From Logic to Geometry

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Logic is the beginning of wisdom not the end

Main Goal Use tools from mathematical logic to better understand classical mathematical structures.

Exploit the interplay of semantics and syntax

Semantics = truth in mathematics structures
Syntax = formal expressions in symbolic first order logic

Mathematical Structures

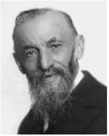
Consider the following structures with the algebraic operations of addition $+$ and multiplication $\cdot$ and distinguished elements 0 and 1

- The *natural numbers* $\mathbb{N}$: 0,1,2,. . . ;
- The *integers* $\mathbb{Z}$: . . . ,-2,-1,0,1,2,. . . ;
- The *rational numbers* $\mathbb{Q}$, integers and quotient of integers $1, -\frac{3}{5}, \frac{22}{7}, \ldots,$
- The real numbers numbers $\mathbb{R}$: all numbers with decimal expansions, $\sqrt{2}$, π, e,. . .
- The complex numbers $\mathbb{C}$: all numbers $a + bi$ where a, b are real and $i^2 = -1$.

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Denote the structures $(\mathbb{N}, +, \cdot, 0, 1)$, $(\mathbb{Z}, +, \cdot, 0, 1), \ldots$

Symbolic Logic

Peano 

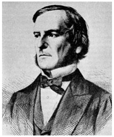
We build up simple formulas using:

- the symbols, $+$, $\cdot$ and $=$
- parenthesis (and)
- constant symbols 0, 1
- variables $x, y, z, x_1, x_2, \ldots$

For example

- $0 + 1 = 1$
- $(1 + 1) \cdot (1 + 1 + 1) = (1 + 1 + 1 + 1 + 1 + 1)$ $\qquad 2 \cdot 3 = 6$
- $y \cdot y = x$ $\qquad y^2 = x$
- $x \cdot x + y \cdot y = 1$ $\qquad x^2 + y^2 = 1$

Symbolic Logic II

Boole 

We build up more complicated formulas using Boolean connectives

- $\wedge$ "and"
- $\vee$ "or"
- $\neg$ "not"
- $\rightarrow$ "implies"

- $x + y = z \wedge x \cdot x + (1 + 1) \cdot y = 0$
- $x = y \rightarrow x + z = y + z$
- $\neg(x \cdot y = 0) \rightarrow \neg(x = 0)$ If $xy \neq 0$, then $x \neq 0$

Symbolic Logic III

Frege 

Quantiifers

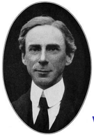
- $\exists$ "there exists"
- $\forall$ "forall"

For example

- $\exists x \; x \cdot x + x + 1 = 0$
- $\exists y \; y \cdot y = x$ — x is a square
- $\forall x \exists y \; y \cdot y = x$ — every element is a square
- $\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (|x - a| < \delta \rightarrow |f(x) - b| < \epsilon)$

$$\lim_{x \rightarrow a} f(x) = b$$

Symbolic Logic IV

Russell  Whitehead 

An important technical point: A *sentence* is a formula where all of the variables are bound in the scope of a quantifier.

Sentences:

$$\forall x \exists y \; y^2 = x$$
$$\exists x \; x^2 = 1 + 1$$

Non Sentences:

$$\exists y \; y^2 = x$$
$$\exists x \; x^2 + y \cdot x + z = 0$$

Theories

Malcev 

Sentences are declarative statements. In any particular structure they are either true or false.

- $\exists x \forall y\ x \cdot y = y$
  - True in $\mathbb{N}$, $\mathbb{Z}$,$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ (take $x = 1$).
- $\forall x \exists y\ x \cdot y = 1$
  - False in $\mathbb{N}$, $\mathbb{Z}$ (take $x = 2$)
  - True in $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.
- $\forall x \exists y\ y^2 = x$
  - False in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ (no $\sqrt{2}$)
  - False in $\mathbb{R}$ (no $\sqrt{-1}$)
  - True in $\mathbb{C}$

The *Theory* of a structure $\mathcal{M}$ is the set of all sentences true in $\mathcal{M}$ and denoted $\mathrm{Th}(\mathcal{M})$.

Definable Sets

A. Robinson 

Formulas with free variable assert a property of the free variables.
$\exists y \; y^2 = x$ asserts "x is a square"

- in $\mathbb{Z}$ or $\mathbb{Q}$ it is true for $x = 9$, but false for $x = 3$
- in $\mathbb{R}$ it is true of any $x \geq 0$ but false for $x = -3$
- in $\mathbb{C}$ it is true for every x.

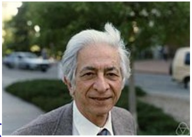
Suppose $\phi(x_1, \ldots, x_n)$ is a formula with free variables $x_1, \ldots, x_n$ and $\mathcal{M}$ is a structure. We say that

$$\{(a_1, \ldots, a_n) : \phi \text{ holds in } \mathcal{M} \text{ of } a_1, \ldots, a_n\}$$

is *definable*.

We also allow parameters.

Examples of Definable Sets in $\mathbb{R}^2$ Seidenberg



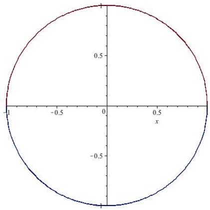
Some definable sets in $\mathbb{R}$.

- $\{(x, y) : x < y\}$ is defined by

$$\exists z \; (z \neq 0 \wedge x + z^2 = y)$$

- the unit circle is defined by

$$x^2 + y^2 = 1$$



Our Main Goals Restated

Let $\mathcal{M}$ be one of our classical mathematical structures.

- Try to understand $\text{Th}(\mathcal{M})$, the complete theory of $\mathcal{M}$.
- Try to understand the definable subsets of $\mathcal{M}^n$.

Hilbert's Program

Hilbert 

Understand $\mathrm{Th}(\mathbb{N})$.

- (Axiomatization Problem) Can we give a simple set of axioms T true about $\mathbb{N}$ such that all true statements can be derived from T by simple logical rules?
- (Decidability Problem) Is there an algorithm which when given a sentence ϕ as input will decide if ϕ is true in $\mathbb{N}$?

Good candidate for axiomatization: *Peano Axioms*

- Basic properties of $+$ and $\cdot$ like $\forall x \forall y \; x(y+1) = xy + x$
- Induction axioms

$$[\phi(0) \wedge \forall x \; (\phi(x) \rightarrow \phi(x+1))] \rightarrow \forall x \; \phi(x)$$

Gödel's Incompleteness Theorem

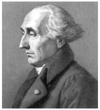
Gödel 

In 1931 Kurt Gödel left Hilbert's Program in ruins.

Theorem (Gödel)

i) There are true sentences about the natural numbers that can not be derived from the Peano axioms.
ii) The same is true for any other possible simple set of axioms
iii) There is no algorithm which when input a sentence ϕ will halt and tell you if ϕ is true in $\mathbb{N}$.

▸ Sketch of Proof

$\mathbb{Z}$ and $\mathbb{Q}$? Lagrange  J. Robinson  Poonen 

Both $\mathrm{Th}(\mathbb{Z})$ and $\mathrm{Th}(\mathbb{Q})$ are undecidable.

- (Lagrange) $\mathbb{N}$ is definable in $\mathbb{Z}$ as
$$\{x : \exists y_1 \exists y_2 \exists y_3 \exists y_4 \; x = y_1^2 + y_2^2 + y_3^2 + y_4^2\}$$
- (J. Robinson 1959) $\mathbb{Z}$ is definable in $\mathbb{Q}$ by a $\forall\exists\forall\exists$ formula
- (Poonen 2010) $\mathbb{Z}$ is definable in $\mathbb{Q}$ by a $\forall\exists$ formula.

- (Park 2012) $\mathbb{Z}$ is definable in $\mathbb{Q}$ by a $\forall$-formula

Park

Hilbert's Tenth Problem MRD  Putnam 

Theorem (Matiyasevich-J. Robinson-Davis-Putnam 1949-70)

There is no algorithm which when given as input a polynomial $f(X_1, \ldots, X_n)$ with coefficients in $\mathbb{Z}$ will always halt and correctly answer whether there is $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ with $f(a_1, \ldots, a_n) = 0$.

Solving Diophantine equations is as hard as deciding if a computer program halts.

Open Question: Is the same true for $\mathbb{Q}$.

Key Lesson: *Quantifiers lead to complexity.*

Model Theory of the Real Field

Tarski 

Theorem (Tarski 193?)

$\mathrm{Th}(\mathbb{R})$ and $\mathrm{Th}(\mathbb{C})$ are decidable.

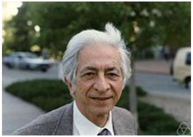
The ordering $x < y$ is definable in $\mathbb{R}$ by $\exists z \neq 0\ x + z^2 = y$.

Theorem (Tarski)

There is an algorithm that transforms any formula ϕ to an equivalent to a quantifier free formula ψ using $<$.

Familiar example: (Quadratic Formula) $\exists x\ x^2 + yx + z = 0$ is equivalent to $y^2 - 4z \geq 0$.

Semialgebraic Sets

Tarski  Seidenberg 

Definition

A subset of $\mathbb{R}^n$ is *semialgerbraic* if it is built up using $\neg, \wedge, \vee$ from sets $\{x \in \mathbb{R}^n : p(x) = 0\}$ and $\{x \in \mathbb{R}^n : q(x) > 0\}$, p and q real polynomials.

Corollary

Definable $=$ *Semialgebraic*

Corollary

The closure of a semialgebraic set is semi algebraic.

$$x \in \mathrm{cl}(A) \Leftrightarrow \forall \epsilon > 0 \exists y \in A \sum (x_i - y_i)^2 < \epsilon$$

Tarski's Problem

Tarski  Macintyre 

Open Problem Suppose we consider the structure $\mathbb{R}_{\exp} = (\mathbb{R}, +, \cdot, \exp)$, where $\exp(x) = e^x$. Is $\mathrm{Th}(\mathbb{R}_{\exp})$ decidable?

A positive answer would show the decidability of hyperbolic geometry. Even deciding equality of terms is difficult. Is

$$e^e = 9e^3 - 6e^2 - 121? \text{ Probably not}$$

Theorem (Macintrye)

Assuming Schanuel's Conjecture there is an algorithm to decide if two terms are equal.

A New Paradigm Macintyre  van den Dries 

Decidability is the wrong problem.

Even the theories we know are decidable are provably intractable.

Our goal should be understanding definable sets.

Simple Consequence of Quantifier Elimination: In $(\mathbb{R}, +, \cdot, 0, 1)$ any definable subset of $\mathbb{R}$ is a finite union of points and intervals.

Definition

We call an structure $(\mathbb{R}, +, \cdot, 0, 1, \ldots)$ *o-minimal* if any definable subset of $\mathbb{R}$ is a finite union of points and intervals.

Remarkable Fact: O-minimality captures many of the good geometric and topological properties of semialgebraic sets.

Digression

CHICAGO SUN-TIMES

July 25, 2012

U of I prof relents, will take ethics training developed by ‘unwise rulers to annoy us’

A University of Illinois math professor who derided states ethics training as childish, petty tyranny and Orwellian ended his four-year boycott by agreeing to pay a fine and submit to the training, a state ethics panel disclosed Monday.

o-minimality

van den Dries 

Definition

We call an structure $(\mathbb{R}, +, \cdot, 0, 1, \dots)$ *o-minimal* if any definable subset of $\mathbb{R}$ is a finite union of points and intervals.

Remarkably o-minimality has remarkable consequence for definable functions and definable subsets of $\mathbb{R}^n$.

Theorem

If $f : \mathbb{R}^n \to \mathbb{R}$ is definable, then we can partition the domain of f into $X_1 \cup \dots \cup X_n$ such that f is continuous (or $\mathcal{C}^m$ on each X_i).

Cell Decomposition Knight  Pillay  Steinhorn 

- An point in $\mathbb{R}$ is a 0-cell
- An interval in $\mathbb{R}$ is a 1-cell
- If $A \subseteq \mathbb{R}^n$ is a k-cell and $f : A \to \mathbb{R}$ is a continuous definable function then

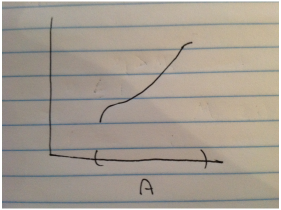
$$\mathrm{graph}(f) = \{(x, y) \in \mathbb{R}^{n+1} : x \in A \wedge y = f(x)\} \text{ is a } k\text{-cell.}$$

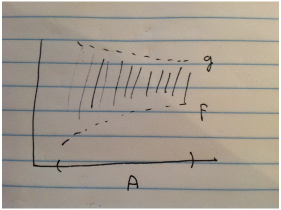
- If $A \subseteq \mathbb{R}^{n+1}$ is a k-cell and $f, g : A \to \mathbb{R}$ are continuous and definable such that $f(x) < g(x)$ for all $x \in A$ then

$$\{(x, y) : f(x) < y < g(x), x \in A\} \text{ is a } k + 1\text{-cell}$$

Cell Decomposition Knight  Pillay  Steinhorn 

Cells in $\mathbb{R}^2$





Theorem (Cell Decomposition–Knight-Pillay-Steinhorn)

If $X \subseteq \mathbb{R}^n$ is definable, then X can be partitioned into finitely many disjoint cells, $X = C_1 \cup \cdots \cup C_m$.
In particular, X has finitely many connected components.

The New Question

Is $\mathbb{R}_{\exp}$ o-minimal?

Note: $\mathbb{R}_{\mathsf{sin}}$ is not o-minimal since

$$\{x : \sin x = 0\} = \{2\pi n : n \in \mathbb{Z}\}.$$

$\mathbb{R}_{\exp}$

Wilkie  Macintyre 

Theorem (Wilkie)

i) If $X \subseteq \mathbb{R}^n$ is definable in $\mathbb{R}_{\exp}$, then there is $V \subseteq \mathbb{R}^{n+m}$ the zero set of a finite set of exponential polynomials such that
$X = \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^m \ (x, y) \in V\}$.
ii) $\mathbb{R}_{\exp}$ is o-minimal.

Theorem (Macintyre-Wilkie)

Assuming Schanuel's Conjecture $\mathrm{Th}(\mathbb{R}_{\exp})$ is decidable.

$\mathbb{R}_{\mathrm{an,exp}}$ van den Dries  Macintyre  Marker 

$\mathbb{R}_{\mathrm{an,exp}}$ Add to $\mathbb{R}_{\mathrm{exp}}$ all analytic functions on compact balls.

Theorem (van-den-Dries, Macintyre, Marker)

i) $\mathbb{R}_{\mathrm{an,exp}}$ *has quantifier elimination with* ln.
ii) $\mathbb{R}_{\mathrm{an,exp}}$ *is o-minimal*

Recently this result has been used in work of J. Pila in number theory.

$\mathbb{R}_{\mathrm{an,exp}}$ van den Dries  Macintyre  Marker 

We were able to use out understanding of $\mathrm{Th}(\mathbb{R}_{\mathrm{an,exp}})$ to construct useful nonstandard models.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is definable in $\mathbb{R}_{\mathrm{exp}}$ then $f(x)$ is eventually less than one of the functions $e^x, e^{e^x}, e^{e^{e^x}}, \ldots$

Hardy's Problem

Hardy 

An LE-function is a composition of exp, ln and algebraic functions.

For functions f arising in many natural mathematical contexts, we can usually find $g \in$ LE such that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

Question: Consider the function f that is the inverse to $(\ln x)(\ln\ln x)$ [i.e., $x = (\ln(f(x)))(\ln\ln(f(x)))$].

Theorem

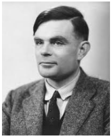
There is no LE-function asymptotic to f.

This is the tip of the iceberg.

There are many more o-minimal expansions of $\mathbb{R}$.

Thank you!

What Can't a Computer Do?

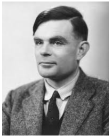


The Halting Problem: Given a computer program P and an input x decide if P halts on input x.

Theorem (Turing)

No computer program can solve the Halting Problem.

What Can't a Computer Do? Proof Sketch



Theorem (Turing)

No computer program can solve the Halting Problem.

Proof.

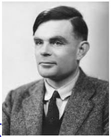
For purposes of contradiction, suppose there is such a program. Write a new program Q that does the following:

- On input P decide if P is a program and if it is decide if P halts on input P.
- If P halts on input P, go into an infinite loop.
- If P does not halt on input P, halt and output 1.

Does Q halt on input Q?

Q halts on input $Q \Leftrightarrow Q$ does not halt on input Q. Contradiction!! □

Proof of Gödel's Theorem

Turing  Gödel 

For each program P and each possible input x we can find a sentence $\phi_{P,x}$ such that P halts on input x if and only if $\phi_{P,x}$ is true in $\mathbb{N}$.

If there was an algorithm to decide what sentences are true in $\mathbb{N}$, then there is an algorithm to answer the halting problem.

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