Math 215

Problem Sampler

1. Construct truth tables for the statements (a) (not A) or B, (b) (not A) and B, (c) $A \Rightarrow B$.

2. Construct truth tables for (a) not (A or B), (b) (not A) and (not B). Deduce that the two statements are equivalent.

3. Find a statement $S(A, B)$ using the two or more of the connectives {and, or, not} that has the following truth table:

$$S(T,T) = F, S(T,F) = F, S(F,T) = T, S(F,F) = F.$$

4. The statement, "neither A nor B" is defined to be "not (A or B)" and is sometimes written symbolically as $A \downarrow B$.

Prove that $A \downarrow A \equiv$ (not A). [Hint: The form in 2(b) is useful for this problem.]

5. Write $(A \downarrow A) \downarrow (B \downarrow B)$ in a much simpler form. [Hint: Use 2(b) again.]

6. The statement "A implies B" written $A \Rightarrow B$ can be defined to be (not$A$) or B.

The statement "B implies A", written $B \Rightarrow A$, would then be "(not$B$) or A".

What is the truth table for $A \Leftrightarrow B$ which is $(A \Rightarrow B)$ and $(B \Rightarrow A)$?

$A \Leftrightarrow B$ can be read, "A if and only if B."

7. If $a < b$ and $c < d$, then $a + c < b + d$.

8. If $a < b$, then $-b < -a$. In this problem, $-x$ is a number with the property that $x + (-x) = 0$.

9. If $1 < a$, then $a < a^2$.

10. If $0 < a < b$, then $a < \sqrt{ab} < \frac{a+b}{2} < b$. [Notice that there are two hypotheses and three conclusions.]

11. Prove that the following two statements are equivalent:

$$A \Rightarrow (B \Rightarrow C) \quad \text{and} \quad (A \text{ and } B) \Rightarrow C.$$

12. Show $\displaystyle\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$.

13. Find a formula for the sum

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} = \sum_{j=1}^{n} \frac{1}{j(j+1)}$$

and prove your formula is correct.

14. Prove $\displaystyle\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$.

15. For non-zero integers a and b we say a divides b if there is an integer q such that

$$b = aq.$$

We express the fact that a divides b, or b is divisible by a, by writing $a|b$. Prove: (a) $a|b$ and $a|c \Rightarrow a|(b+c)$. (b) $a|b$ or $a|c \Rightarrow a|bc$.

16. Which of the following conditions are necessary for the positive integer n to be divisible by 6? If you think the condition is necessary you do not need to give a proof. But if you think the condition is not necessary, give an example of a number n that does not meet the condition but *is* divisible by 6 . (i) 3 divides n. (ii) 9 divides n. (iii) 12 divides n (iv) $n = 12$. (v) 6 divides n^2. (vi) 2 divides n and 3 divides n. (vii) 2 divides n or 3 divides n.

17. Which of the conditions listed above are sufficient for the positive integer n to be divisible by 6? If you think the condition is sufficient you do not need to give a proof. But if you think the condition is not sufficient, give an example of a number n that meets the condition but *is not* divisible by 6.

18. Let the statement $S(n)$ be $n! \geq 3^n$. $S(0)$ is true but $S(1)$ is false. Find the smallest number k greater than 1 such that $S(k)$ is true. Prove by induction that $S(n)$ is true for all $n \geq k$.

19. Prove that, for a positive integer n, a $2^n \times 2^n$ square grid with any one square removed can be covered using L-shaped tiles consisting in three adjacent grid squares.

20. For any sets A and B, recall $A - B = \{x \in A : x \notin B\}$.

Prove $A - B = A - (A \cap B)$. That is, show $A - B \subset A - (A \cap B)$ and $A - B \supset A - (A \cap B)$.

21. Let

$$A = \{(x, y) : 0 \leq x \text{ and } x \leq y\} B = \{(x, y) : 0 \leq y \text{ and } y \leq x\}.$$

Suppose $(x, y) \in A \cap B$. Show that $x = y$ and $0 \leq x$.

22. Let $X = \{a, b, c, d\}$ and $A = \{a, b\}$. Define the function $f : X \longrightarrow \{0, 1\}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Make a table of values for the function f.

23. Let $g : X \longrightarrow \{0, 1\}$ be defined by $g(x) = 1 - f(x)$ where f is the function in the previous problem. (a) What is $g(b)$? (b) Make a table of values for the function g. (c) Which elements of X are in $\{x \in X : g(x) = 1\}$?

24. The sequence of Fibonacci numbers is defined recursively by

$$u_0 = 0, \quad u_1 = 1, \quad \text{and} \quad u_n = u_{n-1} + u_{n-2} \quad \text{for} \quad n \geq 2.$$

Find the value of u_n for $n \leq 8$. As a check, $u_8 = 21$.

Show (by induction) that if $a | u_n$ and $a | u_{n+1}$, then $a = 1$ or $a = -1$.

25. Let $A \subset X$ and $B = X - A$. Prove $A = X - B$.

26. Let $B \subset A \subset X$. Prove $X - A \subset X - B$.

27. Define the composition of the function $f : X \longrightarrow Y$ and the function $g : Y \longrightarrow Z$ to be the function $g \circ f : X \longrightarrow Z$ with $g \circ f(x) = g(f(x))$ for all $x \in X$. Prove that if f is injective and g is injective, then $g \circ f$ is injective.

28. For $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ show that: (a) If $g \circ f$ is injective, then f is injective. (b) If $g \circ f$ is surjective, then g is surjective.

29. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$. Deduce from the results of problem 28 that if $g \circ f$ is injective and $f \circ g$ is surjective, then f is bijective. [You will not need to consider elements $x \in X$ or $y \in Y$ for this problem, you can work with the statements about functions in problem 28 only.]

30. Let $W = \{n \in Z : n \geq 0\}$. Define $f : W \longrightarrow W$ by $f(n) = 2n$. For any real number r, let $\lfloor r \rfloor$ be the largest integer m with $m \leq r$. For example, $\lfloor 3/2 \rfloor = 1$, $\lfloor \sqrt{5} \rfloor = 2$, $\lfloor \pi \rfloor = 3$, and $\lfloor \ln 2 \rfloor = 0$. Define $g : W \longrightarrow W$ by $g(n) = \lfloor n/2 \rfloor$.

(a) Show $g \circ f(n) = n$. (b) Show $f \circ g(n) = \begin{cases} n, & \text{if } n \text{ is even;} \\ n-1, & \text{if } n \text{ is odd.} \end{cases}$ (c) Which of the functions f, g, $f \circ g$, and $g \circ f$ are injective; which are surjective?

31. Show $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$.

32. Let $A = \{a, b, c\}$. List all of the elements of the power set $\mathcal{P}(A)$.

33. Let $A = \{a, b, c\}$ and $B = \{0, 1\}$. (a) Find all functions from A to B. Write down the functions explicitly using a table with a column for each element of A and a row for each function. You do not need to name each function. (b) For each function f in part (a), write down the set $A_f = \{x \in A : f(x) = 1\}$. Use the same table and put A_f in a last column. What can you say about the collection of sets you obtain?

34. Let $S \subset \{1, 2, \ldots, 2n\}$ where S has $n + 1$ elements. Then S contains two numbers such that one divides the other. [Any number can be written uniquely as an odd number times a power of 2, $m = (2k-1)2^j$. Consider the function $f : S \longrightarrow \{1, 2, \ldots, n\}$ defined by $f(m) = k$.]

35. This problem is an extension of problem 29. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ be maps such that $g \circ f$ is injective and $f \circ g$ is surjective. From problem 29 we know that f is bijective. Show that g is also bijective. [Outline: Since f is bijective, the inverse map, f^{-1}, exsits and is bijective. Then $g = (g \circ f) \circ f^{-1}$. Since $g \circ f$ is injective and f^{-1} is injective, problem 27 implies that g is injective. Also $g = f^{-1} \circ (f \circ g)$. Prove a result like problem 27 for surjective maps and deduce that g is surjective.]

36. Consider the game of Brussels Sprouts as described in our notes. Prove that a game of Brussels sprouts that starts with n nodes will have $5n - 2$ moves. Use the Euler formula for plane graphs to accomplish your proof.