GROWTH OF BETTI NUMBERS

BRYAN CLAIR AND KEVIN WHYTE

Introduction

Let $X = \widetilde{X}/\Gamma$ be a finite simplicial complex. We study the growth rate of the Betti numbers of finite regular covers of X. Let $X_i = \widetilde{X}/\Gamma_i$, $i = 1, 2, \ldots$, be a sequence of finite regular coverings of X. It is easy to see that the sequence of Betti numbers $\{b_q(X_i)\}$ can grow at most linearly in $[\Gamma : \Gamma_i]$. A theorem of Lück describes exactly when the linear growth rate is achieved, settling a conjecture of Kahzdan. Lück's Theorem [8] states that

$$b_q^{(2)}(\widetilde{X};\Gamma) = \lim_{i\to\infty} \frac{b_q(X_i)}{[\Gamma : \Gamma_i]}$$

when the Γ_i are a tower of finite index normal subgroups of Γ with $\cap\Gamma_i = \{e\}$.

This shows that linear growth of Betti numbers occurs if and only if the L^2-Betti number, $b_q^{(2)}(\widetilde{X};\Gamma)$, is non zero. Nonzero L^2-Betti number means that the Laplacian in dimension q has kernel, and is therefore equivalent to the existence of non-trivial L^2 harmonic q-cochains on $\widetilde{X}$.

In this paper, we assume $b_q^{(2)}(\widetilde{X};\Gamma)$ vanishes and control the growth of the $\{b_q(X_i)\}$. In particular, the sequence cannot grow arbitrarily close to linear in $[\Gamma : \Gamma_i]$ — there is a universal upper bound, and even better bounds given assumptions on X.

The upper bounds use information about the spectrum of the Laplacian near zero: finite covers with large Betti numbers force the L^2 Laplacian to have lots of small spectrum. For example, the best bounds are achieved when $\widetilde{X}$ has a spectral gap. Our results suggest that one might use finite covers to understand the spectrum of $\widetilde{X}$ and its associated data, such as Novikov-Shubin invariants.

Date: February 27, 2002.

Our main proof follows Lück's in outline, with more involved estimates. There are two key factors in estimating the Betti numbers of covers of X. The first measures how X is "unrolled" in the cover:

Definition. Let Γ be a finitely generated group, with the word metric from some fixed generating set. For any subgroup Γ' of Γ, define short(Γ') to be the length of the shortest non-identity element of Γ'.

It is clear from the proof of Lück's Theorem that instead of a tower of Γ_i, one could allow any sequence of $\Gamma_i \triangleleft \Gamma$ with $\mathrm{short}(\Gamma_i) \to \infty$.

The second ingredient in our estimates is the behavior of the L^2 spectrum of $\widetilde{X}$. Just as Lück's theorem relates linear growth of Betti numbers to the kernel of the L^2 Laplacian, our results relate slower growth rates to the behavior of the spectrum of the L^2 Laplacian near zero.

To say "growth rates" suggests a tower, or at least a sequence of coverings. But the main theorems in this paper are stated as bounds on given finite covers X' of X:

Theorem 0.1. *Let X be a finite simplicial complex, and $\widetilde{X}$ an infinite regular covering with covering group Γ. Suppose that $b_q^{(2)}(\widetilde{X};\Gamma) = 0$, or equivalently that there are no L^2 harmonic q-cochains on $\widetilde{X}$.*

1. *(Spectral Gap) Suppose there is a gap near 0 in the L^2 spectrum of $\widetilde{X}$ in dimension q. Then there are $C > 0$ and $M > 0$ so that for any finite regular cover $X' = \widetilde{X}/\Gamma'$ of X:*
$$b_q(X') \leq C\frac{[\Gamma : \Gamma']}{e^{M\mathrm{short}(\Gamma')}}.$$
2. *(Positive Novikov-Shubin Invariant) If $\widetilde{X}$ has Novikov-Shubin invariant $\alpha_q > 0$, then for any $\varepsilon > 0$ there is a $C_\varepsilon > 0$ so that for any finite regular cover $X' = \widetilde{X}/\Gamma'$ of X:*
$$b_q(X') \leq C_\varepsilon\frac{[\Gamma : \Gamma']}{\left(\mathrm{short}(\Gamma')\right)^{\alpha_q-\varepsilon}}.$$
3. *(General Case) For any X there is a C so that for any finite regular cover $X' = \widetilde{X}/\Gamma'$ of X:*
$$b_q(X') \leq C\frac{[\Gamma : \Gamma']}{\log(\mathrm{short}(\Gamma'))}.$$

The three cases of this theorem are proved later as Theorem 4.1, Theorem 5.2, and Theorem 6.1, respectively. They are all special cases of a more general statement (Proposition 3.4) which involves the spectrum of Δ near 0 in a more technical way.

For a family of covers $X_i = \widetilde{X}/\Gamma_i$ with $\mathrm{short}(\Gamma_i) \to \infty$, Theorem 0.1 can be interpreted as giving sublinear upper bounds on the rate of growth of $b_q(X_i)$ in terms of $[\Gamma : \Gamma_i]$.

We can relate $\mathrm{short}(\Gamma_i)$ directly to $[\Gamma : \Gamma_i]$ if we assume the covers of X "unroll" evenly in all directions, which we make precise in Definition 7.1. We call such a family a *uniform* family. As an example, congruence subgroups of arithmetic groups are uniform.

The uniform assumption is most interesting in the case of spectral gap, where it leads to particularly clean bounds of a form previously studied by Sarnak and Xue ([13],[15]).

Theorem 0.2. *Let X be a finite simplicial complex, and $\widetilde{X}$ an infinite regular covering with covering group Γ. Suppose that Γ has exponential growth, $\{\Gamma_i\}$ is a uniform family of finite index normal subgroups, and that Δ_q on $\widetilde{X}$ has no spectrum below λ_0.*

- *There is some $C > 0$ and $\beta < 1$ so that for all $X_i = \widetilde{X}/\Gamma_i$,*
$$b_q(X_i) < C[\Gamma : \Gamma_i]^\beta.$$
- *Fix $\lambda < \lambda_0$. There is some $C > 0$ and $\beta < 1$ so that for all i,*
$$\#\{\mu \le \lambda \mid \mu \text{ is an eigenvalue of } \Delta_q \text{ on } X_i\} < C[\Gamma : \Gamma_i]^\beta.$$

The second part of Theorem 0.2 is proved by a slight alteration of the main argument for Betti numbers. Although one could make this sort of eigenvalue bound in a more general setting, we only carry out the calculations in the most interesting case.

To produce geometric applications, one needs a good understanding of the spectrum of the Laplacian near zero. The locally symmetric spaces provide a wide class of interesting examples, as the L^2-Betti numbers and Novikov-Shubin invariants are known. For these spaces $\mathrm{short}(\Gamma)$ is essentially the injectivity radius of the cover, and $[\Gamma : \Gamma']$ is the volume of the cover. Specifically, for hyperbolic manifolds:

Theorem 0.3. *Let X be a compact hyperbolic n manifold. There are constants $C > 0$ and $\beta_q > 0$ so that for any finite regular cover X' of X one has*

- *For n odd:*
  - *If $q \neq \frac{n\pm 1}{2}$, then*
$$b_q(X') \leq C\frac{\mathrm{Vol}(X')}{e^{\beta_q \,\mathrm{Inj}(X')}}.$$
  - *For $q = \frac{n\pm 1}{2}$,*
$$b_q(X') \leq C\frac{\mathrm{Vol}(X')\cdot \log \mathrm{Inj}(X')}{\mathrm{Inj}(X')}.$$
- *For n even:*
  - *If $q \neq \frac{n}{2}$, then*
$$b_q(X') \leq C\frac{\mathrm{Vol}(X')}{e^{\beta_q \,\mathrm{Inj}(X')}}.$$
  - *For $q = \frac{n}{2}$, the L^2-Betti number $b_q^{(2)}(\mathbb{H}^n;\Gamma)$ is non-zero.*

We investigate the case $\Gamma = \mathbb{Z}^n$ in depth, using different techniques:

Theorem 0.4. *Suppose $\widetilde{X}$ is a regular $\mathbb{Z}^n$ covering of a finite simplicial complex X, and assume the L^2-Betti number $b_q^{(2)}(\widetilde{X};\mathbb{Z}^n) = 0$. Then there is a constant $C > 0$ so that for any finite cover $X' = \widetilde{X}/\Gamma'$ of X:*
$$b_q(X') \leq C\frac{[\mathbb{Z}^n : \Gamma']}{\mathrm{short}(\Gamma')}.$$

This is similar to the bound Theorem 0.1 would give if $\alpha_q(X)$, the Novikov-Shubin invariants of X, satisfied $\alpha_q \geq 1$. However, Lott [7, Ex. 42] gives examples with $\Gamma = \mathbb{Z}$ and $\alpha_q(X)$ arbitrarily small.

1. Stripes

To understand the growth of Betti numbers of covers it helps to have some examples. In particular, one would like a method for constructing complexes with a given fundamental group where it is easy to describe finite covers and their Betti numbers. The construction described in this section starts with an arbitrary finite complex and attaches pieces which do not change the fundamental group and which contribute in a straightforward way to the Betti numbers of finite covers.

Given complexes X and Y, and a complex Z which includes as a subcomplex of both X and Y, build a complex $X \cup_Z Y$ which is the quotient of the disjoint union of X and Y by the relation which identifies the copies of Z. More generally, given a Z with maps $f : Z \to X$ and $g : Z \to Y$, build a new complex, which we describe as "gluing X and Y along Z", by carrying out the previous construction where we replace X and Y by the mapping cylinders of f and g, which are homotopy equivalent to X and Y and contain Z as a subcomplex.

Assume that $\pi_1(X) = \Gamma$. By Van Kampen's theorem, $\pi_1(X \cup_Z Y)$ is also equal to Γ provided that $\pi_1(Z) \hookrightarrow \pi_1(X)$ and $\pi_1(Z) = \pi_1(Y)$. With these assumptions on fundamental groups, put

$$W = X \cup_Z Y.$$

Let Γ' be a finite index normal subgroup of Γ, and let W' be the cover of W with $\pi_1(W') = \Gamma'$. We can describe W' as the result of a gluing $X' \cup_{Z'} Y'$ where X', Y', and Z' are the induced finite covers of X, Y, and Z.

To understand the homology of W', we need to understand the homology of X', Y', and Z', as well as the maps on homology induced by the inclusions. The special case we want to consider here is where all of the homology comes from the parts we have glued on. So assume that X has dimension n, and that we want to compute $H_q(W')$ for $q > n$. This homology is simply the homology of Y'.

If $\pi_1(Z) = \Gamma$ then we have gained nothing by this construction, as it reduces to the same problem for one of the pieces. However, if $\pi_1(Z)$ is a proper subgroup of Γ then the examples coming from this construction give useful insight into the general problem.

Example 1.1. As a simple example, suppose $\pi_1(Z) = \{e\}$. Then Y' is the disjoint union of $[\Gamma : \Gamma']$ copies of Y. If $H_q(Y) = 0$, then $H_q(W') = 0$ as well, for any Γ'. If $H_q(Y) \neq 0$, then the Betti numbers $b_q(W')$ grow linearly in $[\Gamma : \Gamma']$. Neither case is interesting from the point of view of our problem.

In general, the number of components of Y' is

$$b_0(Y') = [\pi_1(Y) : \pi_1(Y) \cap \Gamma'],$$

each of which is the cover of Y with fundamental group $\pi_1(Y) \cap \Gamma'$. Thus, to understand the Betti numbers of these covers, we need to understand the the Betti numbers of the covers of Y and the number of components. The key source of examples is the following:

Definition 1.2. Put $Z = S^1 \hookrightarrow X$, and $Y = S^1 \times S^q$. The construction of $W = X \cup_Z Y$ is called *gluing a stripe to* X *along* Z. We will often refer to Z simply by the corresponding element of $\Gamma = \pi_1(X)$.

For the stripe construction, the components of the cover Y' of Y are $S^1 \times S^q$, so their q-dimensional homology has rank one. Thus

$$\dim(H_q(W')) = [\pi_1(Z) : \pi_1(Z) \cap \Gamma'].$$

Let γ be a generator of $\pi_1(Z) = \mathbb{Z}$, and let $o(\gamma)$ denote the order of γ in the quotient Γ/Γ'. Then

$$\dim(H_q(W')) = \frac{[\Gamma : \Gamma']}{o(\gamma)}.$$

To get examples of fast sublinear growth of Betti numbers, we want $o(\gamma)$ to grow, but as slowly as possible. The following Lemma demonstrates the key piece of geometry of Γ' that enters our bounds:

Lemma 1.3. *Suppose $\gamma \in \Gamma$ has infinite order. There is a $C > 0$ so that for any complex W built by gluing a stripe along γ, and for any finite index normal Γ':*

$$\dim(H_q(W')) \leq C \frac{[\Gamma : \Gamma']}{\text{short}(\Gamma')}.$$

Proof. By the earlier discussion, this amounts to the claim that the order of γ in Γ/Γ' is at least $\text{short}(\Gamma')/C$. If $\gamma^k = e$ in Γ/Γ' then $\gamma^k \in \Gamma'$. Since γ has infinite order γ^k is a non-trivial element of Γ', and so has length at least $\text{short}(\Gamma')$. On the other hand, the length of γ^k is at most k times the length of γ. Thus the Lemma follows with C equal to the length of γ. □

Example 1.4. Let $\Gamma = \mathbb{Z} \times \mathbb{Z} = \langle a \rangle \times \langle b \rangle$, and take a family of subgroups $\Gamma_i = m_i\mathbb{Z} \times n_i\mathbb{Z}$. Put $X = \mathbb{T}^2$, and glue a q-stripe to X along a. Then $b_q(W_i) = n_i$, while $[\Gamma : \Gamma_i] = m_i n_i$.

If $n_i > m_i$, then $\text{short}(\Gamma_i) = m_i$ and the bound in Lemma 1.3 is achieved. If $m_i > n_i$, however, it is clear that the stripe along a fails to capture the geometry. A stripe along b would give a larger Betti number.

In the previous example, we could have glued two stripes so that for any fixed family of groups $\Gamma_i = m_i\mathbb{Z} \times n_i\mathbb{Z}$, growth of order $[\Gamma : \Gamma_i]/\text{short}(\Gamma_i)$ is achieved.

However, for some families Γ_i, there is no stripe construction that achieves this rate of growth:

Example 1.5. Let $\Gamma = \mathbb{Z} \times \mathbb{Z}$, let $\{p_i\}$ be the sequence of primes which are equivalent to 1 modulo 4. It is well known that such primes are the sum of two squares. Put $p_i = a_i^2 + b_i^2$, and let Γ_i be the lattice spanned by (a, b) and $(-b, a)$ in $\mathbb{Z} \times \mathbb{Z}$, so that $[\Gamma : \Gamma_i] = p_i$. For any $\gamma \in \mathbb{Z} \times \mathbb{Z}$, γ generates Γ/Γ_i unless $\gamma \in \Gamma_i$. Since $\mathrm{short}(\Gamma_i) \to \infty$, any given γ is in only finitely many of the Γ_i. Therefore, the Betti numbers of the Γ_i covers of the complex obtained by gluing a stripe (or any finite number of stripes) are bounded over all i.

We will prove that this example is not a defect of the stripe construction. Indeed, for $\Gamma = \mathbb{Z}^n$, and any family of coverings, the stripes examples achieve the fastest possible growth of Betti numbers (see Remark 2.4). We do not know if this is true for general Γ.

2. The Abelian Case

In this section, we investigate in depth the special case where $\Gamma = \mathbb{Z}^n$. The techniques are different from the general situation considered in Section 3. Here, the key step is an algebraic estimate of rational solutions to a trigonometric polynomial. The results are of a similar form to the general theorems, and illustrate the strength of the stripe construction.

Theorem 2.1. *Suppose $\widetilde{X}$ is a regular $\mathbb{Z}^n$ covering of a finite simplicial complex X, and assume the L^2-Betti number $b_q^{(2)}(\widetilde{X}; \mathbb{Z}^n) = 0$. Then there is a constant $C(X)$ so that for any $\Gamma' < \mathbb{Z}^n$, we have*

$$b_q(X') \leq C(X) \frac{[\mathbb{Z}^n : \Gamma']}{\mathrm{short}(\Gamma')}. \tag{2.1}$$

Here $X' = \widetilde{X}/\Gamma'$.

Suppose that X has a cells in dimension q. All work is done in the fixed dimension q, but the q is suppressed in most of the notation. Put $\Gamma = \mathbb{Z}^n$.

Choosing lifts of cells from X to $\widetilde{X}$, we identify $C_q^{(2)}(\widetilde{X})$ with $\bigoplus_1^a l^2(\Gamma)$. In this basis, Δ is represented by an $a \times a$ matrix B with entries in $\mathbb{Z}[\Gamma]$.

Since $\Gamma = \mathbb{Z}^n$ is abelian, we can form $\det(B) \in \mathbb{Z}[\Gamma]$. The determinant $\det(B)$ will not depend on the choice of lifts of cells, so we will simply write $\det(\Delta)$.

Identify the n-torus $\mathbb{T}^n$ with the irreducible unitary representations of $\mathbb{Z}^n$. Let

$$K = \{\rho \in \mathbb{T}^n \mid \rho(\det(\Delta)) = 0\}.$$

Call K the *pattern* of X on $\mathbb{T}^n$.

We want to calculate the Betti numbers for the finite cover X' of X. The Laplacian on q-chains on X' is also given by the matrix B, now acting on $\bigoplus_1^a l^2(\Gamma/\Gamma')$. The space $l^2(\Gamma/\Gamma')$ splits as a direct sum of irreducible representations ρ, where ρ lies on a lattice Λ of rational points in $\mathbb{T}^n$. Λ is (non-canonically) isomorphic to Γ/Γ', and

$$\Lambda = \{\rho \in \mathbb{T}^n \mid \rho \text{ is trivial on } \Gamma'\}.$$

Lemma 2.2.

$$|\Lambda \cap K| \leq b(X') \leq a \cdot |\Lambda \cap K|$$

Proof. We have

$$\bigoplus_1^a l^2(\Gamma/\Gamma') = \bigoplus_{\rho \in \Lambda} \left(\bigoplus_1^a V_\rho \right),$$

where $V_\rho = \mathbb{C}$. The Laplacian Δ' preserves the a-dimensional space $\bigoplus_1^a V_\rho$, and has kernel there precisely when $\rho(\det(\Delta)) = 0$. The dimension of the kernel is obviously bounded by a. □

The goal, then, is to count certain rational points in $\mathbb{T}^n$ which lie on the pattern of X. The pattern of X is the set of root-of-unity solutions to a polynomial equation, and these are fully described by Conway and Jones [3].

Choose generators $g_1, \ldots, g_n$ for $\mathbb{Z}^n$. Then $\det(\Delta)$ is a polynomial in the g_k (with integer coefficients). For $\rho = (x_1, \ldots, x_n) \in \mathbb{T}^n$, $\rho(g_k) = e^{2\pi i x_k}$. Therefore,

$$\rho(det(\Delta)) = \sum_I A_I e^{2\pi i l_I(x)},$$

where $l_I(x)$ is some integer linear combination of the x_i. It follows from [3] that the rational solutions to

$$\sum_I A_I e^{2\pi i l_I(x)} = 0$$

lie on one of a finite family of rational linear subspaces of $\mathbb{T}_n$.

We have shown that the rational points in the pattern of X are contained in a finite collection of rational linear subspaces. Fix one such space $L \subset \mathbb{T}^n$. If L is not itself a subgroup, then it is a coset of some subgroup $L' < \mathbb{T}^n$. The intersection $\Lambda \cap L$ is either empty or itself a coset of $\Lambda \cap L'$. Therefore,

$$|\Lambda \cap L| \leq |\Lambda \cap L'|.$$

So, to estimate $|\Lambda \cap L|$ we may as well assume that L is a subgroup of $\mathbb{T}^n$.

There is a subgroup $\hat{L} \subset \Gamma$ so that

$$L = \left\{ \rho \in \mathbb{T}^n \mid \rho \text{ is trivial on } \hat{L} \right\}.$$

Note that $\dim(L) + \operatorname{rank}_{\mathbb{Z}} \hat{L} = n$.

Choose any $g \in \hat{L}$. For optimal constants, one should make $||g||$ as small as possible. Let $o(g)$ denote the order of g in Γ/Γ'. We have

$$o(g) \geq \frac{\operatorname{short}(\Gamma')}{||g||}.$$

Construct a representation $\rho : \Gamma \to S^1$ which is trivial on Γ' and sends g to a primitive $o(g)$-th root of unity (this is easy, since Γ/Γ' is a finite abelian group). For $k = 1, 2, \ldots, o(g)$, each representation ρ^k is in Λ, but all assume different values on g. In particular, $\rho^1, \rho^2, \ldots, \rho^{o(g)}$ all lie in different cosets of $\Lambda \cap L < \Lambda$. Therefore

$$|\Lambda \cap L| \leq \frac{|\Lambda|}{o(g)} \leq \frac{||g|| \cdot [\Gamma : \Gamma']}{\operatorname{short}(\Gamma')}.$$

Summing over the various L which appear in K (the pattern of X), we conclude that

$$|\Lambda \cap K| \leq \frac{C_1(X) \cdot [\Gamma : \Gamma']}{\operatorname{short}(\Gamma')}. \tag{2.2}$$

Here $C_1(X)$ is a constant depending on X, or more precisely, depending on the rational linear subspaces appearing in the pattern of X.

Combining (2.2) with Lemma 2.2 completes the proof of Theorem 2.1. □

As a corollary of the proof, we can make an even better statement for $\Gamma = \mathbb{Z}$:

Corollary 2.3. *Suppose $\widetilde{X}$ is a regular $\mathbb{Z}$ covering of a finite simplicial complex X. Put $X_i = \widetilde{X}/i\mathbb{Z}$, the i-fold covering of X. Then exactly one of the following possibilities occurs:*

1. *$b_q^{(2)}(\widetilde{X};\mathbb{Z}) \neq 0$, so the sequence of Betti numbers $b_q(X_i)$ is asymptotically linear in i.*
2. *There is a constant $C(X)$ so that $b_q(X_i) \leq C(X)$ for all i.*

Proof. The torus $\mathbb{T}^1$ is just S^1, so a linear subspace is either a single point or the entire circle. If the pattern of X contains the entire circle, we are in the first case. Otherwise, the pattern of X consists of a finite collection of k points (counted with multiplicities), and so the betti numbers of X_i are bounded by k times the number of cells in X. □

Remark 2.4. For $\Gamma = \mathbb{Z}^n$ and for a given sequence of subgroups $\Gamma_i \subset \Gamma$, we need only consider the stripe construction when looking for large Betti numbers. This is because given $X = \widetilde{X}/\Gamma$, the Betti numbers $b_q(X_i)$ depend only on the rational linear subspaces contained in the pattern of X. We can reproduce these rational linear subspaces by repeatedly gluing stripes to an n-torus.

Example 1.4 then suggests that Theorem 2.1 has the right geometric ingredients for an upper bound, but that these do not completely determine the growth rate. See Example 1.5.

Remark 2.5. The q^{th} Novikov-Shubin invariant for a space with a q-stripe is 1, the same as for $\mathbb{R}$. Compare the bound (2.1) for Γ abelian with the general bound (5.3) for Novikov-Shubin invariant 1.

3. General Bounds

In this section, we derive a general bound on Betti numbers of coverings, in terms of the L^2 spectral density function. The argument follows the proof of Lück's Theorem [8], but we carefully control the estimates throughout.

3.1. **Preliminaries.** Let X be a finite simplicial complex, and $\widetilde{X}$ an infinite regular cover, with covering transformation group Γ. Suppose we have a normal subgroup Γ' of finite index in Γ. Form $X' = \widetilde{X}/\Gamma'$, a covering of X of order $[\Gamma : \Gamma']$.

Let Δ be the combinatorial Laplacian on $C_q^{(2)}(\widetilde{X})$, the space of q-dimensional l^2-cochains. Let Δ' be the Laplacian on q-cochains of X'.

Suppose X has a cells in dimension q. Lifting these cells to $\widetilde{X}$ gives a basis over $l^2(\Gamma)$ for $C_q^{(2)}(\widetilde{X})$. These lifts descend to give a basis over $l^2(\Gamma/\Gamma')$ for $C_q(X')$.

In this basis, Δ is represented by an $a \times a$ matrix B with entries in $\mathbb{Z}[\Gamma]$, acting by right multiplication on $\bigoplus_{j=1}^{a} l^2(\Gamma)$. The same matrix B describes Δ', now acting by right multiplication on $\bigoplus_{j=1}^{a} l^2(\Gamma/\Gamma')$.

Lemma 3.1. *There exists a number $K > 1$ so that $\|\Delta\| \leq K$ and so that for any group Γ' as above, $\|\Delta'\| \leq K$.*

Proof. This is identical to [8, Lemma 2.7]. The number K can be defined from the coefficients of group elements appearing in B. □

Lemma 3.2. *Let*

$$R = \max\{\|g\| \mid g \in \Gamma \text{ appears with nonzero coefficient in some } B_{ij}\}.$$

Then for any polynomial p with $\deg(p) < \frac{\mathrm{short}(\Gamma')}{R}$,

$$\mathrm{Tr}_\Gamma\, p(\Delta) = \mathrm{Tr}_{\Gamma/\Gamma'}\, p(\Delta')$$

Proof. Write

$$\sum_{j=1}^{a} (p(B))_{jj} = \sum_{g \in \Gamma} \lambda_g g \quad (\in \mathbb{Z}[\Gamma]).$$

Then

$$\mathrm{Tr}_\Gamma\, p(\Delta) = \lambda_e$$

and

$$\mathrm{Tr}_{\Gamma/\Gamma'}\, p(\Delta') = \sum_{g \in \Gamma'} \lambda_g.$$

However, B contains group elements of length at most R, so B^n contains group elements of length at most nR. Therefore $p(B)$ contains group elements of length less than $\mathrm{short}(\Gamma')$, and so $\lambda_g = 0$ for all $g \in \Gamma'$, $g \neq e$. □

3.2. Density functions and Betti numbers.

Definition 3.3. Let

$$\mathcal{P}_n = \left\{ \text{Polynomials } p \text{ in one variable} \;\middle|\; \begin{array}{c} \deg p \leq n; \\ p \text{ non-negative on } [0,1]; \\ p(0) = 1 \end{array} \right\}$$

For a probability measure $d\mu$ on $[0,1]$, put

$$J(n,\mu) = \inf_{p\in\mathcal{P}_n} \int_0^1 p(x)d\mu(x).$$

The measure of interest to us is the spectral density function of the Laplacian, suitably rescaled. To define it, let $\{E(\lambda) : \lambda \in [0,\infty)\}$ denote the family of spectral projections of Δ. Then the spectral density function of Δ is

$$F(\lambda) = \mathrm{Tr}_\Gamma\, E(\lambda).$$

Proposition 3.4. *Let*

$$n < \frac{\mathrm{short}(\Gamma')}{R}$$

and

$$\mu(x) = \frac{F(Kx)}{a}.$$

Then

$$b_q(X') \le a[\Gamma : \Gamma']J(n,\mu).$$

Proof. Let $p \in \mathcal{P}_n$. For any $r > 1$, there is some $\lambda > 0$ so that every $x \in [0,\lambda]$ satisfies

$$rp(\frac{x}{K}) \ge 1.$$

Let χ be the characteristic function of the interval $[0,\lambda]$. We have $rp(\frac{x}{K}) \ge \chi(x)$ on $[0,K]$. Then

$$\begin{aligned}\frac{b_q(X')}{[\Gamma:\Gamma']} &\le \mathrm{Tr}_{\Gamma/\Gamma'}\, \chi(\Delta)\\ &\le \mathrm{Tr}_{\Gamma/\Gamma'}\, rp(\frac{\Delta}{K})\\ &= r\, \mathrm{Tr}_\Gamma\, p(\frac{\Delta}{K})\end{aligned}$$

using Lemma 3.2. Sending $r \to 1$,

$$\begin{aligned}b_q(X') &\le [\Gamma:\Gamma']\, \mathrm{Tr}_\Gamma\, p(\frac{\Delta}{K})\\ &= [\Gamma:\Gamma'] \int_0^K p(\frac{\lambda}{K})dF(\lambda)\\ &= a[\Gamma:\Gamma'] \int_0^1 p(x)d\mu(x).\end{aligned}$$

Since $p \in \mathcal{P}_n$ was arbitrary, the proof is done. ∎

3.3. **Choosing a polynomial.** We now proceed to estimate $J(n, \mu)$, with an eye towards using the decay of $\mu(x)$ as $x \to 0$. Fix $z \in (0, 1)$. Let $p \in \mathcal{P}_n$ be bounded by 1 on $[0, z]$ and bounded by M on $[z, 1]$. If $p(z) < M$, the estimates will be strictly improved by moving z to the left, so we may as well restrict attention to the situation where $p(z)$ is the maximal value of p on $[z, 1]$. We have

$$\begin{aligned} J(n, \mu) &\leq \int_0^1 p(x) d\mu(x) \\ &= \int_0^z p(x) d\mu(x) + \int_z^1 p(x) d\mu(x) \\ &\leq \mu(z) + p(z). \end{aligned}$$

To minimize $\mu(z)+p(z)$, we want a polynomial which drops as quickly as possible from $x = 0$ to $x = z$ and then stays low until $x = 1$. Via a linear transformation, this is equivalent to finding a polynomial which is bounded by ± 1 on $[-1, 1]$ and grows as quickly as possible for $x > 1$. Among polynomials of degree n, the Chebyshev polynomial T_n is the optimal solution [12].

The Chebyshev polynomials are defined by $T_n(\cos(\theta)) = \cos(n\theta)$. The first few are $1, x, 2x^2 - 1, 4x^3 - 3x$. We need two facts: first, $T_n(1) = 1$ for all n; second, the Chebyshev polynomials satisfy a recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

which yields

$$T_n(x) = \frac{1}{2}\left[\left(x + \sqrt{x^2 - 1}\right)^n + \left(x - \sqrt{x^2 - 1}\right)^n\right].$$

Now, put

$$l(x) = \left(\frac{-2}{1 - z}\right) x + \left(\frac{1 + z}{1 - z}\right),$$

so $l(1) = -1$, $l(z) = 1$, and $l(0) = \frac{1+z}{1-z}$. Set

$$p_n(x) = \frac{T_n\big(l(x)\big) + 1}{T_n\big(l(0)\big) + 1}.$$

It is easy to see that $p_n \in \mathcal{P}_n$. Therefore

$$\begin{aligned} J(n,\mu) &\le \mu(z) + p_n(z) \\ &\le \mu(z) + \frac{2}{T_n\left(\frac{1+z}{1-z}\right)} \\ &= \mu(z) + \frac{4}{\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^n + \left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^{-n}} \\ &\le \mu(z) + 4\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^{-n} \\ &\le \mu(z) + 4e^{-2n\sqrt{z}}. \end{aligned}$$

This proves:

Proposition 3.5. *For any $n > 0$, probability measure $d\mu$, and $z \in [0,1]$,*

$$J(n,\mu) \le \mu(z) + 4e^{-2n\sqrt{z}} \tag{3.1}$$

To choose z appropriately requires information about μ. We handle three specific cases in the next sections.

4. Spectral Gap

Suppose that the L^2 spectrum for $\widetilde{X}$ has a gap at zero. This means that there exists $\lambda_0 > 0$ with $F(\lambda) = 0$ for all $\lambda < \lambda_0$.

Continuing notation from Section 3, $\mu(x) = F(Kx)/a$ has a gap at zero of size λ_0/K. Letting $z \to \lambda_0/K$ in (3.1) yields

$$J(n,\mu) \le 4e^{-2n\sqrt{\lambda_0/K}}.$$

Combining with Proposition 3.4 proves the following:

Theorem 4.1. *If $\widetilde{X}$ has spectral gap of size λ_0 in dimension q, then*

$$b_q(X') \le 4a\frac{[\Gamma : \Gamma']}{e^{M \operatorname{short}(\Gamma')}}, \tag{4.1}$$

where $M = \frac{2}{R}\sqrt{\frac{\lambda_0}{K}}$. Here R is the maximum length of group elements appearing in the matrix for Δ, and K is a bound on the norms of Δ and Δ_i as in Lemma 3.1.

If Γ has exponential growth, $[\Gamma : \Gamma']$ will be at least exponential in $\operatorname{short}(\Gamma')$. For a family of subgroups $\Gamma_i \triangleleft \Gamma$, a natural assumption is that $\operatorname{short}(\Gamma_i)$ actually does grow like $\log[\Gamma : \Gamma_i]$. This is the assumption of the following Corollary.

Corollary 4.2. *Let $\{\Gamma_i\}$ be a family of normal, finite index subgroups of Γ. Suppose there exists $D > 0$ so that*

$$\text{short}(\Gamma_i) > D \log[\Gamma : \Gamma_i] - \textit{constant}. \tag{4.2}$$

Put $X_i = \widetilde{X}/\Gamma_i$. Then there is some $C > 0$ so that

$$b_q(X_i) < C \cdot [\Gamma : \Gamma_i]^{1-MD}$$

for all i.

This also proves the first part of Theorem 0.2. The estimate (4.2) comes directly from the assumption that Γ has exponential growth and that the family $\{\Gamma_i\}$ is uniform. Applying (4.2) to the next Proposition proves the second part of Theorem 0.2.

Proposition 4.3. *Fix $\lambda < \lambda_0$, and let*

$$E(\lambda) = \{\mu \le \lambda \mid \mu \textit{ is an eigenvalue of } \Delta'_q \textit{ on } X'\}.$$

Then there is a $C > 0$ and $M > 0$ so that

$$\#E(\lambda) < C \frac{[\Gamma : \Gamma']}{e^{M \text{short}(\Gamma')}}.$$

Proof. We continue the notation of Section 3.3, where p_n is a linear transformation of the n^{th} Chebyshev polynomial. Note that p_n is decreasing on $[0, \lambda_0/K]$. Then

$$\#E(\lambda) \cdot p_n(\lambda/K) \le \sum_{\mu \in E(\lambda)} p_n(\mu/K) \le [\Gamma : \Gamma'] \operatorname{Tr}_{\Gamma/\Gamma'} p_n(\Delta'/K).$$

If $n < \text{short}(\Gamma')/R$, Lemma 3.2 applies, and we get

$$\#E(\lambda) \le [\Gamma : \Gamma'] \frac{\operatorname{Tr}_\Gamma p_n(\Delta/K)}{p_n(\lambda/K)}.$$

The spectrum of Δ is empty below λ_0, so

$$\#E(\lambda) \le [\Gamma : \Gamma'] \frac{p_n(\lambda_0/K)}{p_n(\lambda/K)}.$$

Finally, a simple calculation shows that the ratio $\frac{p_n(\lambda_0/K)}{p_n(\lambda/K)}$ decays exponentially in n as $n \to \infty$ (although the base approaches 1 when λ nears λ_0). Replacing n with $\text{short}(\Gamma')/R$ completes the proof. □

5. Positive Novikov-Shubin Invariant

The q^{th} Novikov-Shubin invariant of $\widetilde{X}$ describes the decay of the spectral density function $F(\lambda)$ near $\lambda = 0$. Suppose there is some $C > 0$ so that

$$C^{-1}\lambda^{\alpha_q/2} < F(\lambda) < C\lambda^{\alpha_q/2} \tag{5.1}$$

for small λ. Then α_q is the q^{th} Novikov-Shubin invariant of $\widetilde{X}$. In general, one defines

$$\alpha_q = 2\liminf_{\lambda\to 0^+} \frac{\log(F(\lambda) - F(0))}{\log(\lambda)} \quad \in [0,\infty]$$

when F has no spectral gap. The 2 is a normalization which is discarded by some authors. With our definition, all Novikov-Shubin invariants of $\mathbb{R}^n$ are n.

For the moment, suppose that $\mu(x) < Cx^\beta$ for some $C > 0$ and $\beta > 0$.

To make $J(n,\mu)$ small, we choose z so that the two terms in the bound (3.1) are roughly the same size for large n. We ask that

$$z^\beta = e^{-2n\sqrt{z}}$$

or

$$\beta\log(z) = -2n\sqrt{z}.$$

Substituting $x = z^{-1/2}$ yields

$$x\log(x) = n/\beta,$$

which has the solution $\log(x) = W(n/\beta)$, where W is the Lambert W-function. For large n, $W(y)$ is asymptotic to $\log(y/\log(y))$. Based on this, we choose

$$z = \left(\frac{\log(n/\beta)}{n/\beta}\right)^2.$$

Now plug in to the bound (3.1) for $J(n,\mu)$:

$$\begin{aligned} J(n,\mu) &\le C\left(\frac{\log(n/\beta)}{n/\beta}\right)^{2\beta} + 4\exp\left(-2n\frac{\log(n/\beta)}{n/\beta}\right) \\ &= C\left(\frac{\log(n/\beta)}{n/\beta}\right)^{2\beta} + 4\left(\frac{1}{n/\beta}\right)^{2\beta}. \end{aligned}$$

Therefore, there is some constant C' depending on C and β so that for all n,

$$J(n,\mu) \le C' \cdot \left(\frac{\log(n)}{n}\right)^{2\beta}.$$

Combining this result with Proposition 3.4 gives the following:

Theorem 5.1. *Suppose the q^{th} spectral density function of $\widetilde{X}$ satisfies*

$$F_q(\lambda) < C\lambda^{\beta}$$

for some $\beta > 0$, $C > 0$. Then there is a constant $C_1 > 0$ so that

$$b_q(X') \leq C_1[\Gamma : \Gamma']\left(\frac{\log(\mathrm{short}(\Gamma'))}{\mathrm{short}(\Gamma')}\right)^{2\beta}. \tag{5.2}$$

Now, we interpret this for Novikov-Shubin invariants:

Theorem 5.2. *Suppose $b_q^{(2)}(\widetilde{X};\Gamma) = 0$ and the Novikov-Shubin invariant $\alpha_q(\widetilde{X})$ is positive. Then for every $\varepsilon > 0$, there is some constant C_ε depending on ε and X so that for any finite covering $X' = \widetilde{X}/\Gamma'$ of X,*

$$b_q(X') \leq C_\varepsilon \frac{[\Gamma : \Gamma']}{\left(\mathrm{short}(\Gamma')\right)^{\alpha_q(\widetilde{X})-\varepsilon}}. \tag{5.3}$$

Proof. If $\alpha_q(\widetilde{X}) > 0$, we have a bound of the form

$$F_q(\lambda) < C\lambda^{\beta}$$

for any $\beta < \alpha_q(\widetilde{X})/2$. Since we can always make β a little larger than needed for a given ε, the $\log(\mathrm{short}(\Gamma'))$ term in (5.2) is absorbed into the constant. □

6. Sublogarithmic Decay

The most general estimate known for spectral density functions is

$$F(\lambda) < \frac{a\log(K)}{-\log(\lambda)}.$$

This estimate holds whenever Γ belongs a large class of groups $\mathcal{G}$. The class $\mathcal{G}$ includes all residually finite groups. For details, see [4], [2], [14].

Assume that $\mu(x) < \frac{C}{-\log(x)}$. As in Section 5, we want to choose z so that the two terms in the bound (3.1) are roughly the same size for large n. We ask that

$$\frac{1}{-\log(z)} = e^{-2n\sqrt{z}}.$$

Substitute $x = 2n\sqrt{z}$ to get

$$e^x + 2\log(x) = 2\log(2n),$$

which has the approximate solution $x = \log(2\log(2n))$ for large n. As the precise constants won't matter in the end, choose

$$z = \left(\frac{\log(\log n)}{n}\right)^2.$$

Now plug in to the bound (3.1) for $J(n,\mu)$:

$$J(n,\mu) \leq \frac{C}{-2\big(\log(\log(\log n)) - \log(n)\big)} + 4e^{-\log(\log n)}$$

Therefore, there is some constant C' so that for all n,

$$J(n,\mu) \leq \frac{C'}{\log(n)}. \tag{6.1}$$

Theorem 6.1. *Given Γ and $X = \widetilde{X}/\Gamma$, with $b_q^{(2)}(\widetilde{X};\Gamma) = 0$. There is a constant C so that for any finite covering $X' = \widetilde{X}/\Gamma'$ of X,*

$$b_q(X') \leq C\frac{[\Gamma : \Gamma']}{\log(\mathrm{short}(\Gamma'))}.$$

Proof. If Γ is not residually finite, short(Γ') is uniformly bounded over all Γ', so the theorem is vacuously true. Otherwise, from [8], one has decay of the L^2 spectrum of $\widetilde{X}$ of the form

$$F(\lambda) < \frac{a\log(K)}{-\log(\lambda)},$$

for $\lambda < \varepsilon$, with some $\varepsilon > 0$. Now $\mu(x) < \frac{C}{-\log(x)}$ for some $C > 0$, and we apply (6.1) to Proposition 3.4. ☐

Remark 6.2. If Γ is not residually finite, form $\Gamma_f = \bigcap \Gamma'$, where the intersection is over all finite index normal subgroups of G. Then all finite covers of X are covered by $X_f = \widetilde{X}/\Gamma_f$. The group Γ/Γ_f acts on X_f and is residually finite, so we can still bound Betti numbers of X' in a non-trivial way. The subtle point is that short(Γ') is replaced by the shortest element of Γ'/Γ_f as a subgroup of Γ/Γ_f.

7. Applications And Questions

In applying Theorem 0.1 to specific spaces there are two main obstacles: controlling the L^2 spectral data and calculating the growth of short(Γ') for various covers.

If Γ acts on X properly discontinuously and cocompactly by isometries, then X and Γ are quasi-isometric, hence, up to a constant factor we can measure distances in X rather than in Γ. For any γ in Γ this gives the estimate that the minimal length in Γ of a conjugate of γ is comparable to the minimal distance a point is moved in X, which is the same as the minimal length of a loop in X in the free homotopy class of γ. In this way we can interpret short(Γ') as the minimal length of a nontrivial loop in X/Γ'. If X is non-postively curved then this loop is the shortest closed geodesic, and short(Γ') is comparable to the injectivity radius of X'/Γ'.

Balls of diameter short(Γ') are embedded, so their volume cannot exceed the volume of X/Γ', which is proportional to the index of Γ' in Γ. When Γ has exponential growth, one gets short$(\Gamma') \leq C \log[\Gamma : \Gamma']$, and for $\Gamma = \mathbb{Z}^n$ one has short$(\Gamma') \leq C[\Gamma : \Gamma']^{\frac{1}{n}}$. In the Euclidean case this bound is sharp for covers which have a fundamental domain a cube (for example $\Gamma' = k\mathbb{Z}^n$ for any k), and way off if the fundamental domain is long and thin. Similarly, in general, this estimate is sharp when the covers X/Γ' unroll all directions equally. Precisely, we define:

Definition 7.1. A family $\{\Gamma_i\}$ of finite index normal subgroups of Γ is *uniform* if there is a $C > 0$ so that

$$[\Gamma : \Gamma_i] \leq \mathrm{Vol}(B_\Gamma(C \cdot \mathrm{short}(\Gamma_i)))$$

for all i. Here $B_\Gamma(r)$ is the ball in Γ of radius r, in the word metric.

Say that a family of regular covers $\{X_i\}$ is uniform if the corresponding family of groups is uniform.

As examples of uniform coverings, generalizing the $k\mathbb{Z}^n \subset \mathbb{Z}^n$, we have:

Lemma 7.2. *If Γ is arithmetic, the collection of congruence subgroups $\{\Gamma_n\}$ is uniform.*

Proof. We show this when Γ has exponential growth. The proof for polynomial growth is no more difficult. View Γ as a group of integral matrices. Let m be the largest matrix entry appearing in all generators of Γ. A word in generators of Γ must be of length at least $\log_m(n)$ to have an entry of size n. Therefore, short$(\Gamma_n) \geq \log_m(n)$. As the volume growth of balls in Γ is exponential, we can choose a C so that $\mathrm{Vol}(B_\Gamma(C \cdot \mathrm{short}(\Gamma_n)))$ is larger than any given polynomial in n, and in particular is larger than the index $[\Gamma : \Gamma_n]$. □

All of this discussion allows for geometric estimates on Betti numbers in some cases:

Example 7.3. Suppose X is an n-dimensional manifold of pinched negative curvature, $-1 \leq \kappa < -\left(\frac{n-2d}{n-1}\right)^2$. Then the universal cover $\widetilde{X}$ of X has spectral gap for $|q - n/2| \geq d$ [5]. For such a q, one has $C > 0$ and $M > 0$ so that for all finite regular covers X' of X:

$$b_q(X') < C \frac{\mathrm{Vol}(X')}{e^{M \, \mathrm{Inj}(X')}}$$

For a uniform family of covers $\{X_i\}$, Theorem 0.2 applies and gives $C > 0$, $\beta < 1$ so that

$$b_q(X_i) < C \, \mathrm{Vol}(X_i)^\beta \tag{7.1}$$

for all i.

In particular, (7.1) holds for congruence covers of arithmetic hyperbolic manifolds, outside the middle dimension. Here, Sarnak and Xue [13] have conjectured an upper bound of the form (7.1), with $\beta = 2q/(n-1) + \varepsilon$. For these manifolds, lower bounds of the same type are known. Xue [15] has shown:

$$b_q(X_p) > C_\varepsilon \cdot \mathrm{Vol}(X_p)^{\delta - \varepsilon}$$

for any $\varepsilon > 0$, and with an explicit δ.

For more general symmetric spaces, the spectrum near zero is well understood. The L^2 Betti numbers vanish except possibly in the middle dimension. Near the middle dimension, the spectral density obeys a power law decay, and away from the middle range there is spectral gap. See [11] for the exact results. This allows us to give upper bounds for the growth rates of Betti numbers of locally symmetric manifolds, for example Theorem 0.3.

Adams and Sarnak [1] have precise results for similar questions, calculating multiplicities of representations instead of Betti numbers.

Question 7.4. Is there a $C > 0$ so that for any compact hyperbolic 3-manifold M,

$$b_1(M) \leq C \frac{\mathrm{Vol}(M)}{\mathrm{Inj}(M)}?$$

More generally, for a compact locally symmetric space modeled on G/K, how big can the Betti numbers be in terms of volume and injectivity radius?

For more general spaces, computing the L^2 spectral data is very hard. Even the question of whether all spaces have positive Novikov-Shubin invariants is open. If they do, the bounds for arbitrary spaces in Theorem 6.1 would get significantly better, changing from

$$\frac{[\Gamma : \Gamma']}{\log \operatorname{short}(\Gamma')}$$

to

$$\frac{[\Gamma : \Gamma']}{\operatorname{short}(\Gamma')^{\beta}} \tag{7.2}$$

(for some β depending on X). These results imply that any space violating the bounds (7.2) must have vanishing Novikov-Shubin invariant, and so our results could be used to detect such a space.

Question 7.5. Is there a space X with $b_q^{(2)}(X) = 0$ and a sequence of finite normal covers $X_i = X/\Gamma_i$ so that the q^{th} Betti numbers of X_i grow faster than:

$$\frac{[\Gamma : \Gamma_i]}{\big(\operatorname{short}(\Gamma_i)\big)^{\beta}}.$$

for all $\beta > 0$?

We know of no example with Betti numbers growing anywhere near that quickly:

Question 7.6. Is there a space X with $b_q^{(2)}(X) = 0$ and a sequence of finite normal covers $X_i = X/\Gamma_i$ so that the Betti numbers $b_q(X_i)$ grow faster than any multiple of

$$\frac{[\Gamma : \Gamma_i]}{\operatorname{short}(\Gamma_i)}?$$

References

[1] S. Adams and P. Sarnak. Betti numbers of congruence groups. *Israel J. Math.*, 88(1-3):31–72, 1994.

[2] B. Clair. Residual amenability and the approximation of L^2-invariants. *Michigan J. Math.*, 46, 1999.

[3] J. H. Conway and A. J. Jones. Trigonometric diophantine equations (on vanishing sums of roots of unity). *Acta Arithmetica*, 30(3), 1976.

[4] J. Dodziuk and V. Mathai. Approximating L^2-invariants of amenable covering spaces: A combinatorial approach. *J. Func. Anal.*, 154(2):359–378, 1998.

[5] H. Donnelly and F. Xavier. On the differential form spectrum of negatively curved Riemannian manifolds. *Amer. J. Math.*, 106:169–185, 1984.

[6] M. Farber. Von Neumann categories and extended L^2 cohomology. *K-Theory*, 15:347–405, 1998.

[7] J. Lott. Heat kernels on covering spaces and topological invariants. *J. Diff. Geom.*, 35:471–510, 1992.

[8] W. Lück. L^2-torsion and 3-manifolds. In *Low-dimensional topology (Knoxville, TN, 1992*, p. 75-107, Internat. Press, Cambridge, MA, 1994.

[9] W. Lück. Approximating L^2-invariants by their finite dimensional analogues. *Geom. and Func. Anal.*, 4(4):455–481, 1994.

[10] W. Lück. L^2-invariants: theory and applications to geometry and K-theory. Submitted to Grundlehren der math. Wissenschaften in Springer Verlag, 2002.

[11] M. Olbrich. L^2-invariants of locally symmetric spaces. Preprint, 2000. *http://front.math.ucdavis.edu/math.DG/0009039.*

[12] T. Rivlin. *An Introduction to the Approximation of Functions.* Dover, 1981. Corrected reprint of the 1969 original.

[13] P. Sarnak and X. Xue. Bounds for multiplicites of automorphic forms. *Duke Math. J.*, 64, 1991.

[14] T. Schick. L^2-determinant class and approximation of L^2-betti numbers. *Trans. AMS*, 353(8), 2001.

[15] X. Xue. On the Betti numbers of a hyperbolic manifold. *Geom. and Func. Anal.*, 2(1):126–135, 1992.

Department of Mathematics, Saint Louis University, St. Louis, MO 63103
E-mail address: bryan@slu.edu

Department of Mathematics, University of Illinois at Chicago, Chicago, IL 60607
E-mail address: kwhyte@uic.edu