SUPER TWIST SPINNING

Louis H. Kauffman*

1. INTRODUCTION

This paper describes a generalized twist-spinning construction for spherical knots in codimension two. The construction is a combination of super-spinning and twist-spinning, hence the terminology super twist-spinning.

Just as twist-spun knots are fibered by theorem of Zeeman [6], [2], so are super twist-spun knots fibered via a generalization of the Zeeman Theorem. (Theorem 3.1 of this paper.)

Super twist-spinning is closely related to the knot-product construction of [3], [4]. Section 2 will review this background. Section 3 then sketches super twist-spinning and the proof of the fibration theorem. Section 4 concludes with a discussion of results and open problems about these spinning constructions.

The knot product construction was originally devised as a generalization of branched covering constructions-including a geometric construction of the link of the sum of two isolated algebraic singularities

* Research partially supported by NSF Grant DMS-8701772

(see [5]). An isolated singularity gives rise to a smooth manifold embedded in codimendion two by intersection with a sphere about the singularity. Thus a trefoil knot es the link of the singularity at the origin of the variety.

$$z_1^2 + z_2^3 = 0,$$

in two complex variables. The unit sphere suffices in this case, so that the trefoil is explicitly described by the equations

$$z_1^2 + z_2^3 = 0,$$

$$|z_1|^2 + |z_2|^2 = 1$$

In general if $f = f(z_1, z_2, \ldots, z_n)$ and $g = g(w_1, w_2, w_3, \ldots, w_m)$ are isolated singularities in complex n and m-space, respectively, then f+g is an isolated singularity in complex n+m space. If L(f) denotes the intersection of the variety of f with a small sphere about the singularity then L(f), L(g) and L(f+g) are the links of these singularities, each embedded in the corresponding sphere. The knot product construction describes the embedding and construction of L(f+g) in terms of the embeddings of L(f) and L(g). In fact, the product construction is a purely knot theoretic construction, more general than the case of the algebraic singularities to which it applies. It produces a product $K \otimes L$ embedded in S^{n+m+1} whenever $K \subset S^n$ and $L \subset S^m$ with L fibered (see section 1).

The super twist spinning construction associates to a spherical knot $K \subset S^n$ and a fibered knot $L \subset S^m$ a new spherical knot $Spin_L(K)$ in S^{n+m}. As we shall see in section three, the idea for super twist spinning is very sinple. This idea can be used as a starting point for motivating the knot product construction.

While the idea requires a bit more fleshing out, as we do in section 3, it can be stated as follows:

Twist spinning is based on the idea of turning the knot as it is spun. Spinning consists in crossing the knot (minus a disc) with a circle. Super-spinning consists in crossing the knot with a high dimensional sphere S^m.

To combine twist spinning and super spinning we need to know by how much to twist the knot for each point on the spinning sphere S^m. If S^m is a fibered knot with fiber L, then on S^m - L there is a fibration $p : S^m - L \rightarrow S^1$.

Thus one can spin by the amount p(x) for x in the complement of L. Now, *specific properties of twist spinning* (the triviality of the 1-twist spin) allow us to fill in this spinning process and create the super twist spun knot $\mathrm{Spin}_L(K) \subset S^{n+m}$.

The super twist spin of a given knot K turns as it spins, governed by the fibration of the complement of L. The complement of the super twist spun knot $\mathrm{Spin}_L(K)$ fibers over the circle with fiber the knot product punc $K \otimes L$, the result of removing a ball from the knot product $K \otimes L$. This results becomes the usual theorem of Zeeman when the "fibered knot L" is taken to a be a mapping of the circle to itself of degree a (the empty knot of degree a).

2.RECALL OF KNOT PRODUCT AND BRANCHED FIBRATIONS

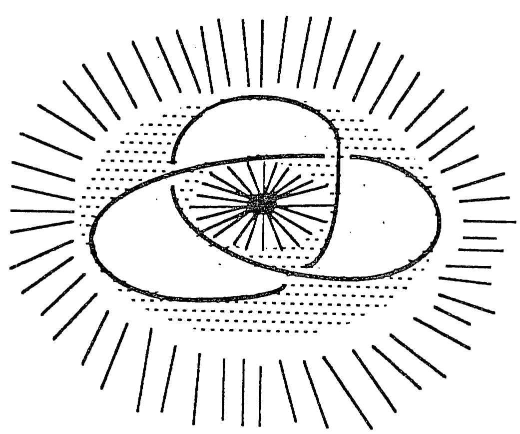
Throughout this paper a *knot* refers to a (connected) codimension-two differentiable submanifold of a sphere. Thus, in the case where the ambient sphere has dimension three, a knot is the familiar classical knot-an embedded circle in three dimendional space.

A *link* consist in a embedding of mutually disjoint connected submanifolds in a sphere. Thus, in dimensional three a link is an embedding of a collection of circles.

Unless otherwise specified, I shall use the term *knot* to mean *knot or link* when working in demension three.

A knot is said to be *fibered* if its complement can be smoothly fibered over the circle. It is assumed that each component of the knot or link has a trivial normal bundle, and that the fibration restrics to projection to the fiber of the normal circle bundle for an appropriate choice of trivialization. (See [4] for discussion of uniquenes and implications of the choices of trivialization.)

The trefoil (see Figure 1) is a classical example of a fibered knot. The fibration may be described by choosing a spanning surface as shown en Figure 1. The surface consists of three twisted bands within a solid torus, and two disks exterior to the torus. To describe the fibration it is sufficient to explain a family of surfaces that disjointly fill up all of the three-sphere minus the knot. These surfaces can be regarded as each having the knot as boundary. In the case in question, this is seen by rotating the places where the boundaries of the exterior disks join the solid torus (and rotating the disks as well so that they exchange places after a rotation of 180 degrees). The bands inside the solid torus are also rotated,with a screw motion, to match the movement of the outer disks. The result is a moving family of surfaces in the three-sphere satisfying the conditions for the fibration. See [6] for a very good discussion of the fibration.



Fiber surface for the trefoil

Figure 1

The product construction [2], [3] associated to two knots (in possibly different dimensional spheres) a new knot in a sphere whose dimension is one more than the sum of the dimensions of the given knots. One knot must be fibered for the construction to be well-defined. Given knots $K \subset S^n$ and $L \subset S^m$ (with L fibered) we shall define the *product* $K \otimes L \subset S^{n+m+1}$.

The basis for this construction is the following observation: Given any knot $K \subset S^n$, there is a map $k : S^n \to D^2$ (the two dimensional disk), transversal to the origin, so that K is the inverse image of the origin under this map. In fact if K has a tubular neighborhood in the form $K \times D^2$, then k can be taken to be projection on the D^2 factor on this neighborhood. With E (K) denoting the complement

$$E(K) = \text{Closure } (S^n - K \times D^2),$$

we can assume the maps E (K) to the unit circle, and that k restricted to E (K) is the fibering in the case that K is a fibered knot.

Call $k : S^n \to D^2$ a *generator* for the knot K.

Given a generator k for K, we can also form the *cone* on k denoted ck.This is the map $ck : D^{n+1} \to D^2$ defined by the equation

$$ck\ (tu) = tk\ (u)$$

where t lies between 0 and 1, and u is a point on the sphere S^n.

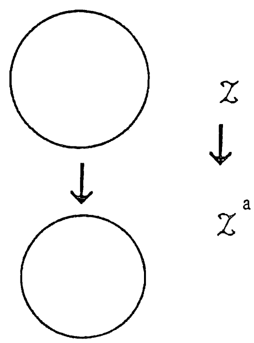
Note that ck restricted to the inverse image of the circle is just the map from (K) $\to S^1$. Thus, if K is a fibered knot, then ck is a fibration on the complement of the inverse image of zero, and the inverse image of zero is homeomorphic to the cone on the knot K.

When K is a fibered knot, I say that $ck : D^{n+1} \to D^2$ is a *branched fibration* over D^2 branched along zero, with fiber F (the fiber of the fibered knot K).

This formalism is intended to include the case n = 1 and the map of degree a, a : $D^2 \to D^2$ given by the formula

$$z \to z^a = a(z)$$

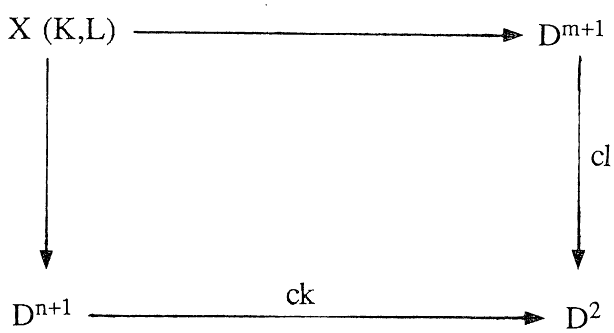
where z is a complex variable parametrizing the disk. In this case the associated fibered knot is the *empty knot of degree a* whose fibration is the map of the circle to itself of degree a.



The empty knot of degree a

Figure 2

Given a knot K and a fibered knot L with coned generators ck : $D^{n+1} \to D^2$ and cl : $D^{m+1} \to D^2$ we form the pullback $X(K,L) \subset D^{n+1} \times D^{m+1}$ via the diagram



$$X(K,L) = \{(x,y) \in D^{n+1} \times D^{m+1} | ck(x) = cl(y)\}.$$

and define the *product* $K \otimes L$ to be the boundary of this pull-back in the boundary of the product of the two balls:

$$K \otimes L = \partial(X(K,L)) \subset \partial\left(D^{n+1} \times D^{m+1}\right) = S^{n+m+1}.$$

Note that in the case where $L = [a]$ is the empty knot of degree a, the product $K \otimes [a]$ is the a-fold cyclic branched cover of S^n branched along K. The product construction gives a natural embedding of this branched cover in codimension two (in S^{n+2} when K is in S^n).

The simplest example is the product $[a] \otimes [b] \subset S^3$, a torus link of type (a,b). In general, products of empty knots $[a_1] \otimes [a_2] \otimes \ldots \otimes [a_m]$ yield the Brieskorn manifolds (see [5]) :

$$[a_1] \otimes [a_2] \otimes \cdots \otimes [a_m] \cong \Sigma(a_1, a_2, \ldots, a_m)$$

$$\Sigma(a_1, a_2, \ldots, a_m) = S^{2m-1} \cap \text{Variety}\left(Z_1^{a_1} + Z_2^{a_2} + \ldots + Z_m^{a_m}\right).$$

In general, the mapping $K \to K \otimes [2] \otimes [2]$ for spherical knots takes sphericals to sphericals, and gives a specific realization of the isomorphism of the Levine concordance groups in higher dimensions [2]. There are many other examples, and the full structure of the product construction remains to be investigated.

Remark : It is interesting to see a direct "cut and paste" description of the manifold $K \otimes L$. It is given by the formula

$$K \otimes L = (D^{n+1} \times L) \cup (E_K \times_{S^1} E_L) \cup (K \times D^{m+1})$$

where $\quad E_K \times_{S^1} E_L = \{(x, y) \in E_K \times E_L \mid k(x) = \ell(y) \in S^1\}.$

Note that $\quad \partial\left(E_K \times_{S^1} E_L\right) \cong (K \times E_L) \cup (E_K \times L)$

While $$\partial\left(D^{n+1} \times L\right) = \left(E_K \times L\right) \cup \left(K \times D^2 \times L\right)$$

and $$\partial\left(K \times D^{m+1}\right) = \left(K \times D^2 \times L\right) \cup \left(K \times E_L\right)$$

This decomposition can be easily read from our pull-back description.

3. SUPER TWIST SPINNING

In order to define super twist spinning of knots, it is necessary to recall the notions of spinning, and of twist spinning. Given a knot $K \subset S^n$, one can excise an unknotted disk pair (D^n, D^{n-2}) so that

$$(S^n, K) = \left(D^n, K_1\right) \cup \left(D^n, D^{n-2}\right).$$

In the case of a classical knot S^3, K_1 is a (knotted) arc running from the north pole to the south pole of a ball D^3. Standard spinning swings this arc around to trace out a 2-sphere in four-dimensional space -- we add some trivial pieces to close the construction. Thus one defines $\left(S^{n+1}, \mathrm{Spin}(K)\right)$ by the formula

$$\left(S^{n+1}, \mathrm{Spin}(K)\right) = \left(D^n \times S^1, K_1 \times S^1\right) \cup \left(S^{n-1} \times D^2, S^{n-3} \times D^2\right).$$

The second piece of the formula fills in the spun arcs to form the embedded sphere.

Twist spinning is a combination of spinning and twisting the arc at the same time. Since spinning involves the angular parameter on the circle S^1, we can use this parameter in the factor $K_1 \times S^1$ to twist the embedding to form

$$\left[\bigcup_{x \in S^1} \left(u(x)K_1, x\right)\right] \cup \left[S^{n-3} \times D^2\right]$$

where $u(x)K_1$ is the image of K_1 under a rotation of the ball D^n about the axis through the north and south poles -- u (x) is a function of the angle x in S^1. In particular, if we take u (x) = ax so that the arc twists a times as if spins once around, then this is *a-twist spinning*. With malice aforethought, I let $SPIN_{[a]}$ (K) denote the a-twist spin of K.

Zeeman [6] proved the beautiful

Theorem. The a-twist spin of a spherical knot is a fibered knot, and that fiber is a punctured a-fold cyclic branched cover of the ambient sphere of the original knot, branched along the knot. In particular, the 1-twist spin of a knot is trivial.

In [2] we showed another way to see this fact by exhibiting an interchange between the axis (trivial knot) in S^{n+1} and the 1-twist spin of any knot in S^n. This was then used to give an alternate proof of the Zeeman Theorem.

In dimension 4, the twist spinning construction has been used [1] to construct multiple distinct sharing the same conplement.

Another variant of the spinning construction is *super spinning*. Here the product with the circle is replaced by a product with a sphere S^m of arbitrary dimension. Thus we form the super-spun pair:

$$\left(S^{n-1}, \mathrm{Super}(K)\right) = \left(D^n \times S^m, K_1 \times S^m\right) \cup \left(S^{n-1} \times D^{m+1}, S^{n-3} \times D^{m+1}\right).$$

Super twist spinning-informal discussion

Finally, the main invention of this paper, super twist spinning, is a simultaneous generalization of super spinning and twist spinning. The data for super twist spinning consists in a spherical knot (S^n, K), and a fibered knot (S^m, K) -- not necessarily spherical. Note that (S^m, L), being fibered, comes equipped with a mapping $l : E\,(L) \rightarrow S^1$. Hence to each point x in E (L) there is an associated rotation l (x). Consider therefore the space obtained by formnig

$$A = \left[\bigcup_{x \in E_L} \left(l(x)K_1, x\right)\right]$$

The boundary of A consist in $S^{n-3} \times E(L)$ union with $\{(\theta K_{1'}, (\theta, p))\}$ with θ in S^1, and p in L. (The boundary of E (L) is $S^1 \times L$, and the fiber map restricts to projection to the S^1 factor for this boundary).

We may regard S^{n+m} as

$S^{n+m} = D^n \times S^m/{=}{=}$where (a,b') for any (a,b') in S^m and a in the sphere S^{n-1}. In this picture

A is identifield with A/== and

has boundary simply

$$B = \left(\left(\bigcup_{\theta \in S^1} (\theta K_{1'}, \theta)\right) \times L/==\right) = \mathrm{Spin}_{[1]}(K) \times L.$$

Since $\mathrm{Spin}_{[1]}$ (K) is a trivial knot there is a ball $D_1^n \subset S^{n+1} \subset S^{n+m}$ with boundary $\mathrm{Spin}_{[1]}$ (K). Hence $D_1 \times L$ has boundary B, and we can form

$$\mathrm{Spin}_L(K) = A \cup (D_1 \times L) \subset S^{n+m}$$

This is the supertwist spin of K relative to L.

The idea behind the super twist spin is simply this : rotate K according to the angle of the fibration for L. Fill in along the tubular neighborhood of L by using the fact that the standard 1-twist spin of K is trivial.

In order to make this description more precise, lets return to the decompositon

$$S^{n+m} = (D^n \times S^m) \cup (S^{n-1} \times D^{m+1})$$

Then $S^{n+m} = (D^n \times E_L) \cup (S^{n-1} \times D^{m+1}) \cup (D^n \times D^2 \times L)$

Thus we have the branched fibration (see section 2)

$$\begin{array}{ccccc} (D^n \times E_L) & \cup & (S^{n-1} \times D^{m+1}) & & \\ \downarrow 1 & \downarrow \pi & \downarrow cl & & \\ (D^n \times S^1) & \cup & (S^{n-1} \times D^2) & = & S^{n+1} \end{array}$$

In other, we have that

$$\left(D^n \times E_L\right) \cup \left(S^{n-1} \times D^{m+1}\right) = b\left(S^{n+1}, S^{n-1}; L\right)$$

where b (X,Y; L) denotes the branched fibration of X along Y-using the generator cl : $D^{m+1} \to D^2$ corresponding to the fibered link L. Thus

$$S^{n+m} = b\left(S^{n-1}, S^{n-1}; L\right) \cup \left(D^{n+2} \times L\right)$$

The L-super twist spin of $K \subset S^n$ is then *defined* to be

$$\mathrm{Spin}_L(K) = \pi^{-1}\left(\mathrm{Spin}_{[1]}(K)\right) \cup \left(D^n \times L\right) \subset S^{n+m}$$

where $\mathrm{Spin}_{[1]}(K) \subset S^{n+1}$ denotes the 1-twist spin of K, and the embedding $D^n \times L \to D^{n+2} \times L$ uses the fact that $\mathrm{Spin}_{[1]}(K)$ bounds a ball in S^{n-1}. In the case of branched coverings, this method of defined a-twist spinning was used [2] to give an alternate proof of the Zeeman theorem. Here it provides us with a canonical description of the super twist construction.

The Theorem of Zeeman generalizes to

THEOREM 3.1. Let K be a spherical knot in S^n and L be a fibered knot in S^m. Then the super twist spun knot $Spin_L$ (K) is a spherical fibered knot in S^{n+m} with fiber punc K⊗L where K⊗L in the knot product as constructed in [4] and section 2 of this paper.(punc (M) is the manifold with boundary obtained from a manifold M by removing the interior of a ball in the manifold).

Remark. The same line of heuristic reasoning by which we gave a description of the super twist spinning yields an insight for this theorem as well : Visualize the knotted ball pair (D^n, K_1). While K is not necessarily a fibered knot, nevertheless we can imagine D^n as decomposed into singular fibers F(x) with x running over the circle such that each fiber has a boundary K_1 union a ball D^{n-2} running from "north to south" on the boundary of D^n. Each D^{n-2} has boundary S^{n-3} and S^{n-3} is the boundary of the interior ball K_1 in D^n. (Think of a three-ball, knotedd arc pair). The super twist spin is arranged so that these fibers are spun around as well to create a manifold whose boundary is the super twist spun knot. A close look shows that this manifold spun from the fibers in D^n via the fibration of the knot L, is the punctured knot product K⊗L. The argument in this form is exactly analogous to Zeeman's original proof of his fibration theorem, with branched fibrations replacing the branched coverings.

Proof of 3.1. Recall that we have formally defined the pair $\left(S^{n+m}, \mathrm{Spin}_L(K)\right)$ via branched fibrations so that $S^{n+m} = b\left(S^{n+1}, S^{n-1}; L\right) \cup \left(D^{n+2} \times L\right)$ where $b\left(S^{n+1}, S^{n-1}; L\right)$ is the branched fibration described by the diagram

$$\begin{array}{ccc} b\left(S^{n+1}, S^{n-1}; L\right) & \rightarrow & D^{m+1} \\ \downarrow \pi & & \downarrow c\ell \\ S^{n+1} & \rightarrow & D^2 \end{array}$$

That is, $b\left(S^{n+1}, S^{n-1}; L\right)$ is the branched fibration (via L) of S^{n+1} branching along the standard unknottted sphere $S^{n-1} \subset S^{n+1}$. Since $\mathrm{Spin}_{[1]}(K) \subset S^{n+1}$ with $\mathrm{Spin}_{[1]}(K) \cap S^{n-1} = S^{n-3}$, the standard unknotted sphere in S^{n-1}, we form $\mathrm{Spin}_L(K)$ by restricting this branched fibration to $\mathrm{Spin}_{[1]}(K)$.

That is

$$\mathrm{Spin}_L(K) = b\left(\mathrm{Spin}_{[1]}(K), S^{n-3}; L\right) \cup \left(D^n \times L\right)$$

where $D^n \times L$ embeds in $D^{n+2} \times L$ to extend the embedding

$$\partial\left(b\left(S^{n+1}, S^{n-1}; L\right)\right) = S^{n-1} \times L \supset \partial\left(b\left(\mathrm{Spin}_{[1]}(K) S^{n-3}; L\right)\right) =$$

$$= \mathrm{Spin}_{[1]}(K) \times L$$

Here $\mathrm{Spin}_{[1]}(K) \times L \subset S^{n+1} \times L$ is the product with L of the given embedding $\mathrm{Spin}_{[1]}(K) \subset S^{n+1}$. Since we already know that $\mathrm{Spin}_{[1]}(K) \cong S^{n+1}$ and that the embedding $\mathrm{Spin}_{[1]}(K) \subset S^{n+1}$ bounds a ball in S^{n+1} we can let $\mathrm{Spin}_{[1]}(K) \subset S^{n+1}$ be represented by $\mathfrak{J}: S^{n-1} \to S^{n+1}$ extending to $\mathfrak{J}': D^n \to S^{n+1}$.

Let $\mathfrak{J}'': D^n \to D^{n+2}$ be the embedding obtained by deforming $\mathfrak{J}'$ slightly (relative to $\mathfrak{J}(S^{n-1})$) into the interior of D^{n+2}. Then $\mathfrak{J}''$ gives the specific embedding $D^n \times L \to D^{n+2} \times L$ needed to complete the definition of the embedding $\mathrm{Spin}_L(K) \subset S^{n+m}$ via the decomposition explained above.

To prove the fibration theorem I use the interchangeability of $\mathrm{Spin}_{[1]}(K)$ and the axis S^{n-1} in S^{n+1} (see [2]). This means that we can interchange the roles of $\mathrm{Spin}_{[1]}(K)$ and axis S^{n+1} in the definition of $\mathrm{Spin}_L(K)$. After this interchange we have

$$S^{n+m} = b\left(S^{n+1}, \mathrm{Spin}_{[1]}(K); L\right) \cup \left(D^{n+2} \times L\right) \supset \mathrm{Spin}_L(K) =$$

$$= b\left(S^{n-1}, S^{n-3}; L\right) \cup \left(D^n \times L\right)$$

That is, $\mathrm{Spin}_L(K)$ *is obtained by taking the inverse image of the axis sphere* S^{n-1} in S^{n+1} *under the branched fibration of* S^{n+1} *along the 1-spun* $\mathrm{Spin}_{[1]}(K) \subset S^{n+1}$.

In this interchange picture, the embedding $D^n \times L \to D^{n+2} \times L$ is induced by the embedding $S^{n-1} \to S^{n+1}$ that represents $\mathrm{Spin}_{[1]}(K)$.

We are now in a position to prove the fibration theorem. Observe that the pair $\left(S^{n+1}, S^{n-1}\right)$ is fibered with fibers diffeomorphic to D^n, intersecting $\mathrm{Spin}_{[1]}(K)$ transversally in copies of $K_1 \subset D^n$ where K_1 denotes the result of excising a trivially embedded D^{n-2} from $K \subset S^n$. This fibering lifts under the branched fibration to a fibering of the knot $\left(S^{n+m}, \mathrm{Spin}_L(K)\right)$ with fibers $b\left(D^n, K_1; L\right) \cup \left(D^n \times L\right)$. It is easy to see that

$$b\left(D^n, K_1; L\right) \cup \left(D^n \times L\right) \cong \mathrm{punc}\left(K \otimes L\right).$$

This completes the proof.

Remark. The exchange shown in Figure 3 where a knotted hole attached to an unknotted arc is the basic geometry behind the interchangability of the 1-twist spin and the standard axis. It is also the picture of how an apparently unknotted branch set becomes the knotted branch set relevant to the fibering in the Zeeman theorem and its generalization.

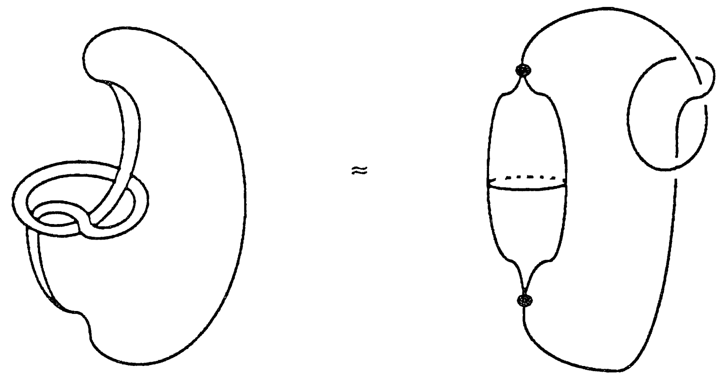


Figure 3

4. QUESTIONS

This paper has been a sketch of ideas and construction related to knot products, spinning and super twist spinning. We have traced the line of ideas to show that the product construction and its associated notions of fibered knots and branched fibrations could have arisen entirely in geometric knot theory in response to a natural generalization of twist spinning to super twist spinning.

The product construction itself has its origins in the study of links of algebraic singularities. In all cases of these constructions there is much possibility for a deeper investigation of examples. While we are quite familiar with certain knot products that arise from the sum of singularities, the general behaviour of even the 5-knots in 7-space produced by products of classical knos is largely unexplored.

Similarly, it would be very interesting to know more about the super twist spun 4-knots in 6-space produced by the spinning of classical knots by classical fibered knos and links.

Just as twist spinning has produced highly significant examples in dimension four, it is likely that super twist spinning will produce significant examples in higher dimensions.

A specific question: Are there distinct super twist spun knots with homeomorphic complements ?

REFERENCES

[1] C. Mc A. Gordon. Knots in the 4-sphere. Comment. Math. Helvetici. *39* (51) (1977), 585-596.

[2] L. Kauffman and D. Goldsmith.Twist spinning re-visited. Trans. Amer. Math. Soc. *239* (1978), 229-251.

[3] L. Kauffman. Products of knots. Bull. Amer. Math. Soc. *80* (1974), 1104-1107.

[4] L. Kauffman and W.D.Neumann. Products of knots, branched fibrations and sums of singularities. Topology *16* (1977), 369-393.

[5] J. Milnor. *Singular Points of Complex Hypersurfaces.* Annals of Mathematics Study # 61 (1968). Princeton University Press, Princeton, New Jersey.

[6] E.C.Zeeman. Twisting spun knots. Trans. Amer. Math. Soc. *115* (1965), 471-495.

Louis H. Kauffman
Department of Mathematics, Statistics and Computer Science
The University of Illinois at Chicago
Chicago, Illinois 60680, USA

and

Institut des Hautes Scientifiques
35, route de Chartres
91440 Bures sur Yvette, FRANCE

CONTRIBUCIONES MATEMATICAS

en

HOMENAJE

al profesor

D. ANTONIO PLANS SANZ DE BREMOND

Zaragoza 1990