Axioms for Ordered Fields

Basic Properties of Equality

- $x = x$
- if $x = y$, then $y = x$
- if $x = y$ and $y = z$, then $x = z$
- for any function $f(x_1, \ldots, x_n)$, if $x_1 = y_1, \ldots, x_n = y_n$ then $f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$.

Similarly, if a predicate is true of $x_1, \ldots, x_n$, and $x_1 = y_1, \ldots, x_n = y_n$, then the predicate is also true of $y_1, \ldots, y_n$.

Axioms about Addition and Multiplication Axioms

i) (Commutativity) $a + b = b + a$ and $ab = ba$;

ii) (Associativity) $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$;

iii) (Distributivity) $a(b + c) = ab + ac$;

iv) (Zero Rule) $a + 0 = a = 0 + a$;

v) (Unity) $a \cdot 1 = a$;

vi) (Subtraction) The equation $a + x = 0$ has a unique solution $x = -a$. Similarly, the equation $a + x = b$ has a unique solution $b - a$.

vii) (Division) If $a \neq 0$, then the equation $ax = b$ has a unique solution $x = b/a = ba^{-1}$.

Order Axioms

viii) (Trichotemy) Either $a = b$, $a < b$ or $b < a$;

ix) (Addition Law) $a < b$ if and only if $a + c < b + c$;

x) (Multiplication Law) If $c > 0$, then $ac < bc$ if and only if $a < b$. If $c < 0$, then $ac < bc$ if and only if $b < a$;

xi) (Transitivity) If $a < b$ and $b < c$, then $a < c$.

Axioms i)–xi) are true in the real numbers $\mathbb{R}$ and the rational numbers $\mathbb{Q}$.

Axioms i)–vi) and viii)–x) are true in the integers $\mathbb{Z}$.