WHEN IS A GROUP ACTION DETERMINED BY IT'S ORBIT STRUCTURE?

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ABSTRACT. We present a simple approach to questions of topological orbit equivalence for actions of countable groups. For example, for any action of a countable group Γ on a topological manifold where the fixed sets for any element are contained in codimension two submanifolds, every orbit equivalence is equivariant. Even in the presence of larger fixed sets, for actions preserving rigid geometric structures our results force sufficiently smooth orbit equivalences to be equivariant. For instance, if a countable group Γ acts on $\mathbb{T}^n$ and the action is C^1 orbit equivalent to the standard action of $SL_n(\mathbb{Z})$ on $\mathbb{T}^n$, then Γ is isomorphic to $SL_n(\mathbb{Z})$ and the actions are isomorphic. (The same result holds if we replace $SL_n(\mathbb{Z})$ by a finite index subgroup.) We also show that preserving a geometric structure is an invariant of smooth orbit equivalence and give an application of our ideas to the theory of hyperbolic groups.

In the course of proving our theorems, we generalize a theorem of Sierpinski which says that a connected Hausdorff compact topological space is not the disjoint union of countably many closed sets. We prove a stronger statement that allows "small" intersections provided the space is locally connected. This implies that for any continuous action of a countable group Γ on a connected, locally connected, locally compact, Hausdorff topological space, where the fixed set of every element is "small", every orbit equivalence is equivariant.

INTRODUCTION

Many interesting properties of a dynamical system are properties of the orbits: minimality, periodic points, etc. In this paper we study the inverse problem; to what extent does the orbit structure determine the action?

In the measurable category, the topic of orbit equivalence of group actions has been studied in great detail. For amenable groups, measurable orbit equivalence of ergodic measure preserving actions is shown to be a trivial relation in [CFW]. For non-amenable groups, many interesting rigidity theorems have been proven, starting with Zimmer, and continuing with work of Adams, Furman, Gaboriau, and Monod-Shalom [A, F1, Ga, MS, Z]. All of these results depend heavily on deep machinery, whether it is Zimmer's cocycle superrigidity and Ratner's measure classification theorem or the theories of L^2 or bounded cohomology. In this note, we prove analogues

and variants of results known in the measurable category in the topological and smooth categories. The most striking fact is that here we can prove analogous theorems by almost entirely elementary methods and without assuming an invariant measure or volume for the action.

Definition 0.1. *Let X and X' be topological spaces and Γ and Γ' groups. Two actions (X, Γ) and (X', Γ') are* topologically orbit equivalent *if there is a homeomorphism f from X to X' which maps the orbit relation for Γ to the orbit relation for Γ'. The map f is called a* topological orbit equivalence.

Since we only consider topological orbit equivalence in this paper, we will use the phrase orbit equivalent and orbit equivalence to mean topological orbit equivalent and topological orbit equivalence.

Definition 0.2. *Let a group Γ act on a topological space X. We say that the action is C^0 OE rigid if any orbit equivalence from (X', Γ') to (X, Γ) is equivariant.*

Furthermore, if X is a smooth manifold and Γ acts smoothly, we say the action is C^k OE rigid if any orbit equivalence which is a C^k diffeomorphism is equivariant.

Given a group Γ acting on a space X and $\gamma \in \Gamma$, we let $\text{Fix}(\gamma)$ be the set of γ fixed points.

Theorem 0.3. *Let Γ be a countable group acting on a connected manifold X. For every $\gamma \in \Gamma$ assume that $\text{Fix}(\gamma)$ is contained in a submanifold of codimension two. Then the Γ action on X is C^0 OE rigid.*

Both free actions and complex analytic actions trivially satisfy the hypotheses of the theorem. See section 3 for more examples. After proving our results, we discovered some prior results for $\mathbb{Z}$ actions; see [BT, Remark 3.4] and [K, GPS]. All of these results follow from Theorem 3.4 below. This theorem is proved using the generalization of Sierpinski's theorem mentioned in the abstract.

Even in cases where fixed sets are larger, for example codimension one submanifolds, in the presence of a rigid geometric structure we can show that any non-equivariant orbit equivalence is not (very) smooth. For example:

Theorem 0.4. *Let $\Gamma < SL_n(\mathbb{Z})$ be a subgroup of finite index. Then the standard Γ action on $\mathbb{T}^n$ is C^1 OE rigid.*

In this setting our techniques, combined with a few simple observations, can easily identify all self orbit equivalences of the action of Γ on $\mathbb{T}^n$.

In addition, we see that preserving a geometric structure is an invariant of sufficiently smooth orbit equivalences. See section 2 for definitions and discussion.

After essentially completing the work presented here, we discovered the large body of work on orbit equivalence of $\mathbb{Z}$ actions and the associated C^* algebras, see for example [BH, BT, GPS] as well as the references mentioned

above. It would be interesting to study the analogous questions regarding associated group transformation C^* algebras and orbit equivalences for groups larger than $\mathbb{Z}$.

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1. Basic Techniques

Given a space X and a group Γ acting on X, we define $\mathrm{Aut}(X,\Gamma)$ to be the group of homeomorphisms of X preserving the orbit equivalence relation, and $\mathrm{Inn}(X,\Gamma)$ to be the subgroup of homeomorphisms which send every x to a point in Γx. Clearly $\Gamma \subset \mathrm{Inn}(X,\Gamma)$. When X is a smooth manifold, $\mathrm{Aut}^k(X,\Gamma)$ and $\mathrm{Inn}^k(X,\Gamma)$ refer to the subgroups of C^k-diffeomorphisms in $\mathrm{Aut}(X,\Gamma)$ and $\mathrm{Inn}(X,\Gamma)$. Below we abuse notation by letting $\mathrm{Inn}^0(X,\Gamma) = \mathrm{Inn}(X,\Gamma)$ and $\mathrm{Aut}^0(X,\Gamma) = \mathrm{Aut}(X,\Gamma)$ whether X is smooth or not.

The prototypical result of the sort we are after is that $\Gamma = \mathrm{Inn}^k(X,\Gamma)$. When this holds, it implies that for (Y,Γ') orbit equivalent to (X,Γ), there is an isomorphism $\Gamma' \to \Gamma$ which makes the given orbit equivalence equivariant. In particular, it follows that $\mathrm{Aut}^k(X,\Gamma) = N_{\mathrm{Diff}^k(X)}(\Gamma)$, the normalizer of Γ in $\mathrm{Diff}^k(X)$. Actions for which this holds are determined by their orbit structure in the strongest possible sense.

We note that there is a short exact sequence:

$$1 \to Z_{\mathrm{Diff}^k(X)}(\Gamma) \to N_{\mathrm{Diff}^k(X)}(\Gamma) \to \mathrm{Aut}(\Gamma) \cap \mathrm{Diff}^k(X) \to 1$$

where $Z_{\mathrm{Diff}^k(X)}(\Gamma)$ denotes the centralizer in $\mathrm{Diff}^k(X)$ of Γ and $\mathrm{Aut}(\Gamma) \cap \mathrm{Diff}^k(X)$ denotes those automorphisms of Γ which can be realized as restrictions of inner automorphisms of $\mathrm{Diff}^k(X)$. It is frequently possible to identify both $Z_{\mathrm{Diff}^k(X)}(\Gamma)$ and $\mathrm{Aut}(\Gamma) \cap \mathrm{Diff}^k(X)$ explicitly and so obtain an exact description of $\mathrm{Aut}^k(X,\Gamma)$.

Given X and Γ, and an inner orbit equivalence f, define the sets

$$T_\gamma = \{x \,|\, f(x) = \gamma x\}$$

Clearly these sets are closed, and by definition they cover X. The intersection of T_γ and $T_{\gamma'}$ is contained in the fixed set of $\gamma^{-1}\gamma'$. On each T_γ, f agrees with translation by γ, so f is, in this sense "piecewise Γ". Conversely, anything which is piecewise in Γ is an inner orbit equivalence. The following example was shown to us by Scot Adams. A very similar example occurs as [BT, Example 4.3].

Example 1.1. *Let $X = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $\Gamma = SL_2(\mathbb{Z})$. Let f be the identity for $0 \leq x \leq \frac{1}{2}$, and let $f(x,y) = (x, y+2x)$ for $\frac{1}{2} \leq x \leq 1$. It is easy to check that this is a homeomorphism of the torus, and it is clearly an inner orbit equivalence.*

This example is piecewise $SL_2(\mathbb{Z})$ in a much nicer sense than the above. For a general X and Γ we define the piecewise Γ maps, $\mathrm{PInn}(X,\Gamma)$ as those inner orbit equivalences for which a finite collection of T_γ cover X. When the orbit structure is not rigid, one can still hope that $\mathrm{Inn}(X,\Gamma) = \mathrm{PInn}(X,\Gamma)$ as the later are somewhat more easily understood.

Let $X = S^1 = \mathbb{R} \cup \{\infty\}$, and let Γ be the group of rational affine maps. Let f be the map which is the identity outside of $(0,2)$, sends $\frac{1}{n}$ to $\frac{1}{n+1}$ for all $n \in \mathbb{N}$, and which is affine on the intervals $[\frac{1}{n+1}, \frac{1}{n}]$. This is a (Lipschitz) inner orbit equivalence which is not in $\mathrm{PInn}(X,\Gamma)$. This is approximately as bad as it gets for countable groups.

Lemma 1.2. *If X is a complete metric space and Γ is countable, then $\cup_\Gamma Int(T_\gamma)$ is an open dense subset of X.*

Proof. This is essentially the Baire category theorem. Let U be any non-empty open set in X. If U is disjoint from $\cup_\Gamma Int(T_\gamma)$ then $U = \cup_G \partial(T_\gamma)$, which is a countable union of closed, nowhere dense sets. This is impossible by the Baire category theorem, hence U intersects $\cup_\Gamma Int(T_\gamma)$, which proves density. Openness is obvious. $\square$

Thus, on an open dense set, every inner orbit equivalence behaves locally like an element of Γ. This is surprisingly powerful in many cases.

2. Geometric Structures

Many of the most natural examples of group actions preserve some extra structure. One unified treatment of these extra structures is via "geometric structures a la Gromov", which are a generalization of a reduction of the structure group of the frame bundle. Let D^k be the group of k-jets of diffeomorphisms of $\mathbb{R}^n$ preserving the origin, and $P^k(M)$ the kth order frame bundle of M, viewed as a principle D^k bundle.

Definition 2.1. *A* geometric structure *on a manifold M consists of the following data:*

(1) *a manifold V*
(2) *a D^k action on V*
(3) *a D^k equivariant map $\omega : P^k(M) \to V$.*

We frequently refer to ω as the geometric structure leaving V and k implicit. We say two geometric structures are of the same type *if the k's are the same and the V's are isomorphic as D^k spaces.*

Basic examples are familiar: metrics, volume forms, symplectic structures, affine connections, projective connections, etc. The reader unhappy with the above definition will lose little by considering only the above list. Note that we are not assuming that our geometric structures are A-structures in the sense of Gromov, as is necessary for most applications.

Theorem 2.2 (Orbit Equivalences Preserve geometric Structures). *Let M be a compact manifold with a C^k-action of a countable group Γ, preserving a geometric structure ω of order k. Then $\mathrm{Inn}^k(M,\Gamma) \subset \mathrm{Aut}^k(\omega)$.*

Proof. Let f be the orbit equivalence of M. Let ω' be the pullback of ω, in other words the composition of the map $f^k : P^k(M) \to P^k(M)$ with the map $P^k(M) \to V$ defining ω. By the key lemma, f agrees locally on an open dense set with elements of Γ. Since Γ preserves ω, this means that ω and ω' are equal on an open dense set. By continuity, they are equal. $\quad\square$

Corollary 2.3. *Let M be a compact manifold with Γ action preserving a kth-order geometric structure. Let (N,Γ') be any manifold with an action C^k orbit equivalent to (M,Γ). The Γ' action on N preserves a geometric structure of the same kind as ω.*

Proof. The rigid structure on N preserved by Γ' is the pullback of ω via the C^k orbit equivalence $f : N \to M$. Since, for $\gamma' \in \Gamma'$, the diffeomorphism $f \circ \gamma' \circ f^{-1}$ of M is in $\mathrm{Inn}^k(M,\Gamma)$, the result follows. $\quad\square$

This implies, for example, that one can recognize whether an action is volume preserving or symplectic just from the orbit structure. In some cases one can see how to do this, for example, one can try to build a measure by counting the fraction of periodic orbits in a set. If the finite orbits are equidistributed, this will recover the volume. Of course this does not always work. It is unclear how to even begin to build a symplectic form out of the orbits, although the Arnold conjectures give a way to recognize that certain orbit structures cannot arise from symplectic actions.

When the automorphism group of a geometric structure is smaller, for example for isometric or affine structures, Theorem 2.2 is very strong. One class of such structures are the ones Gromov calls rigid. For our purposes the most convenient formulation of this is:

Definition 2.4. *If ω is a geometric structure on M, ω is called* rigid *(of order r) if the action of $\mathrm{Aut}^r(\omega)$ is free and proper on $P^r(M)$.*

This notion of a rigid geometric structure is equivalent to the one defined in [Gr], as shown there on pages 69-70.

Theorem 2.5 (Actions Preserving Rigid Structures Are Rigid). *Let Γ be a countable group acting on a compact manifold M. Assume the action preserves a rigid geometric structure, ω, of order r. Then $\Gamma = \mathrm{Inn}^r(M,\Gamma)$.*

Proof. Since ω is rigid, $\mathrm{Aut}^r(\omega)$ is a Lie group which acts on M, freely and properly on $P^r(M)$. By theorem 2.2, any inner orbit equivalence is an element of $\mathrm{Aut}^r(\omega)$. Suppose $g \in \mathrm{Aut}^r(\omega)$ preserves Γ orbits. Then $M = \cup_\Gamma T_\gamma$, with $gm = \gamma m$ for all $m \in T_\gamma$. If T_γ has non-empty interior then there is some open set on which g and γ agree, in particular, they agree at some point of $P^r(M)$. Since the $\mathrm{Aut}^r(\omega)$ action is free on $P^r(M)$, this implies $\gamma = g$. $\quad\square$

Corollary 2.6. *Let Γ be a countable group acting on a compact manifold M. Assume the action preserves a rigid geometric structure, ω, of order r. Then $\mathrm{Aut}^r(M, \Gamma)$ consists only of equivariant maps, and therefore $\mathrm{Aut}^r(M, \Gamma) = N_{\mathrm{Diff}^r(X)}(\Gamma)$.*

Recall that $N_{\mathrm{Diff}^r(X)}(\Gamma)$ is an extension of $Z_{\mathrm{Diff}^r(X)}(\Gamma)$ by a subgroup of $\mathrm{Aut}(\Gamma)$. For many Γ, the group $\mathrm{Out}(\Gamma)$ is known to be finite, for example all irreducible lattices in semi-simple Lie groups not locally isomorphic to $SL_2(\mathbb{R})$, and all one ended hyperbolic groups. In addition, for sufficiently hyperbolic actions the centralizer can often be shown to be trivial. For such groups the corollary implies Γ is finite index in $\mathrm{Aut}^r(M, \Gamma)$. In specific cases one can often identify the group $\mathrm{Aut}^r(M, \Gamma)$ completely. For example, if we consider the torus $\mathbb{T}^n$ with it's natural affine structure, which is a rigid structure of order 1, Theorem 2.5 implies:

Theorem 2.7. *Let $\Gamma < SL_n(\mathbb{Z})$ be of finite index. For the standard action of Γ on $\mathbb{T}^n$, the group $\mathrm{Aut}^1(\mathbb{T}^n, \Gamma)$ is of finite index in $GL_n(\mathbb{Z})$.*

For $n \geq 3$ one can use Mostow rigidity to calculate $\mathrm{Aut}(SL_n(\mathbb{Z}))$, but by considering the action on $H^1(M, \mathbb{Z})$ it is easy to see that no automorphism not in $GL_n(\mathbb{Z})$ can be realized as a diffeomorphism of the torus. In particular, the theorem holds even for $n = 2$, despite the fact that $\mathrm{Aut}(SL_2(\mathbb{Z}))$ is quite large. Easy dynamical arguments using the presence of hyperbolic matrices in Γ show that the centralizer of the action is trivial.

Note that Theorem 0.4 of the introduction follows easily from this one.

One can state much more general results concerning affine actions of lattices on homogeneous spaces. Here one can use Ratner's theorem to identify centralizers and Mostow rigidity to identify automorphisms. If the action is sufficiently hyperbolic, the use of Ratner's theorem can be replaced with techniques deriving from [HPS]. In particular, most of the results of [F2] can be reproven in the C^1 (rather than measurable) category by these methods.

The differentiability hypotheses in Theorem 2.5 are necessary in general. The group of C^0 orbit equivalences of $SL_n(\mathbb{Z})$ action on the torus is large. See section 4 for some discussion of this case. This is not simply a case of continuous versus differentiable, as the higher order derivatives are also needed:

Example 2.8. *Let $X = S^1 = \mathbb{R} \cup \{\infty\}$ and $\Gamma = SL_2(\mathbb{Z}[\frac{1}{2}])$ where the action is given by restricting the Moebius action of $SL_2(\mathbb{R})$. Let f be the map which is the identity for $x < 0$, $f(x) = \frac{2x}{2-x}$ on $[0, 1]$, $6 - \frac{4}{x}$ on $[1, 2]$, and $x + 2$ for $x > 2$. This map is piecewise Γ and is C^1 but not C^2. Since the action of Γ preserves the projective structure, a rigid structure of order 2, one sees that the differentiability assumptions are needed. This example can be suspended to give similar examples on $\mathbb{R}P^n$ for all n.*

3. SMALL FIXED SETS

The last section explored geometric reasons for an orbit structure to be rigid. There are also topological/dynamical reasons. The prototype is the case of free actions. If Γ acts freely on M, then the decomposition $M = \cup_\Gamma T_\gamma$ is a decomposition into a countable disjoint union of closed sets. If M is connected, compact and Hausdorff this implies, by a theorem of Sierpinski [Si], that all but one of these sets is empty. Thus, for any free action the orbit structure is rigid. We generalize that result to cases where the fixed sets are small. It is interesting to note that in the context of the previous section, and of lattices in Lie groups, the elements with big fixed sets are unipotents. Thus, in this case, the presence of unipotents in a subgroup is an obstruction to rigidity.

Throughout this section, we let the topological space X be connected, locally connected, Hausdorff, and locally compact.

A subset A in X *locally disconnects* if there is a connected open set U such that $U \setminus A$ is not connected.

A subset Z of X will be called *small* if it is closed, has empty interior, and does not locally disconnect.

- Submanifolds of codimension 2 and higher are small.
- X has no cut points iff points are small.

Lemma 3.1. *A finite union of small sets in small.*

Proof. If A and B are small, then clearly $C = A \cup B$ is closed with empty interior. Suppose C disconnects a connected open set U, so that $U \setminus C = V \cup W$, disjoint open sets. If $B \subset A$ this contradicts A being small. Let b be a point of $B \setminus A$, and let N be a tiny connected open neighborhood of b which is disjoint from A. Since B is small, it cannot disconnect N, so $N \setminus B$ is contained in a single component of $U \setminus C$. Let $\hat{V}$ (resp. $\hat{W}$) be V (resp. W) union those points of $B \setminus A$ whose neighborhoods only intersect V (resp. W). These are disjoint open sets whose union is $U \setminus A$. This contradicts the fact that A is small. $\square$

Theorem 3.2. *Let X be connected, locally connected, Hausdorff and locally compact. Let Y_i be a countable collection of closed sets which cover X. If each intersection $Y_i \cap Y_j$ is small, then $Y_i = X$ for some i.*

Proof. Without loss of generality, we may assume that $Int(Y_i) \cap Y_j$ is empty whenever $i \neq j$ (just successively replace Y_i by $Y_i \setminus (Y_i \cap \cup_{j>i} Int(Y_j))$ for $i = 1, 2, \ldots$). Assume for contradiction that no $Y_i = X$.

Lemma 3.3. *There is a pre-compact open set U in X which is disjoint from Y_1 and has non-empty intersection with infinitely many Y_j.*

Proof. Fix $j > 1$. If $\partial Y_j \subset Y_1$ then $X \setminus (Y_1 \cup Y_j) = Int(Y_j) \cup (Y_j)^c$, which are disjoint open sets. As intersections are small, this implies that $Int(Y_j)$ is empty, and hence $Y_j = \partial Y_j \subset Y_1$. If this holds for all $j > 1$ then $X = Y_1$, contradiction.

Choose a $j > 1$ for which ∂Y_j is not contained in Y_1 and let x be a point in $\partial Y_j \setminus Y_1$. Let U be a pre-compact, connected neighborhood of x which misses Y_1. If $U \cap Y_i$ is non-empty for infinity many i then the proof is complete. Assume $U \subset Y_2 \cup Y_3 \cup \ldots Y_n$. Since $U \subset \text{Int}(Y_2 \cup Y_2 \cup \ldots Y_n)$, it is immediate that $U \subset \bigsqcup_{l=2}^n \text{Int}(Y_l) \bigsqcup \cup_{1 < i < j \leq n} (Y_i \cap Y_j)$. It follows that

$$U \setminus (\cup_{1 < i < k \leq n}(Y_i \cap Y_k)) \subset \bigsqcup_{1 < i \leq n} (\text{Int}(Y_i))$$

By Lemma 3.1 the left hand side is connected, hence contained in $\text{Int}(Y_k)$ for some k. This implies $U \subset Y_k$, which implies that $x \in \text{Int}(Y_k)$ which is disjoint from Y_j and contradicts the choice of x.

$\square$

Let U_1 be the set produced by the lemma. Apply the lemma to U_1 and the cover $Y_2 \cup U_1, Y_3 \cup U_1, \ldots$ to get U_2 with $\bar{U}_2 \subset U_1$ and U_2 disjoint from Y_2. Continue inductively to get a sequence U_i of pre-compact open sets with $\bar{U}_i \subset U_{i-1}$ and U_i disjoint from Y_j for $j \leq i$. The intersection of the U_i is equal to the intersection of the $\bar{U}_i$ and hence is non-empty. On the other hand, this intersection is disjoint for all the Y_i, which is a contradiction. $\square$

Theorem 3.4 (Small Fixed Sets Implies Rigid). *Let Γ be a countable group acting on X, a connected, locally connected, locally compact Hausdorff topological space. If $\text{Fix}(\gamma)$ is small for all $\text{id} \neq \gamma \in \Gamma$ then $Inn(X, \Gamma) = \Gamma$, and every orbit equivalence is equivariant.*

Proof. Let $f \in \text{Inn}(\Gamma)$. Recall that $T_\gamma = \{x \in X | f(x) = \gamma x\}$. For distinct γ and γ' the intersection $T_\gamma \cap T_{\gamma'} \subset \text{Fix}(\gamma^{-1}\gamma')$ which is small by assumption. By Theorem 3.2 there is $\gamma \in \Gamma$ such that $T_\gamma = X$. $\square$

These results have many applications. For example:

Corollary 3.5. *Let $\Gamma \subset SL_n(\mathbb{Z})$ consist entirely of matrices diagonalizable over $\mathbb{C}$. Then the orbit structure of Γ acting on $\mathbb{T}^n$ is rigid.*

As a simple consequence of Corollary 3.5, for any standard hyperbolic action of $\mathbb{Z}^{n-1}$ on $\mathbb{T}^n$ we have $\text{Aut}(\mathbb{T}^n, \mathbb{Z}^{n-1}) = S_n \ltimes \mathbb{Z}^{n-1}$ where S_n is the symmetric group acting on $\mathbb{T}^n$ by permuting coordinates.

Corollary 3.6. *Let $\Gamma \subset SL_n(\mathbb{R})$ be any countable group. Then the orbit structure of Γ acting on S^{n-1} is rigid.*

Interestingly, this result is false if one allows orientation reversing elements, see example 2.8. That problem does not occur over $\mathbb{C}$.

Corollary 3.7. *Let $\Gamma \subset PGL_n(\mathbb{C})$ be any countable group. Then the orbit structure of Γ acting on $\mathbb{C}P^{n-1}$ is rigid.*

There are also applications outside actions on manifolds. For example:

Corollary 3.8. *Let Γ be a word hyperbolic group which does not split over any virtually cyclic group. The orbit structure of Γ on $\partial\Gamma$ is rigid. Furthermore if Γ and Γ' are hyperbolic groups such that the boundary actions are orbit equivalent then Γ and Γ' are commensurable.*

Proof. Since the group does not split over a finite subgroup, the boundary is connected by Stallings' Ends theorem [S]. Likewise, since the group does not split over any virtually infinite cyclic group, the boundary has no cut points, by a theorem of Bowditch [B]. Since the fixed set of any element of Γ acting on the boundary is a pair of points, this implies the fixed sets do not locally disconnect. The boundary is locally connected by [BM]. Thus the hypotheses of theorem 3.4 are satisfied, so $\mathrm{Inn}(\partial\Gamma, \gamma) = \Gamma$ and every orbit equivalence is equivariant. By [P], $\mathrm{Out}(\Gamma)$ is finite, so Γ is finite index in $\mathrm{Aut}(\partial\Gamma, \Gamma)$. $\square$

It would be interesting to know if similar results hold for $CAT(0)$ groups. The main obstacle seems to be determining when the boundary of a flat can locally disconnect the boundary of the group. This might imply something like a splitting over an abelian subgroup. However the situation is already unclear for cutpoints, see e.g [Sw].

4. Some Interesting Groups

At the boundary of rigidity, the groups of orbit equivalences can be very interesting in their own right. For example, Thompson's group can be conjugated to a piecewise $PSL_2(\mathbb{Z})$ action on the circle ([Gh]). An interesting family of groups arise in the case of $SL_n(\mathbb{Z})$ acting on the n-torus. Let Γ_n be the group of homeomorphisms of $\mathbb{T}^n$ for which there is a decomposition of $\mathbb{T}^n$ into finitely many pieces on which the homeomorphism agrees with an element of $Gl_n(\mathbb{Z})$.

Lemma 4.1. *Γ_n is the full group of Lipschitz orbit equivalences of the $SL_n(\mathbb{Z})$ action on $\mathbb{T}^n$.*

Proof. If f is any Lipschitz orbit equivalence, then by the Lemma 1.2, $\mathbb{T}^n = \cup_{SL_n(\mathbb{Z})} T_\gamma$, with the union of the interiors dense. Since f is Lipschitz, there are only finitely many elements of $SL_n(\mathbb{Z})$ that can agree with f on an open dense set. Thus there are $\gamma_1, \ldots, \gamma_n$ in $SL_n(\mathbb{Z})$ such that the union T_{γ_i} is $\mathbb{T}^n$. Since the intersections of two of these sets is contained in the fixed set of some $\gamma_i \gamma_j{}^{-1}$, there is a finite union of sub-tori such that f restricted to any complementary component is in $SL_n(\mathbb{Z})$. $\square$

This group is definitely bigger than $GL_n(\mathbb{Z})$, see example 1.1.

Unlike Thompson's group, these groups are not simple. There is a (split) surjection from Γ_n to $GL_n(\mathbb{Z})$ given by the action on $H^1(\mathbb{T}^n)$. Explicitly, if f is equal to A_i on the set X_i, then the image of f in $GL_n(\mathbb{Z})$ is $\Sigma A_i vol(X_i)$. Further, by considering the germ of f at 0 one gets a homomorphism to

piecewise $Sl_n(\mathbb{Z})$ acting on the $n - 1$ sphere (by the double cover of the standard projective action).

Lemma 4.2. *Any subgroup of Γ_n is residually finite.*

This follows from the interpretation as Lipschitz orbit equivalences. There are maps to finite permutation groups given by restriction to any finite orbit. Since finite orbits are dense, the only thing in the kernel of all such maps is the identity.

Question 4.3. *Are the groups Γ_n finitely generated?*

Question 4.4. *Is Γ_n the full group of orbit equivalences of $\mathbb{T}^n$ with the standard $SL_n(\mathbb{Z})$ action?*

If the answer to Question 4.3 is no, one might try to find interesting finitely generated subgroups of Γ_n.

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