Math 215: Introduction to Advanced Mathematics
Problem Set 12

Due: Friday December 8

1) Suppose I is a countable set and that for each $i \in I$ we have a countable set A_i. Let $f_i : \mathbb{N} \to A_i$ be a surjection. Let

$$A = \bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}.$$

Let $F : I \times \mathbb{N} \to A$ be the function $F(i, n) = f_i(n)$.

a) Prove that F is a surjection.

b) Prove that A is countable.

This excersise shows that a *countable union of countable sets is countable.*

2) a) Prove that the interval $(0, 1)$ is equipotnet with the interval (a, b). [Note: the interval $(c, d) = \{x \in \mathbb{R} : c < x < d\}$.]

b) Prove that the interval $(0, 1)$ is equipotent with the interval $(0, +\infty)$.

c) Prove that the interval $(0, +\infty)$ is equipotent with $\mathbb{R}$. Conclude that $(0, 1)$ is equipotent with $\mathbb{R}$.

3) If A is any set we define $A^2 = A \times A$ and

$$A^n = \underbrace{A \times \ldots \times A}_{n-\text{times}}.$$

We also think of $A^n = \{(a_1, \ldots, a_n) : n \in A\}$. Let $S(A) = \bigcup_{n \in \mathbb{N}} A^n$. Then $S(A)$ is the set of all finite sequences from A.

a) Prove that if A is countable, then A^n is countable for all n. [Hint: This should be an easy induction.]

b) Prove that if A is countable, then $S(A)$ is countable.

c) Let $\mathbb{Q}[X]$ be the set of all polynomials with rational coefficients. Prove that $\mathbb{Q}[X]$ is countable. [Hint: Show there is a bijection to $S(\mathbb{Q})$.

d) We say that $\alpha \in \mathbb{R}$ is an *algebraic number* if there is a nonzero polynomial $p(X) \in \mathbb{Q}[X]$ such that $p(\alpha) = 0$. For example $\sqrt{2}$ is algebraic since if we take $p(X) = X^2 - 2$, then $p(\alpha) = 0$. If α is not an algebraic number we say that α is *transcendental*.

Prove that the set of algebraic numbers is countable. [Hint: For $p \in \mathbb{Q}[X]$ nonzero, let $A_p = \{x \in \mathbb{R} : p(x) = 0\}$. For example if $p(X) = X^2 - 2$, then $A_p = \{\sqrt{2}, -\sqrt{2}\}$. You may use the fact that each A_p is finite–in fact $|A_p|$ is at most the degree of p. Note that the set of algebraic number is $\bigcup_{p \in \mathbb{Q}[X]-\{0\}} A_p$.]

e) Prove that there is a transcendental real number.

4) (Bonus 5pt) [Cantor-Schröder-Bernstein Theorem] Suppose $f : X \to Y$ and $g : Y \to X$ are injections. Prove that there is a bijection $h : X \to Y$. [Hint: Let $X_0 = X$, $Y_0 = Y$, $X_{n+1} = \overrightarrow{g}(Y_n)$ and $Y_{n+1} = \overrightarrow{f}(X_n)$. Let $X_\infty = \bigcap_{n=0}^\infty X_n$ and $Y_\infty = \bigcap_{n=0}^\infty Y_n$. Let

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_\infty \cup \bigcup_{n=1\infty} A_{2n} - A_{2n+1} \\ g^{-1}(x) & \text{otherwise} \end{cases}.$$

Prove that h is a bijection.]