omega_* \longrightarrow \mathbb{Q}$, and hence an alg homom $\Omega_* \otimes \mathbb{Q} \longrightarrow \mathbb{Q}$.

<u>Pf:</u> Need only check behaviour on products. $M \times M'$ has total Pont class $p \times p'$ modulo elts of order 2, so $K(p \times p') = K(p) \times K(p')$ $\therefore$

$$\langle K(p \times p'), \mu \times \mu' \rangle = (-1)^{mm'} \langle K(p), \mu \rangle \langle K(p'), \mu' \rangle$$

$$\Rightarrow \quad K[M \times M'] = K[M] K[M'] \qquad //$$

$$\left(\tau \times \tau' \simeq \tau(M \times M') \simeq (\pi_1^* \tau) \oplus (\pi_2^* \tau') \right), \; \pi_1, \pi_2 \text{ the projs})$$

<u>Theorem (Hirzebruch).</u> Let $\{L_K\}$ be the mult sequence of polynomials corresponding to $f(t) = \sqrt{t} / \tanh \sqrt{t}$.

then $\boxed{\sigma(M^{4K}) = L[M^{4K}]}$

Proof. It suffices to check it for $L_K(\mathbb{C}P^{2K})$.

Here $P = (1+a^2)^{2K+1}$

$$L(1+a^2+0+\cdots) = \sqrt{a^2}/\tanh\sqrt{a^2}$$

$$\therefore L(P) = (a/\tanh a)^{2K+1}$$

Hence the L genus $\langle L(P), \mu\rangle$ = coefficient of a^{2K} in $L(P)$ above.

We check this by residues:

Let $u = \tanh(z) = \dfrac{e^z - e^{-z}}{e^z+e^{-z}}$

$$\Rightarrow du = \frac{(e^z+e^{-z})(e^z+e^{-z}) - (e^z-e^{-z})(e^z-e^{-z})}{(e^z+e^{-z})^2}dz$$

$$du = (1-u^2)\,dz$$

$$\therefore dz = \frac{du}{1-u^2} = (1+u^2+u^4+\cdots)\,du$$

$$(\text{Coeff of } a^{2K}) = \frac{1}{2\pi i}\oint \frac{dz}{z^{2K+1}}\left(\frac{z}{\tanh z}\right)^{2K+1}$$

$$= \frac{1}{2\pi i}\oint \frac{dz}{(\tanh z)^{2K+1}}$$

$$= \frac{1}{2\pi i}\oint \frac{(1+u^2+u^4+\cdots)\,du}{u^{2K+1}} = 1$$

Hence $L[\mathbb{C}P^{2K}] = 1 = \sigma(\mathbb{C}P^{2K})$.

This completes the proof. //

Some facts:

$$\sqrt{t}/\tanh\sqrt{t} = 1 + \tfrac{1}{3}t - \tfrac{1}{45}t^2 + - \cdots + (-1)^{K+1}\, 2^{2K} B_K \frac{t^K}{(2K)!} \cdots$$

B_K = K^{th} Bernoulli number.

The first few L-polyns: $L_1 = \frac{1}{3}P_1$

$L_2 = \frac{1}{45}(7P_2 - P_1^2)$

$L_3 = \frac{1}{945}(62P_3 - 13P_1P_2 + 2P_1^3)$

Remarks: Let's go over the method of calculating the parts of a multiplicative sequence again:

Let K be a mult seq $\leftrightarrow f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \lambda_3 t^3 + \cdots$.

Thus $K(1+t) = f(t)$

$K(1+\sigma_1+\sigma_2+\cdots+\sigma_n) = f(t_1)f(t_2)\cdots f(t_n)$

So if $\sigma_k = k^{th}$ elem symm fcn in $t_1,\ldots,t_n$ then

$$K(1+\sigma_1+\sigma_2+\cdots+\sigma_n) = \prod_{i=1}^{n}\left(\sum_{k_i=0}^{\infty}\lambda_{k_i}t_i^{k_i}\right), \quad (\lambda_0 = 1)$$

$$= \sum_{k_1,k_2,\ldots,k_n=0}^{\infty}\lambda_{k_1}\lambda_{k_2}\cdots\lambda_{k_n}t_1^{k_1}t_2^{k_2}\cdots t_n^{k_n}.$$

Let $S_K(\sigma_1,\ldots,\sigma_n) = \sum t_1^{k_1}\cdots t_n^{k_n}$ as before, when K is a partition of n $(|K| = \sum k_i = n$ ~~[illegible]~~$)$.

$\lambda_K = \lambda_{k_1}\lambda_{k_2}\cdots\lambda_{k_n}$.

Then the above shows that

$$K_n(\sigma_1,\ldots,\sigma_n) = \sum_{|K|=n}\lambda_K S_K(\sigma_1,\ldots,\sigma_n)$$

and $K = 1 + K_1 + K_2 + \cdots$.

Table

$S_1(\sigma_1) = \sigma_1$

$\begin{cases} S_2(\sigma_1,\sigma_2) = \sigma_1^2 - 2\sigma_2 \\ S_{1,1}(\sigma_1,\sigma_2) = \sigma_2 \end{cases}$

$\begin{cases} S_3(\sigma_1,\sigma_2,\sigma_3) = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 \\ S_{1,2}(\sigma_1,\sigma_2,\sigma_3) = \sigma_1\sigma_2 - 3\sigma_3 \\ S_{1,1,1}(\sigma_1,\sigma_2,\sigma_3) = \sigma_3 \end{cases}$

$\begin{cases} S_4(\sigma_1,\ldots,\sigma_4) = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 4\sigma_1\sigma_3 - 4\sigma_4 \\ S_{1,3} = \sigma_1^2\sigma_2 - 2\sigma_2^2 - \sigma_1\sigma_3 + 4\sigma_4 \\ S_{2,2} = \sigma_2^2 - 2\sigma_1\sigma_3 + 2\sigma_4 \\ S_{1,1,2} = \sigma_1\sigma_3 - 4\sigma_4 \\ S_{1,1,1,1} = \sigma_4 \end{cases}$

---

e.g. $S_2(\sigma_1,\sigma_2) = \sum t_i^2 = t_1^2 + \cdots + t_n^2 = (\sum t_i)^2 - 2(\sum t_i t_j) = \sigma_1^2 - 2\sigma_2$ //

From this it is easy to calculate that

$$K_1(\sigma_1) = \lambda_1 S_1(\sigma_1) = \lambda_1\sigma_1$$

$$K_2(\sigma_1,\sigma_2) = \lambda_2(\sigma_1^2 - 2\sigma_2) + \lambda_1^2\sigma_2 = \lambda_2\sigma_1^2 + (\lambda_1^2 - 2\lambda_2)\sigma_2$$

$$K_3(\sigma_1,\sigma_2,\sigma_3) = \lambda_3\sigma_1^3 + (\lambda_1\lambda_2 - 3\lambda_3)\sigma_1\sigma_2 + (\lambda_1^3 + 3\lambda_3 - 3\lambda_1\lambda_2)\sigma_3$$

To obtain the specifics of L_1, L_2, L_3 we must consider the series $f(t) = \sqrt{t}/\tanh(\sqrt{t})$.

The Bernoulli Numbers are defined to be coefficients in the series

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \frac{B_1}{2!}z^2 - \frac{B_2}{4!}z^4 + \frac{B_3}{6!}z^6 - + \ldots$$

The first few values are

$$B_1 = 1/6,\ B_2 = \frac{1}{30},\ B_3 = \frac{1}{42},\ B_4 = \frac{1}{30},\ B_5 = \frac{5}{56},\ \ldots$$

To obtain them, let $g(z) = z/(e^z - 1)$ and suppose $g(z) = \sum_{n=0}^{\infty} a_n z^n/n!$.

Then $z = (e^z - 1)g(z) = \left(\sum_{k=1}^{\infty} z^k/k!\right)\left(\sum_{n=0}^{\infty} a_n z^n/n!\right)$

$$\therefore\ 1 = \left(\sum_{k=0}^{\infty} z^k/(k+1)!\right)\left(\sum_{n=0}^{\infty} a_n z^n/n!\right)$$

$\Rightarrow$ a) $a_0 = 1$

b) $\displaystyle\sum_{k+n=N} \frac{a_n}{(k+1)!\,n!} = 0, \quad N > 0$

Thus $\dfrac{a_0}{2} + a_1 = 0 \Rightarrow a_1 = -1/2$

$$\frac{a_0}{3!} + \frac{a_1}{2!\,1!} + \frac{a_2}{1!\,2!} = 0$$

$$\Rightarrow \frac{1}{6} - \frac{1}{4} + \frac{a_2}{2} = 0$$

$$\Rightarrow a_2 = 1/6 \quad \therefore\ \underline{B_1 = 1/6}$$

etc....

Now $\dfrac{x}{\tanh x} = \dfrac{x(e^{x}+e^{-x})}{e^{x}-e^{-x}} = \dfrac{x(e^{2x}+1)}{e^{2x}-1}$

$$= \frac{2x}{e^{2x}-1} + x$$

$$= 1 + \frac{B_1}{2!}(2x)^2 - \frac{B_2}{4!}(2x)^4 + \frac{B_3}{6!}(2x)^6 - + \cdots$$

$$\therefore \frac{\sqrt{t}}{\tanh\sqrt{t}} = 1 + \frac{B_1}{2!}\cdot 2^2 t - \frac{B_2}{4!}2^4 t^2 + \frac{B_3}{6!}2^6 t^3 - + \cdots$$

$$\frac{2^2 B_1}{2!} = \frac{4}{2\cdot 6} = \frac{1}{3}$$

$$\frac{2^4 B_2}{4!} = \frac{2^4}{2^3\cdot 3\cdot 30} = \frac{1}{45}$$

$$\frac{2^6 B_3}{6!} = \frac{2^6}{6!\cdot 42} = \frac{2^6}{6\cdot 5\cdot 2^3\cdot 3\cdot 7\cdot 6} = \frac{2}{3\cdot 5\cdot 3\cdot 7\cdot 3}$$

$$= \frac{2}{945}$$

So $\dfrac{\sqrt{t}}{\tanh\sqrt{t}} = 1 + \dfrac{t}{3} - \dfrac{t^2}{45} + \dfrac{2t^3}{945} - + \cdots$

Exercise: Verify that if L = mult sequence corres to $\sqrt{t}/\tanh\sqrt{t}$, then

$$L_1(\sigma_1) = \tfrac{1}{3}\sigma_1$$

$$L_2(\sigma_1,\sigma_2) = \tfrac{1}{45}(7\sigma_2 - \sigma_1^2)$$

$$L_3(\sigma_1,\sigma_2,\sigma_3) = \tfrac{1}{945}(62\sigma_3 - 13\sigma_1\sigma_2 + 2\sigma_1^3)$$

Example: Let F be the Brieskorn (Milnor) fiber corresponding to $f(z) = z_0^3 + z_1^5 + z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^2$

Let $\Sigma = \partial F$ so $\Sigma = S^{13} \cap V(f(z))$.

We know from earlier discussions

that Σ is a homotopy sphere); hence Σ is homeomorphic to S^{11} ($\dim F = 12$) and we may form the (topological) manifold $M = F \cup_{\Sigma} D^{12}$.

Previous calculation $\Rightarrow$ $\sigma(M) = -8$.

Claim: 1) M has no possible smooth structure.

2) Σ is an exotic sphere.

Proof: We know $\tilde{H}_*(F) = 0$ except for $* = 6$

If M is smoothable then the Hirzebruch Theorem applies and we have

$$\sigma(M) = L[M].$$

But $\tilde{H}_*(M) = 0$ for $* \neq 6, 12$

$\therefore P_1 = P_2 = 0$

so $\sigma(M) = L_3[M]$

$$= \frac{62}{945} P_3[M].$$

Hence $-8 = \frac{62}{945} P_3[M]$

$P_3[M] \in \mathbb{Z} \Rightarrow 62 \mid 8$ contradiction.

Hence M is not smoothable.

If $\Sigma \cong S^{11}$ then we could give M a smooth str by smoothly pasting D^{12} to F and using the given diff str on F.

Therefore Σ is exotic.

QED