1 Introduction

My main area of research is K-theory and specifically, I am studying relations between algebraic and topological K-theory of complex analytic spaces. In general, K-theory is the study of a topological space through the structure of the category of vector bundles over that space. While algebraic K-theory contains a great deal of information about this category, it is notoriously difficult to compute.

Grothendieck introduced K-theory [5] to formulate his generalization of the Hirzebruch Riemann-Roch theorem as a relative statement about proper morphisms of regular algebraic varieties. Since the push-forward of a locally free sheaf with respect to a proper morphism no longer has to be locally free, it is essential to work with the category of coherent sheaves instead, which is invariant under proper push-forwards. A central fact used in Grothendieck's proof is that any coherent sheaf on a regular algebraic variety has a global resolution by locally free sheaves. This is no longer the case for complex manifolds even though there are still local resolutions.

Baum, Fulton, and MacPherson [4] and later Gillet [6], generalized Grothendieck's theorem to a covariant natural transformation (with respect to proper maps) between the algebraic K-theory spectrum of a possibly singular algebraic variety to its topological K-homology spectrum, which is easier to compute.

In [1], Atiyah employed ideas from theory of operator algebras to develop a K-homology theory for spaces such as complex analytic spaces, which are not smooth manifolds. One way to obtain a K-homology spectrum for a C^*-algebra A, is to use the K-theory spectrum of the C^*-algebra $\mathfrak{Q}(A)$ called the *Paschke dual* of A [14]. However the definition of the Paschke dual depends on the choice of a representation of A and is only functorial up to homotopy. In regards to complex analytic spaces, Levy constructs a Riemann-Roch transformation (with respect to proper maps) [10, 11]. Nevertheless this construction does not elucidate fully the interaction between algebraic and topological K-theory.

I outline below the method I have used to define a Riemann-Roch transformation from the algebraic K-theory spectrum to the topological K-homology spectrum of a complex manifold. This method may be generalized to complex analytic spaces, and could unify Grothendieck's approach to the Riemann-Roch theorem with the Atiyah-Singer index theorem [2].

To define this transformation, let A be a C^*-algebra. We give a completely functorial construction of a K-theory spectra whose shifted homotopy groups are the topological K-homology groups of A. To achieve this, we introduce the *Paschke category* $(\mathfrak{D}/\mathfrak{C})_A$ of A, which is an exact C^*-category, functorial in A. Hence we can apply Waldhausen's $S.$-construction on this category [19] to obtain the designated spectrum. Subsequently we realize the Dolbeault complex in this category as an exact sequence in the Paschke category and show that this process induces a map of spectra

$$\tau_X^D : \mathbb{K}^{alg}(X) \to \mathbb{K}(Ch(\mathfrak{D}/\mathfrak{C})_{C_0(X)}) \tag{1.1}$$

wherein for an exact category $\mathcal{A}$, $Ch(\mathcal{A})$ denotes the category of bounded acyclic chain complexes in $\mathcal{A}$. Next we construct a natural map of spectra $\mathbb{K}^{top}(Ch(\mathcal{D}/\mathcal{C})_A) \to \Omega\mathbb{K}^{top}((\mathfrak{D}/\mathfrak{C})_A)$. When X is a complex manifold and the C^*-algebra $A = C_0(X)$, the composition of this map with τ_X^D introduced in 1.1, will give our Riemann-Roch transformation, which we will denote by τ_X.

Theorem 1.1. *Let $f : X \to Y$ be a proper smooth morphism of complex manifolds. It follows from our definition of τ that the diagram below commutes up to homotopy:*

$$\begin{array}{ccc} \mathbb{K}^{alg}(X) & \xrightarrow{Rf_*} & \mathbb{K}^{alg}(Y) \\ \downarrow{\scriptstyle \tau_X} & & \downarrow{\scriptstyle \tau_Y} \\ \mathbb{K}^{top}(X) & \xrightarrow{f_*} & \mathbb{K}^{top}(Y). \end{array} \tag{1.2}$$

Also, τ behaves well with respect to restriction to open subsets.

In Section 2, I will define the Paschke category, and discuss its K-theory. Then I will describe in greater detail the ideas involved in the Riemann-Roch transformation in Section 3. Finally, in Section 4, I will describe why I believe this new approach could lead to fruitful results in both algebraic and topological sides of K-theory, and most importantly, can further illuminate our understanding of the intimate relations between them.

2 K-theory and Paschke Category

For a compact manifold X, topological K-theory $K^0_{top}(X)$ is defined [3] as the abelian group generated by the isomorphism classes of vector bundles over X, with the group action defined by taking the direct sum of the vector bundles. Then one defines $K^n_{top}(X)$ as the topological K-theory of the smash product $S^n \wedge X$ of the n-dimensional sphere with X. This process is contravariant, as we can pull-back vector bundles, and defines a cohomology theory which corresponds to the spectrum BU. The dual homology theory (originally defined via Spanier-Whitehead duality [17]) corresponding to BU is called the *topological K-homology.* One way to expand the scope of topological K-theory and K-homology is to use theory of C^*-algebras.

Recall that a *C^*-algebra* is a complex normed algebra which is complete with respect to the given norm, and has an involutive $*$-operator with the expected compatibility between all three of these structures. For example for a manifold X, the algebra $C_0(X)$ of continuous complex valued functions which vanish at infinity with the supremum norm and taking conjugation as the $*$-operator is a commutative C^*-algebra. Morphisms between two C^*-algebras which preserve the structure are called $*$-morphisms. We call a $*$-morphism ρ from A to the C^*-algebra of bounded operators on a Hilbert space $\mathfrak{B}(H)$ a *representation.* Also, for representations $\rho_i : A \to \mathfrak{B}(H_i), i = 1, 2$ we say a bounded operator $T : H_1 \to H_2$, is *pseudo-local* if $T\rho_1(a) - \rho_2(a)T$ is always compact, and we say it is *locally compact* if both $T\rho_1(a), \rho_2(a)T$ are compact.

We say that the category $\mathcal{A}$ is *enriched* in the category $\mathcal{B}$, if for each two objects of $\mathcal{A}$ the morphisms between them is an object in the category $\mathcal{B}$. We can define a C^*-category as a category enriched in the category of Banach spaces, with an involutive $*$-operation on the morphisms. Any C^*-algebra can be considered as a C^*-category with one object whose morphism space is the mentioned C^*-algebra.

Quillen defined *exact categories* as additive categories with a suitable class of short exact sequences, and defined K-theory of exact categories [15]. For example for a complex manifold X, category of holomorphic vector bundles on X, or category of coherent analytic sheaves on X are both exact categories, and *algebraic K-theory* spectrum $\mathbb{K}^{alg}(X)$ is defined as the K-theory spectrum of the category of coherent analytic sheaves on X. As a matter of fact, this defines a covariant functor with respect to proper morphisms of complex analytic spaces. Waldhausen generalized Quillen's construction to categories with two classes of morphisms, called the cofibrations (resembeling admissible monomorphisms) and the weak equivalences [19]. These categories are now called Waldhausen categories, and he defined K-theory space of a Waldhausen category $\mathcal{A}$ by defining a simplicial category $S.\mathcal{A}$, then constructing a topological space by restricting to weak equivalences $wS.\mathcal{A}$ of this category, and then considering the loop space $\Omega|wS.\mathcal{A}|$ of its *goemetric realization.*

For a C^*-algebra A we define the *Paschke category* $(\mathfrak{D}/\mathfrak{C})_A$ of A to be the C^*-category whose objects are representations of A, and morphisms between two representations are pseudo-local operators modulo locally compact operators. We define an exact structure on this category by simply saying that a chain complex is exact, if there is a contracting homotopy. Precomposing a representation of a C^*-algebra B with a $*$-morphism $f : A \to B$, induces an exact pull-back functor $f^* : (\mathfrak{D}/\mathfrak{C})_B \to (\mathfrak{D}/\mathfrak{C})_A$, and this process is functorial.

Much of the machinery used by Waldhausen also works in the setting of topological categories [12], and in particular for C^*-categories. Hence, we can apply Waldhausen's construction on the exact C^*-category $(\mathfrak{D}/\mathfrak{C})_A$, and then apply fat geometric realization [16] to obtain a topological space $\Omega\|wS.(\mathfrak{D}/\mathfrak{C})_A\|$ whose homotopy groups are the K-theory groups of the Paschke category. Then we prove the following.

Theorem 2.1. *Let A be a separable C^*-algebra and let $i \geq 1$. Then the topological K-theory groups $K_i((\mathfrak{D}/\mathfrak{C})_A)$ of the Paschke category of A are isomorphic to topological K-homology groups $K^{1-i}(A)$ of A.*

The proof revolves around a certain class of representations called the *ample representations.* These form a strictly cofinal subcategory (in the sense of [19]) of the Paschke category, and hence their K-theory spectrum is homotopy equivalent to that of the Paschke category. Moreover by Voiculescu's theorem [18], the C^*-algebra of automorphisms of any ample representation is independent of the choice of the ample representation, and is called the *Paschke dual* $\mathfrak{Q}(A)$ of A. This means that $K_i((\mathfrak{D}/\mathfrak{C})_A) = \pi_i(\mathfrak{Q}(A))$ which is equal to $K^{1-i}(A)$ by [14].

The fact that pull-back maps agree under this isomorphism is a result of the additivity theorem [15][19].

3 The Riemann-Roch transformation

By theorem 2.1, we can use the Paschke category to obtain a map to the topological K-homology spectrum. We proceed to define the Riemann-Roch transformation for complex manifolds in three steps, in which of course the first step is the hardest.

1. Let X be a complex manifold and let E be a holomorphic vector bundle on it. We can consider the Dolbeault complex $\mathcal{A}^{0,*}(E)$ with coefficients in E. Consider the L^2-integrable sections of the $\mathcal{A}^{0,i}(E)$ with respect to a choice of hermitian metrics, which comes together with the natural representations ρ^i given by point-wise multiplication. This gives a sequence of objects in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C_0(X)}$. The Dolbeault operator $D_E = \bar{\partial}_E + \bar{\partial}_E^*$ is unbounded, however, it is an essentially self-adjoint and elliptic differential operator, and hence one can apply functional calculus to it with respect to the bounded function $\chi(t) = \frac{t}{\sqrt{1+t^2}}$ to obtain the pseudo-local (and in particular bounded) operator $\chi(D_E)$ which has the same index theoretic properties as D_E. This determines an exact sequence

$$0 \to \rho^0 \xrightarrow{\chi_0(D_E)} \rho^1 \xrightarrow{\chi_1(D_E)} \dots \xrightarrow{\chi_{n-1}(D_E)} \rho^n \to 0$$

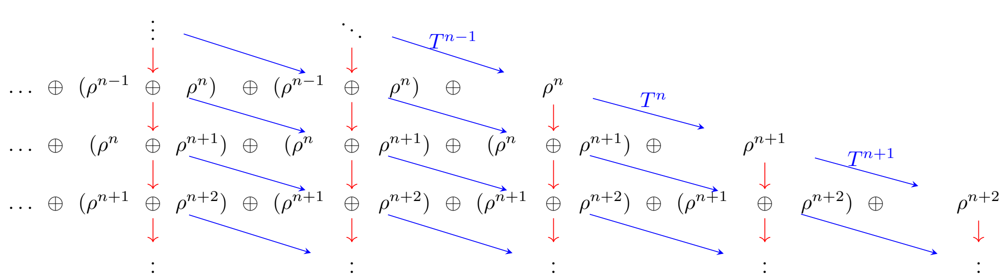
   in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C_0(X)}$. If X is a compact manifold, then by the Hodge decomposition theorem, the above sequence of Hilbert spaces is quasi-isomorphic to the Dolbeault complex. One can also show that the choice of the hermitian metric on the bundle E does not affect the objects (up to natural isomorphisms) over relatively compact open subsets. By theorem 2.1 and by descent properties of topological K-homology, the local maps glue to give a global map

$$\tau_X^D : \mathbb{K}^{alg}(X) \to \mathbb{K}(Ch(\mathfrak{D}/\mathfrak{C})_{C_0(X)})$$

   wherein for an exact category $\mathcal{A}$, $Ch(\mathcal{A})$ denotes the category of bounded acyclic chain complexes in $\mathcal{A}$.

2. Let $Bi(\mathcal{A})$ denote the category of bounded acyclic *double* chain complexes in $\mathcal{A}$. For example, start with a chain complex, and copy each differential to obtain a double chain complex. This induces a natural functor $\Delta : Ch(\mathcal{A}) \to Bi(\mathcal{A})$. For a discrete exact category $\mathcal{A}$, Grayson in [7] constructs a homotopy equivalence of spectra between the homotopy cofiber of $\Delta : \mathbb{K}(Ch(\mathcal{A})) \to \mathbb{K}(Bi(\mathcal{A}))$ and the loop space $\Omega\mathbb{K}(\mathcal{A})$ of the K-theory spectrum of $\mathcal{A}$. By going through the arguments one can still define the map $\tau_{\mathcal{A}}^G$ as a natural map of spectra from the homotopy cofiber of $\Delta : \mathbb{K}(Ch(\mathcal{A})) \to \mathbb{K}(Bi(\mathcal{A}))$ to $\Omega\mathbb{K}(\mathcal{A})$ for a topological exact category $\mathcal{A}$.

3. The last ingredient we need is given by a generalization of the process used by Higson in [8]. We define an exact functor τ_A^H which sends an acyclic chain complex $(\rho^\cdot, T^\cdot)$ in the Paschke category $(\mathfrak{D}/\mathfrak{C})_A$ to the double chain complex displayed below, where the double chain complex is considered with the diagonal arrows as the top chain complex, and the vertical arrows as the bottom chain complex. All the maps with no label are the trivial ones.



By composing these three maps we obtain a map from the K-theory spectrum of the category of locally free sheaves on the complex manifold X to its topological K-homology spectrum. This induces a Riemann-Roch transformation τ_X from algebraic K-theory spectrum of X to its topological K-homology spectrum.

By carefully following the construction above, one can show that τ commutes up to homotopy with the push-forward with respect to proper smooth morphisms, and also restriction to open subsets. This is the content of theorem 1.1.

Question 1. Let $\iota : X \to Y$ be a closed embedding of complex manifolds. Does the diagram 1.2 still commute up to homotopy? In other words, is $\tau_Y R\iota_*$ homotopic to $\iota_* \tau_X$ where $R\iota_*$ is the derived pushforward of coherent analytic sheaves and ι_* is the pull-back map on the Paschke category induced by the restriction map $\iota^* : C_0(Y) \to C_0(X)$?

Currently, I aim to resolve the problem in the case when ι is inclusion of a point. Then I plan to reduce question 1 to inclusion of X as the zero section in its normal bundle in Y, and reduce that to inclusion of $\mathbb{C}^n$ in $\mathbb{C}^{n+m}$ as a subspace. Then I will investigate if this can resolve the Riemann-Roch problem in full generality.

4 Future directions

For a pair of C^*-algebras A, B, the *bivariant KK-groups* $KK(A, B)$ are defined as the abelian group generated by objects called the *Fredholm modules*, with relations induced by direct sum, homotopy, and unitary equivalence [9]. A Fredholm module is a triple (ρ, H, F), where H is a Hilbert B-module (i.e. the inner product takes values in the C^*-algebra B instead of the complex numbers, and there is a continuous action of B on H), and ρ is a representation of A into B-linear bounded operators on H, and F is a pseudo-local B-linear bounded operator on H, so that $F - F^*$ and $F^2 - Id_H$ are locally compact. When $B = \mathbb{C}$ this gives K-homology of the C^*-algebra A, and when $A = \mathbb{C}$, this gives the K-theory of the C^*-algebra B.

One of the most powerful properties of Kasparov bivariant KK-theory is the Kasparov product, which as a special case for C^*-algebras A, B, C, gives a bilinear composition product

$$KK(A, B) \times KK(B, C) \to KK(A, C).$$

The construction of the product relies on Kasparov's technical theorem [9]. This brings me to my first objective.

Objective 4.1. For nuclear C^*-algebras A, B, construct a biexact functor $(\mathfrak{D}/\mathfrak{C})_A \times (\mathfrak{D}/\mathfrak{C})_B \to (\mathfrak{D}/\mathfrak{C})_{A \otimes B}$.

Such a functor would in turn induce a product structure on K-homology spectra of nuclear C^*-algebras. Compare the product structure to Kasparov product.

It is immediate that Kasparov's technical theorem is very close to what we need in here. However, the technical theorem revolves around certain classes of bounded operators on a single Hilbert space and is not functorial, so it would not directly help here. I would like to investigate the possibility of obtaining a *functorial (Kasparov's) technical theorem*, and use it to define a product structure.

The definition of the Paschke category given above can be generalized in different directions to more general exact C^*-categories, e.g. similar to [13] one can define the Paschke category as a category of $*$-functors between certain C^*-categories.

Objective 4.2. Define a *bivariant Paschke category* whose K-theory groups are isomorphic to the bivariant KK-groups.

I intend to explore the exciting possibility of using categorical machinery of algebraic K-theory to obtain new results for KK-theory, or find new insights for existing properties.

Finally, in a slightly different direction, it is easy to observe the similarities between the technical details used in the first step (as above) of defining the Riemann-Roch transformation, and the details of defining a class corresponding to a Dirac operator on a $spin^{\mathbb{C}}$-manifold in Kasparov K-homology. Following this thread of ideas, I wonder if it is possible to:

Objective 4.3. Replicate the whole process of the Riemann-Roch transformation for the Dolbeault operator (as explained above) for a Dirac operator on a *spin*$^{\mathbb{C}}$-manifold. Does such a transformation still make the diagram 1.2 commute?

If this can be achieved, then there will be a new natural interplay between the Grothendieck Riemann-Roch theorem and the Atiyah-Singer index theorem.

References

[1] Michael F Atiyah. Global theory of elliptic operators. In *Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969)*, pages 21–30, 1970.

[2] Michael F Atiyah and Isadore M Singer. The index of elliptic operators on compact manifolds. *Bulletin of the American Mathematical Society*, 69(3):422–433, 1963.

[3] Michael Francis Atiyah and DW Anderson. *K-theory*, volume 2. WA Benjamin New York, 1967.

[4] Paul Baum, William Fulton, and Robert MacPherson. Riemann-roch for singular varieties. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 45(1):101–145, 1975.

[5] Armand Borel and Jean-Pierre Serre. Le théorème de riemann-roch. *Bulletin de la Société mathématique de France*, 86:97–136, 1958.

[6] Henri Gillet. Riemann-roch theorems for higher algebraic k-theory. *Advances in Mathematics*, 40(3):203–289, 1981.

[7] Daniel Grayson. Algebraic -theory via binary complexes. *Journal of the American Mathematical Society*, 25(4):1149–1167, 2012.

[8] Nigel Higson and John Roe. *Analytic K-homology*. OUP Oxford, 2000.

[9] Gennadii Georgievich Kasparov. The operator k-functor and extensions of cˆ*-algebras. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 44(3):571–636, 1980.

[10] Roni N Levy. The riemann-roch theorem for complex spaces. *Acta Mathematica*, 158(1):149–188, 1987.

[11] Roni N Levy. Riemann-roch theorem for higher bivariant k-functors. In *Annales de l'institut Fourier*, volume 58, pages 571–602. Chartres: L'Institut, 1950-, 2008.

[12] Paul D Mitchener. *Symmetric Waldhausen K-theory spectra of topological categories*. Institut for Matematik og Datalogi, SDU-Odense Universitet, 2001.

[13] Paul D Mitchener. Kk-theory of c*-categories and the analytic assembly map. *K-theory*, 26(4):307–344, 2002.

[14] William Paschke. K-theory for commutants in the calkin algebra. *Pacific Journal of Mathematics*, 95(2):427–434, 1981.

[15] Daniel Quillen. Higher algebraic k-theory: I. In *Higher K-theories*, pages 85–147. Springer, 1973.

[16] Graeme Segal. Categories and cohomology theories. *Topology*, 13(3):293–312, 1974.

[17] EH Spanier. A first approximation to homotopy theory. In *Algebraic and Classical Topology*, pages 99–104. Elsevier, 1962.

[18] Dan Voiculescu. A non-commutative weyl-von neumann theorem. *Rev. Roum. Math. Pures et Appl.*, 21:97–113, 1976.

[19] Friedhelm Waldhausen. Algebraic k-theory of spaces. *Algebraic and geometric topology (New Brunswick, NJ, 1983)*, 1126:318–419, 1985.