Take Home Exam -- Math 215 -- Fall 2011

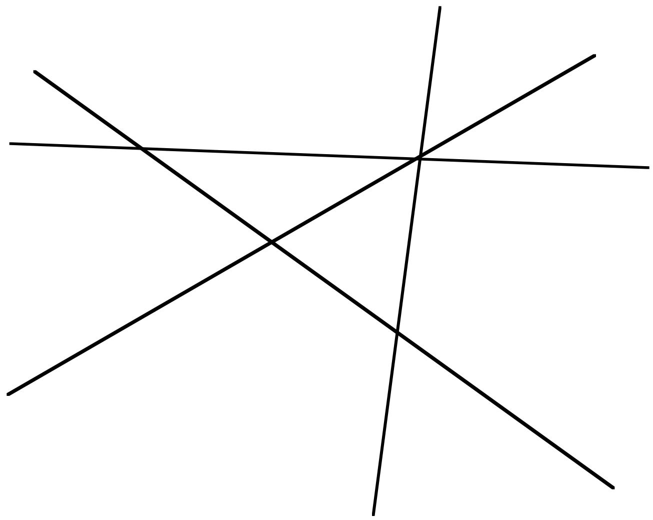
1. Prove by induction that
$1^2 + 2^2 + 3^2 + ... + n^2 = n(n+1)(2n+1)/6$
for n = 1, 2, 3,... .

2. Let Un be the n-th Fibonacci number as defined by Eccles in Definition 5.4.2. Show by induction that
$U_1{}^2 + U_2{}^2 + ... + U_n{}^2 = U_n U_{n+1}$ for n = 1,2,3,... .
After giving your proof by induction, explain the meaning of this equation carefully by showing how it follows from the construction of starting with a square, and then adding a square to the side of the first square, then adding a square to the longer side of the resulting rectangle and continuing in that fashion for a number of steps.

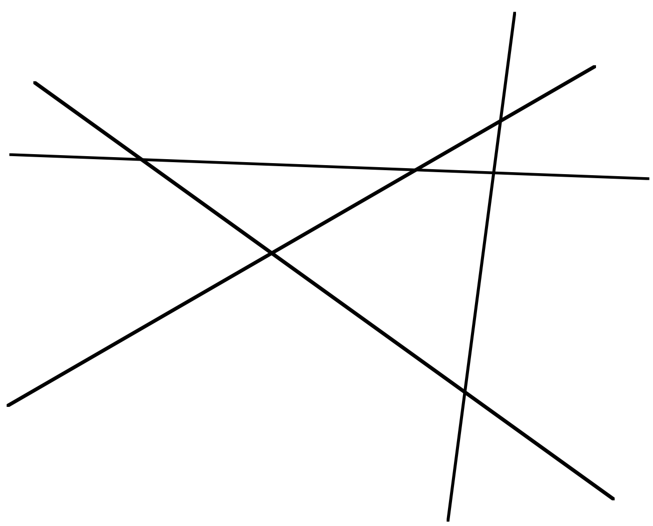
3. Prove that if a square is odd then it is one more than a multiple of 8. For example 9 = 1 + 8, 25 = 1 + 8 x 3,
49 = 1 + 8 x 6. Prove the general result.

4. Read the notes posted on the website titled "Conway's Army". Learn to play the game indicated in the notes and read the proof about this game that is in the notes. Write a detailed summary, in your own words, of the proof that it is impossible to reach the fifth level. *Your account of this should be complete and logical so that someone could read it without looking at the original article*. If there are aspects of the proof that you wish to illustrate or elaborate upon, include these in your presentation.

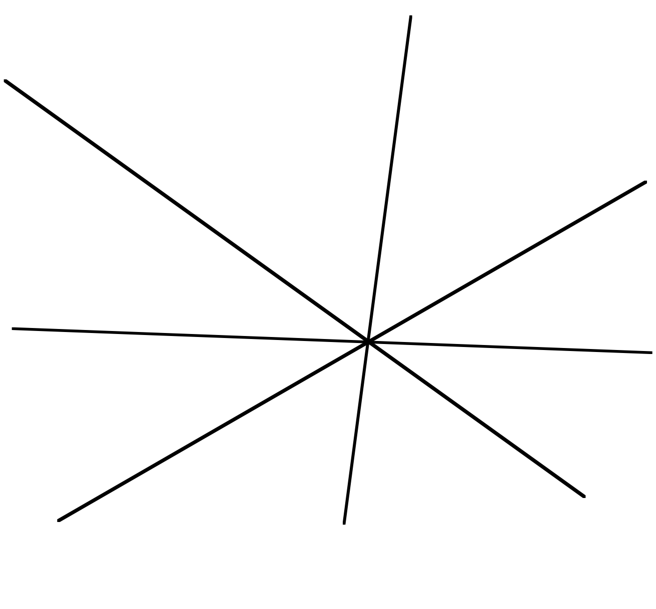
5. You will recall that we proved by an induction argument that the number of regions in the plane created by n lines that are in general position (so that all intersections are between exactly two lines and no two lines are parallel) is 1 + n(n+1)/2. Please give a careful proof of this result. Having done that, can you conjecture a formula for the number of regions produced by lines that intersect multiply?
A case of such a thing would be like the illustration below.



Note that in this case we got 10 regions instead of 1 + (4 x 5)/2 = 11 regions as in the picture below.



An extreme case would be



with only 8 regions. So as you see there would be different types of nodes. Some nodes would be an intersection of two lines. Some would be an itersection of three or four or more lines. Can you guess a formula for the resulting number of regions? Can you prove that your formula is true?

6. Let $N = \{1,2,3,4,...\}$ be the natural numbers.
Let $g: N \text{------>} N$ be defined by
$g(n) = n/2$ if n is even,
$g(n) = 3n + 1$ if n is odd.
Note that $g(1) = 4$, $g(4) = 2$, $g(2) = 1$. So if you repeatedly apply g to 1 it will keep coming back to 1. If you apply g to some other number it may reach 1. For example, $g(6) = 3$, $g(3) = 10$, $g(10) = 5$, $g(5) = 16$, $g(16) = 8$, $g(8) = 4$, $g(4) = 2$, $g(2) = 1$. After that it will cycle. So we can agree to STOP all computations when you hit the value 1. Ok? Now try this out on 7. That is compute $g(7)$, $g(g(7))$, ... until you get 1. (You will get 1 eventually). It has been conjectured that this process will always hit 1 no matter what natural number you use to start the process. Try it on 27, $g(27)$, $g(g(27))$... (It gets big before it gets down to 1 as you will see.). This is called the Collatz Process. Investigate the Collatz Process with some examples of your own. If you can prove that it will always hit 1 eventually, you will have done better than any mathematician since the 1940's when the problem first appeared.

7. Let x be a solution to $x = a + b/x$. Note that this means (multiply both sides by x) that $x^2 = ax + b$, so that x is a solution to a quadratic equation.
Now suppose that x is a solution to $x = a + b/(c + d/x)$. Show that this x will also be a solution to a quadratic equation. Find the exact form of the quadratic equation and the solution. (Given that a,b,c,d are all real numbers).
Generalize this result and prove by induction that if $a_1, a_2, ..., a_n$ are non-zero real numbers and x satisfies the equation
$x = a_1 + a_2/(a_3 + a_4/(a_5 + a_6/(a_7 + ... + a_{n-2}/(a_{n-1} + a_n/x)...)))$,
then x satisfies a quadratic equation.

8. (a) Let N be the natural numbers. Let $F;N \text{----->} P(N)$ be defined by the equation
$F\{1\} = \{\ \}$ (the empty set) and otherwise,
$F(n) = \{p \mid p \text{ is a prime divisor of } n\}$.

Describe C = {n | n is not a member of F(n)}.
(b) Let X be any set. Prove that no mapping F: X ------> P(X) is surjective. Here we are asking you to recount the proof that is based on constructing the set C = { x in X | x is not in F(x) }. Please write a clear proof that if C = F(z) for some z in X then there is a contradiction.
(c) Let QU denote the collection of all sets where every set in QU has sets as its members and these sets also have sets as members. *You can assume that no set in QU is a member of itself.*
By definition, if X is a set in QU, then the members of X are also in QU. Thus X is also a subset of QU. So we can define a mapping F: QU ------> P(QU) by F(X) = X. Apply Cantor's proof as in part (b) and deduce that QU cannot be a set.

9. Let Ax denote "For all x" and Ex denote "There exists an x".
Show that (Ex, x belongs to P) v (Ex, x belongs to Q) is logically equivalent to Ex, [(x belongs to P) v (x belongs to Q)].
Show that (Ax, x belongs to P) v (Ax, x belongs to Q) is NOT logically equivalent to Ax, [(x belongs to P) v (x belongs to Q)].
Draw Venn Diagrams to illustrate your reasoning.

10. Choose a Theorem whose proof you like and give a concise but complete account of the Theorem and its Proof.