4 Complete Theories

Definition 4.1 A satisfiable theory T is *complete* if $T \models \phi$ or $T \models \neg\phi$ for all $\mathcal{L}$-sentences ϕ.

It is easy to see that T is complete if and only if $\mathcal{M} \equiv \mathcal{N}$ for any $\mathcal{M}, \mathcal{N} \models T$. If $\mathcal{M}$ is an $\mathcal{L}$-structure, then $\mathrm{Th}(\mathcal{M})$ is a complete theory, but it may be difficult to figure out if $\phi \in \mathrm{Th}(\mathcal{M})$.

When we are trying to understand $\mathrm{Th}(\mathcal{M})$ for a particular structure $\mathcal{M}$ we will often do this by looking for easy to understand theory T such that $\mathcal{M} \models T$ and T is complete. If $T \models \phi$, then $\mathcal{M} \models \phi$.. On the other hand, if $T \not\models \phi$, then, since T is complete, $T \models \neg\phi$ and, as before, $\mathcal{M} \models \neg\phi$ so $\mathcal{M} \not\models \phi$. Thus we would have

$$\mathcal{M} \models \phi \Leftrightarrow T \models \phi$$

In this section, will give one useful test to decide if a theory is complete.

Categoricity

We know from the Löwenheim-Skolem Theorem (Theorem 2.9) that if a theory has an infinite model it has arbitrarily large models. Thus the theory of an infinite structure can not capture the structure up to isomorphism. Sometimes though knowing the theory and the cardinality determines the structure.

Definition 4.2 T is κ-categorical if and only if any two models of T of cardinality κ are isomorphic.

• Let $\mathcal{L}$ be the empty language. Then the theory of an infinite set is κ-categorical for all cardinals κ.

• Let $\mathcal{L} = \{E\}$, where E is a binary relation, and let T be the theory of an equivalence relation with exactly two classes, both of which are infinite. It is easy to see that any two countable models of T are isomorphic. On the other hand, T is not κ-categorical for $\kappa > \aleph_0$. To see this, let $\mathcal{M}_0$ be a model where both classes have cardinality κ, and let $\mathcal{M}_1$ be a model where one class has cardinality κ and the other has cardinality $\aleph_0$. Clearly, $\mathcal{M}_0$ and $\mathcal{M}_1$ are not isomorphic.

Let $\mathcal{L} = \{+, 0\}$ be the language of additive groups and let T be the $\mathcal{L}$-theory of nontrivial torsion-free divisible Abelian groups. The axioms of T are the axioms for Abelian groups together with the axioms

$$\exists x \; x \neq 0,$$

$$\forall x (x \neq 0 \rightarrow \underbrace{x + \ldots + x}_{n-\text{times}} \neq 0)$$

and

$$\forall y \exists x \; \underbrace{x + \ldots + x}_{n-\text{times}} = y$$

for $n = 1, 2, \ldots$.

Proposition 4.3 *The theory of torsion-free divisible Abelian groups is κ-categorical for all $\kappa > \aleph_0$.*

Proof We first argue that models of T are essentially vector spaces over the field of rational numbers $\mathbb{Q}$. Clearly, if V is any vector space over $\mathbb{Q}$, then the underlying additive group of V is a model of T. On the other hand, if $G \models T$, $g \in G$, and $n \in \mathbb{N}$ with $n > 0$, we can find $h \in G$ such that $nh = g$. If $nk = g$, then $n(h - k) = 0$. Because G is torsion-free there is a unique $h \in G$ such that $nh = g$. We call this element g/n. We can view G as a $\mathbb{Q}$-vector space under the action $\frac{m}{n} g = m(g/n)$.

Two $\mathbb{Q}$-vector spaces are isomorphic if and only if they have the same dimension. Thus, models of T are determined up to isomorphism by their dimension. If G has dimension λ, then $|G| = \lambda + \aleph_0$. If κ is uncountable and G has cardinality κ, then G has dimension κ. Thus, for $\kappa > \aleph_0$ any two models of T of cardinality κ are isomorphic.

Note that T is not $\aleph_0$-categorical. Indeed, there are $\aleph_0$ nonisomorphic models corresponding to vector spaces of dimension $1, 2, 3, \ldots$ and $\aleph_0$.

A similar argument applies to the theory of algebraically closed fields. Let ACF_p be the theory of algebraically closed fields of characteristic p, where p is either 0 or a prime number.

Proposition 4.4 ACF_p *is κ-categorical for all uncountable cardinals κ.*

Proof Two algebraically closed fields are isomorphic if and only if they have the same characteristic and transcendence degree (see, for example Lang's *Algebra* X §1). An algebraically closed field of transcendence degree λ has cardinality $\lambda + \aleph_0$. If $\kappa > \aleph_0$, an algebraically closed field of cardinality κ also has transcendence degree κ. Thus, any two algebraically closed fields of the same characteristic and same uncountable cardinality are isomorphic.

Vaught's Test

Categoricity give a very simple test for completeness.

Theorem 4.5 (Vaught's Test) *Suppose every model of T is infinite, $\kappa \geq \max(|\mathcal{L}|, \aleph_0)$ and T is κ-categorical. Then T is complete.*

Proof Suppose not. Let ϕ be an $\mathcal{L}$-sentence such that $T \not\models \phi$ and $T \not\models \neg\phi$. Let $T_0 = T \cup \{\phi\}$ and $T_1 = T \cup \{\neg\phi\}$. Each T_i has a model, thus since T has only infinite models, each T_i has an infinite model. By the Löwenheim-Skolem theorem there is $\mathcal{A}_i \models T_i$ where $\mathcal{A}_i$ has cardinality κ. Since T is κ-categorical, $\mathcal{A}_0 \cong \mathcal{A}_1$ and hence by 1.10, $\mathcal{A}_0 \equiv \mathcal{A}_1$. But $\mathcal{A}_0 \models \phi$ and $\mathcal{A}_1 \models \neg\phi$, a contradiction.

The assumption that T has no finite models is necessary. Suppose that T is the $\{+, 0\}$-theory of Abelian groups, where every element has order 2.

Exercise 4.6 Show that T is κ-categorical for all $\kappa \geq \aleph_0$. [Hint: Models of T are essentially vector spaces over $\mathbb{F}_2$.]

However, T is not complete. The sentence $\exists x \exists y \exists z\ (x \neq y \wedge y \neq z \wedge z \neq x)$ is false in the two-element group but true in every other model of T.

Vaught's Test implies that all of the categorical theories discussed above are complete. In particular, the theory of algebraically closed fields of a fixed characteristic is complete. This result of Tarski has several immediate interesting consequences.

Definition 4.7 We say that an $\mathcal{L}$-theory T is *decidable* if there is an algorithm that when given an $\mathcal{L}$-sentence ϕ as input decides whether $T \models \phi$.

Lemma 4.8 *Let T be a recursive complete satisfiable theory in a recursive language $\mathcal{L}$. Then T is decidable.*

Proof Start enumerating all finite sequence of strings of $\mathcal{L}$-symbols. For each one, check to see if it is a derivation of $\Delta \vdash \phi$ or $\Delta \vdash \neg\phi$. If it is then check to see if all of the sentences in Δ are in T. If so output yes if $\Delta \vdash \phi$ and no if $\Delta \vdash \neg\phi$. If not, go on to the next string. Since T is complete, the Completeness Theorem implies there is a finite $\Delta \subseteq T$ such that $\Delta \vdash \phi$ or $\Delta \vdash \neg\phi$. Thus our search will halt at some stage.

Informally, to decide whether ϕ is a logical consequence of a complete satisfiable recursive theory T, we begin searching through possible proofs from T until we find either a proof of ϕ or a proof of $\neg\phi$. Because T is satisfiable, we will not find proofs of both. Because T is complete, we will eventually find a proof of one or the other.

Corollary 4.9 *For $p = 0$ or p prime, ACF_p is decidable. In particular, $\mathrm{Th}(\mathbb{C})$, the first-order theory of the field of complex numbers, is decidable.*

The completeness of ACF_p can also be thought of as a first-order version of the Lefschetz Principle from algebraic geometry.

Corollary 4.10 *Let ϕ be a sentence in the language of rings. The following are equivalent.*

i) ϕ is true in the complex numbers.

ii) ϕ is true in every algebraically closed field of characteristic zero.

iii) ϕ is true in some algebraically closed field of characteristic zero.

iv) There are arbitrarily large primes p such that ϕ is true in some algebraically closed field of characteristic p.

v) There is an m such that for all $p > m$, ϕ is true in all algebraically closed fields of characteristic p.

Proof The equivalence of i)–iii) is just the completeness of ACF_0 and v)$\Rightarrow$ iv) is obvious.

For ii) $\Rightarrow$ v) suppose that $\mathrm{ACF}_0 \models \phi$. There is a finite $\Delta \subset \mathrm{ACF}_0$ such that $\Delta \vdash \phi$. Thus, if we choose p large enough, then $\mathrm{ACF}_p \models \Delta$. Thus, $\mathrm{ACF}_p \models \phi$ for all sufficiently large primes p.

For iv) $\Rightarrow$ ii) suppose $\mathrm{ACF}_0 \not\models \phi$. Because ACF_0 is complete, $\mathrm{ACF}_0 \models \neg\phi$. By the argument above, $\mathrm{ACF}_p \models \neg\phi$ for sufficiently large p; thus, iv) fails.

Ax found the following striking application of Corollary 4.10.

Theorem 4.11 *[Ax] Every injective polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ is surjective.*

Proof Remarkably, the key to the proof is the simple observation that if k is a finite field, then every injective function $f : k^n \to k^n$ is surjective. From this observation it is easy to show that the same is true for $\mathbb{F}_p^{\mathrm{alg}}$, the algebraic closure of the p-element field.

Claim Every injective polynomial map $f : (\mathbb{F}_p^{\mathrm{alg}})^n \to (\mathbb{F}_p^{\mathrm{alg}})^n$ is surjective.

Suppose not. Let $\overline{a} \in \mathbb{F}_p^{\mathrm{alg}}$ be the coefficients of f and let $\overline{b} \in (\mathbb{F}_p^{\mathrm{alg}})^n$ such that $\overline{b}$ is not in the range of f. Let k be the subfield of $\mathbb{F}_p^{\mathrm{alg}}$ generated by $\overline{a}, \overline{b}$. Then $f|k^n$ is an injective but not surjective polynomial map from k^n into itself. But $\mathbb{F}_p^{\mathrm{alg}} = \bigcup_{n=1}^{\infty} \mathbb{F}_{p^n}$ is a locally finite field. Thus k is finite, a contradiction.

Suppose that the theorem is false. Let $X = (X_1, \ldots, X_n)$. Let

$$f(X) = (f_1(X), \ldots, f_n(X))$$

be a counterexample where each $f_i \in \mathbb{C}[X]$ has degree at most d. There is an $\mathcal{L}$-sentence $\Phi_{n,d}$ such that for K a field, $K \models \Phi_{n,d}$ if and only if every injective polynomial map from K^n to K^n where each coordinate function has degree at most d is surjective. We can quantify over polynomials of degree at most d by quantifying over the coefficients. For example, $\Phi_{2,2}$ is the sentence

$$\forall a_{0,0}\forall a_{0,1}\forall a_{0,2}\forall a_{1,0}\forall a_{1,1}\forall a_{2,0}\forall b_{0,0}\forall b_{0,1}\forall b_{0,2}\forall b_{1,0}\forall b_{1,1}\forall b_{2,0}$$

$$\Big[\big(\forall x_1\forall y_1\forall x_2\forall y_2((\sum a_{i,j}x_1^i y_1^j = \sum a_{i,j}x_2^i y_2^j \wedge \sum b_{i,j}x_1^i y_1^j = \sum b_{i,j}x_2^i y_2^j) \rightarrow$$
$$(x_1 = x_2 \wedge y_1 = y_2))\big) \rightarrow \forall u\forall v\exists x\exists y \sum a_{i,j}x^i y^j = u \wedge \sum b_{i,j}x^i y^j = v\Big].$$

By the claim $\mathbb{F}_p^{\mathrm{alg}} \models \Phi_{n,d}$ for all primes p. By Corollary 4.10, $\mathbb{C} \models \Phi_{n,d}$, a contradiction.

We will return to the model theory of algebraically closed fields in §6.

There are other interesting applications of Vaught's Test. Let $\mathrm{L} = \{<\}$ and let DLO be the theory says we have a dense linear order with no top or bottom element. Then $\mathbb{Q} \models \mathrm{DLO}$ and $\mathbb{R} \models \mathrm{DLO}$.

Theorem 4.12 (Cantor) *Any two countable models of* DLO *are isomorphic. Thus* DL0 *is $\aleph_0$-categorical. Since* DLO *has no finite models it is complete.*

It follows the $(\mathbb{R}, <) \equiv (\mathbb{Q}, <)$. Thus we can not express the fact that $\mathbb{R}$ is complete. DLO is not κ-categorical for any uncountable cardinal κ. Indeed, if κ is uncountable there are 2^κ non-isomorphic models of cardinality κ.

5 Quantifier Elimination

In model theory we try to understand structures by studying their definable sets. Recall that if $\mathcal{M}$ is an $\mathcal{L}$-structure, then $X \subseteq M^n$ is *definable* if there is an $\mathcal{L}$-formula $\phi(v_1, \ldots, v_n, w_1, \ldots, w_m)$ and $b_1, \ldots, b_m \in M$ such that

$$X = \{\overline{a} \in M^n : \mathcal{M} \models \phi(\overline{a}, \overline{b})\}.$$

The study of definable sets is often complicated by quantifiers. For example, in the structure $(\mathbb{N}, +, \cdot, <, 0, 1)$ the quantifier-free definable sets are defined by polynomial equations and inequalities. Even if we use only existential quantifiers the definable sets become complicated. By the Matijasevič–Robinson–Davis–Putnam solution to Hilbert's 10th problem [?], every recursively enumerable subset of $\mathbb{N}$ is defined by a formula

$$\exists v_1 \ldots \exists v_n \ p(x, v_1, \ldots, v_n) = 0$$

for some polynomial $p \in \mathbb{N}[X, Y_1, \ldots, Y_n]$. As we allow more alternations of quantifiers, we get even more complicated definable sets.

Not surprisingly, it will be easiest to study definable sets that are defined by quantifier-free formulas. Sometimes formulas with quantifiers can be shown to be equivalent to formulas without quantifiers. Here are two well-known examples. Let $\phi(a, b, c)$ be the formula

$$\exists x \ ax^2 + bx + c = 0.$$

By the quadratic formula,

$$\mathbb{R} \models \phi(a, b, c) \leftrightarrow [(a \neq 0 \wedge b^2 - 4ac \geq 0) \vee (a = 0 \wedge (b \neq 0 \vee c = 0))],$$

whereas in the complex numbers

$$\mathbb{C} \models \phi(a, b, c) \leftrightarrow (a \neq 0 \vee b \neq 0 \vee c = 0).$$

In either case, ϕ is equivalent to a quantifier-free formula. However, ϕ is not equivalent to a quantifier-free formula over the rational numbers $\mathbb{Q}$.

For a second example, let $\phi(a, b, c, d)$ be the formula

$$\exists x \exists y \exists u \exists v \ (xa + yc = 1 \wedge xb + yd = 0 \wedge ua + vc = 0 \wedge ub + vd = 1).$$

The formula $\phi(a, b, c, d)$ asserts that the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible. By the determinant test,

$$F \models \phi(a, b, c, d) \leftrightarrow ad - bc \neq 0$$

for any field F.

Definition 5.1 We say that a theory T has *quantifier elimination* if for every formula ϕ there is a quantifier-free formula ψ such that

$$T \models \phi \leftrightarrow \psi.$$

Our goal in this section is to give a very useful model theoretic test for elimination of quantifiers. In the next section we will show that this method can be applied to the theory of algebraically closed fields and develop some rich consequences. We begin by introducing some preliminary tools.

Diagrams

We begin by giving a way to construct $\mathcal{L}$-embeddings.

Definition 5.2 Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure. Let $\mathcal{L}_M$ be the language where we add to $\mathcal{L}$ constant symbols m for each element of M. The *atomic diagram* of $\mathcal{M}$ is $\{\phi(m_1, \ldots, m_n) : \phi$ is either an atomic $\mathcal{L}$-formula or the negation of an atomic $\mathcal{L}$-formula and $\mathcal{M} \models \phi(m_1, \ldots, m_n)\}$. We let $\mathrm{Diag}(\mathcal{M})$ denote the atomic diagram of $\mathcal{M}$

Lemma 5.3 *Suppose that $\mathcal{N}$ is an $\mathcal{L}_M$-structure and $\mathcal{N} \models \mathrm{Diag}(\mathcal{M})$; then, viewing $\mathcal{N}$ as an $\mathcal{L}$-structure, there is an $\mathcal{L}$-embedding of $\mathcal{M}$ into $\mathcal{N}$.*

Proof Let $j : M \to N$ be defined by $j(m) = m^{\mathcal{N}}$; that is, $j(m)$ is the interpretation of this constant symbol m in $\mathcal{N}$. If m_1, m_2 are distinct elements of M, then $m_1 \neq m_2 \in \mathrm{Diag}(\mathcal{M})$; thus, $j(m_1) \neq j(m_2)$ so j is an embedding. If f is a function symbol of $\mathcal{L}$ and $f^{\mathcal{M}}(m_1, \ldots, m_n) = m_{n+1}$, then $f(m_1, \ldots, m_n) = m_{n+1}$ is a formula in $\mathrm{Diag}(\mathcal{M})$ and $f^{\mathcal{N}}(j(m_1), \ldots, j(m_n)) = j(m_{n+1})$. If R is a relation symbol and $\overline{m} \in R^{\mathcal{M}}$, then $R(m_1, \ldots, m_n) \in \mathrm{Diag}(\mathcal{M})$ and $(j(m_1), \ldots, j(m_n)) \in R^{\mathcal{N}}$. Hence, j is an $\mathcal{L}$-embedding.

Quantifier Elimination Tests

Theorem 5.4 *Suppose that $\mathcal{L}$ contains a constant symbol c, T is an $\mathcal{L}$-theory, and $\phi(\overline{v})$ is an $\mathcal{L}$-formula. The following are equivalent:*

i) There is a quantifier-free $\mathcal{L}$-formula $\psi(\overline{v})$ such that $T \models \forall \overline{v}\ (\phi(\overline{v}) \leftrightarrow \psi(\overline{v}))$.

ii) If $\mathcal{M}$ and $\mathcal{N}$ are models of T, $\mathcal{A}$ is an $\mathcal{L}$-structure, $\mathcal{A} \subseteq \mathcal{M}$, and $\mathcal{A} \subseteq \mathcal{N}$, then $\mathcal{M} \models \phi(\overline{a})$ if and only if $\mathcal{N} \models \phi(\overline{a})$ for all $\overline{a} \in \mathcal{A}$.

Proof i)$\Rightarrow$ ii) Suppose that $T \models \forall \overline{v}\ (\phi(\overline{v}) \leftrightarrow \psi(\overline{v}))$, where ψ is quantifier-free. Let $\overline{a} \in \mathcal{A}$, where $\mathcal{A}$ is a common substructure of $\mathcal{M}$ and $\mathcal{N}$ and the latter two structures are models of T. In Proposition 1.8, we saw that quantifier-free formulas are preserved under substructure and extension. Thus

$$\begin{aligned}
\mathcal{M} \models \phi(\overline{a}) \quad &\Leftrightarrow \quad \mathcal{M} \models \psi(\overline{a}) \\
&\Leftrightarrow \quad \mathcal{A} \models \psi(\overline{a}) \quad \text{(because } \mathcal{A} \subseteq \mathcal{M}) \\
&\Leftrightarrow \quad \mathcal{N} \models \psi(\overline{a}) \quad \text{(because } \mathcal{A} \subseteq \mathcal{N}) \\
&\Leftrightarrow \quad \mathcal{N} \models \phi(\overline{a}).
\end{aligned}$$

ii) $\Rightarrow$ i) First, if $T \models \forall \overline{v}\ \phi(\overline{v})$, then $T \models \forall \overline{v}\ (\phi(\overline{v}) \leftrightarrow c = c)$. Second, if $T \models \forall \overline{v}\ \neg\phi(\overline{v})$, then $T \models \forall \overline{v}\ (\phi(\overline{v}) \leftrightarrow c \neq c)$.

Thus, we may assume that both $T \cup \{\phi(\overline{v})\}$ and $T \cup \{\neg\phi(\overline{v})\}$ are satisfiable.

Let $\Gamma(\overline{v}) = \{\psi(\overline{v}) : \psi$ is quantifier-free and $T \models \forall \overline{v}\ (\phi(\overline{v}) \rightarrow \psi(\overline{v}))\}$. Let $d_1, \ldots, d_m$ be new constant symbols. We will show that $T \cup \Gamma(\overline{d}) \models \phi(\overline{d})$. Then, by compactness, there are $\psi_1, \ldots, \psi_n \in \Gamma$ such that

$$T \models \forall \overline{v}\ \left(\bigwedge_{i=1}^{n} \psi_i(\overline{v})\ \rightarrow \phi(\overline{v}) \right).$$

Thus

$$T \models \forall \overline{v}\ \left(\bigwedge_{i=1}^{n} \psi_i(\overline{v})\ \leftrightarrow \phi(\overline{v}) \right)$$

and $\bigwedge_{i=1}^{n} \psi_i(\overline{v})$ is quantifier-free. We need only prove the following claim.

Claim $T \cup \Gamma(\overline{d}) \models \phi(\overline{d})$.

Suppose not. Let $\mathcal{M} \models T \cup \Gamma(\overline{d}) \cup \{\neg\phi(\overline{d})\}$. Let $\mathcal{A}$ be the substructure of $\mathcal{M}$ generated by $\overline{d}$.

Let $\Sigma = T \cup \mathrm{Diag}(\mathcal{A}) \cup \phi(\overline{d})$. If Σ is unsatisfiable, then there are quantifier-free formulas $\psi_1(\overline{d}), \ldots, \psi_n(\overline{d}) \in \mathrm{Diag}(\mathcal{A})$ such that

$$T \models \forall \overline{v}\ \left(\bigwedge_{i=1}^{n} \psi_i(\overline{v}) \rightarrow \neg\phi(\overline{v}) \right).$$

But then

$$T \models \forall \overline{v}\ \left(\phi(\overline{v}) \rightarrow \bigvee_{i=1}^{n} \neg\psi_i(\overline{v}) \right),$$

so $\bigvee_{i=1}^{n} \neg\psi_i(\overline{v}) \in \Gamma$ and $\mathcal{A} \models \bigvee_{i=1}^{n} \neg\psi_i(\overline{d})$, a contradiction. Thus, Σ is satisfiable.

Let $\mathcal{N} \models \Sigma$. Then $\mathcal{N} \models \phi(\overline{d})$. Because $\Sigma \supseteq \mathrm{Diag}(\mathcal{A})$, $\mathcal{A} \subseteq \mathcal{N}$, by Lemma 5.3 i). But $\mathcal{M} \models \neg\phi(\overline{d})$; thus, by ii), $\mathcal{N} \models \neg\phi(\overline{d})$, a contradiction.

The proof above can easily be adapted to the case where $\mathcal{L}$ contains no constant symbols. In this case, there are no quantifier-free sentences, but for each sentence we can find a quantifier-free formula $\psi(v_1)$ such that $T \models \phi \leftrightarrow \psi(v_1)$.

The next lemma shows that we can prove quantifier elimination by getting rid of one existential quantifier at a time.

Lemma 5.5 *Let T be an $\mathcal{L}$-theory. Suppose that for every quantifier-free $\mathcal{L}$-formula $\theta(\overline{v}, w)$ there is a quantifier-free formula $\psi(\overline{v})$ such that $T \models\ \exists w\ \theta(\overline{v}, w) \leftrightarrow \psi(\overline{v})$. Then, T has quantifier elimination.*

Proof Let $\phi(\overline{v})$ be an $\mathcal{L}$-formula. We wish to show that $T \models \forall \overline{v}\ (\phi(\overline{v}) \leftrightarrow \psi(\overline{v}))$ for some quantifier-free formula $\phi(\overline{v})$. We prove this by induction on the complexity of $\phi(\overline{v})$.

If ϕ is quantifier-free, there is nothing to prove. Suppose that for $i = 0, 1$, $T \models \forall \overline{v}\ (\theta_i(\overline{v}) \leftrightarrow \psi_i(\overline{v}))$, where ψ_i is quantifier free.

If $\phi(\overline{v}) = \neg\theta_0(\overline{v})$, then $T \models \forall \overline{v}\ (\phi(\overline{v}) \leftrightarrow \neg\psi_0(\overline{v}))$.

If $\phi(\overline{v}) = \theta_0(\overline{v}) \wedge \theta_1(\overline{v})$, then $T \models \forall v\ (\phi(\overline{v}) \leftrightarrow (\psi_0(\overline{v}) \wedge \psi_1(\overline{v})))$.

In either case, ϕ is equivalent to a quantifier-free formula.

Suppose that $T \models \forall \overline{v}(\theta(\overline{v}, w) \leftrightarrow \psi_0(\overline{v}, w))$, where ψ_0 is quantifier-free and $\phi(\overline{v}) = \exists w \theta(\overline{v}, w)$. Then $T \models \forall \overline{v}\ (\phi(\overline{v}) \leftrightarrow \exists w\ \psi_0(\overline{v}, w))$. By our assumptions, there is a quantifier-free $\psi(\overline{v})$ such that $T \models \forall \overline{v}\ (\exists w\ \psi_0(\overline{v}, w) \leftrightarrow \psi(\overline{v}))$. But then $T \models \forall \overline{v}\ (\phi(\overline{v}) \leftrightarrow \psi(\overline{v}))$.

Combining Theorem 5.4 and Lemma 5.5 gives us the following simple, yet useful, test for quantifier elimination.

Corollary 5.6 *Let T be an $\mathcal{L}$-theory. Suppose that for all quantifier-free formulas $\phi(\overline{v}, w)$, if $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{A}$ is a common substructure of $\mathcal{M}$ and $\mathcal{N}$, $\overline{a} \in A$, and there is $b \in M$ such that $\mathcal{M} \models \phi(\overline{a}, b)$, then there is $c \in N$ such that $\mathcal{N} \models \phi(\overline{a}, c)$. Then, T has quantifier elimination.*

Theories with Quantifier Elimination

We conclude with several observations about theories with quantifier elimination.

Definition 5.7 An $\mathcal{L}$-theory T is *model-complete* $\mathcal{M} \prec \mathcal{N}$ whenever $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{M}, \mathcal{N} \models T$.

Stated in terms of embeddings: T is model-complete if and only if all embeddings are elementary.

Proposition 5.8 *If T has quantifier elimination, then T is model-complete.*

Proof Suppose that $\mathcal{M} \subseteq \mathcal{N}$ are models of T. We must show that $\mathcal{M}$ is an elementary submodel. Let $\phi(\overline{v})$ be an $\mathcal{L}$-formula, and let $\overline{a} \in M$. There is a quantifier-free formula $\psi(\overline{v})$ such that $\mathcal{M} \models \forall \overline{v}\ (\phi(\overline{v}) \leftrightarrow \psi(\overline{v}))$. Because quantifier-free formulas are preserved under substructures and extensions, $\mathcal{M} \models \psi(\overline{a})$ if and only if $\mathcal{N} \models \psi(\overline{a})$. Thus

$$\mathcal{M} \models \phi(\overline{a}) \Leftrightarrow \mathcal{M} \models \psi(\overline{a}) \Leftrightarrow \mathcal{N} \models \psi(\overline{a}) \Leftrightarrow \mathcal{N} \models \phi(\overline{a}).$$

There are model-complete theories that do not have quantifier elimination. Let us just point out the following test for completeness of model-complete theories.

Proposition 5.9 *Let T be a model-complete theory. Suppose that there is $\mathcal{M}_0 \models T$ such that $\mathcal{M}_0$ embeds into every model of T. Then, T is complete.*

Proof If $\mathcal{M} \models T$, the embedding of $\mathcal{M}_0$ into $\mathcal{M}$ is elementary. In particular $\mathcal{M}_0 \equiv \mathcal{M}$. Thus, any two models of T are elementarily equivalent.

We will use Proposition 5.9 below in cases where Vaught's test does not apply.

We have provided a number of proofs of quantifier elimination without explicitly explaining how to take an arbitrary formula and produce a quantifier free one. In all of these cases, one can give explicit effective procedures. After the fact, the following lemma tells us that there is an algorithm to eliminate quantifiers.

Proposition 5.10 *Suppose that T is a decidable theory with quantifier elimination. Then, there is an algorithm which when given a formula ϕ as input will output a quantifier-free formula ψ such that $T \models \phi \leftrightarrow \psi$.*

Proof Given input $\phi(\overline{v})$ we search for a quantifier-free formula $\psi(\overline{v})$ such that $T \models \forall \overline{v}\ (\phi(\overline{v}) \leftrightarrow \psi(\overline{v}))$. Because T is decidable this is an effective search. Because T has quantifier elimination, we will eventually find ψ.

6 Algebraically Closed Fields

We now return to the theory of algebraically closed fields. In Proposition 4.4, we proved that the theory of algebraically closed fields of a fixed characteristic is complete. We begin this section by showing that algebraically closed fields have quantifier elimination. For convenience we will formulate ACF in the language $\mathcal{L} = \{+, -, \cdot, 0, 1\}$. We add $-$ to the language, so that substructures are integral domains. Without $-$ we would have weaker structures that are a bit more cumbersome to deal with.

Theorem 6.1 ACF *has quantifier elimination.*

Proof

Suppose K and L are algebraically closed fields and A is an integral domain with $A \subseteq K \cap L$. By Corollary 5.6, we need to show that if $\phi(v, \overline{w})$ is a quantifier free formula, $\overline{a} \in A$, $b \in K$ and $K \models \phi(b, \overline{a})$, then there is $c \in L$ such that $L \models \phi(c, \overline{a})$.

Let F be the algebraic closure of the fraction field of A . We, may without loss of generality, assume that $F \subseteq K \cap L$. It will be enough to show that , $\overline{a} \in F$, and $K \models \phi(b, \overline{a})$ for some $b \in K$, then there is $c \in F$ such that $F \models \phi(c, \overline{a})$, for then, by Proposition 1.8, $L \models \phi(c, \overline{a})$.

We first note that ϕ can be put in disjunctive normal form, namely there are atomic or negated atomic formulas $\theta_{i,j}(\overline{v}, w)$ such that:

$$\phi(\overline{v}, w) \leftrightarrow \bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} \theta_{i,j}(\overline{v}, w).$$

Because $K \models \phi(\overline{a}, b)$, $K \models \bigwedge_{j=1}^{m} \theta_{i,j}(\overline{a}, b)$ for some i. Thus, without loss of generality, we may assume that ϕ is a conjunction of atomic and negated atomic formulas. In our language atomic formulas $\theta(v_1, \ldots, v_n)$ are of the form $p(\overline{v}) = 0$, where $p \in \mathbb{Z}[X_1, \ldots, X_n]$. If $p(X, \overline{Y}) \in \mathbb{Z}[X, \overline{Y}]$, we can view $p(X, \overline{a})$ as a polynomial in $F[X]$. Thus, there are polynomials $p_1, \ldots, p_n, q_1, \ldots, q_m \in F[X]$ such that $\phi(v, \overline{a})$ is equivalent to

$$\bigwedge_{i=1}^{n} p_i(v) = 0 \wedge \bigwedge_{i=1}^{m} q_i(v) \neq 0.$$

If any of the polynomials p_i are nonzero, then b is algebraic over F. In this case, because F is algebraically closed, $b \in F$. Thus, we may assume that $\phi(v, \overline{a})$ is equivalent to

$$\bigwedge_{i=1}^{m} q_i(v) \neq 0.$$

But $q_i(X) = 0$ has only finitely many solutions for each $i \leq m$. Thus, there are only finitely many elements of F that do not satisfy F. Because algebraically closed fields are infinite, there is a $c \in F$ such that $F \models \phi(c, \overline{a})$.

Corollary 6.2 ACF *is model-complete and* ACF_p *is complete where* $p = 0$ *or* p *is prime.*

Proof Model-completeness is an immediate consequence of quantifier elimination.

The completeness of ACF_p was proved in Proposition 4.4, but it also follows from quantifier elimination. Suppose that $K, L \models ACF_p$. Let ϕ be any sentence in the language of rings. By quantifier elimination, there is a quantifier-free sentence ψ such that

$$\mathrm{ACF} \models \phi \leftrightarrow \psi.$$

Because quantifier-free sentences are preserved under extension and substructure,

$$K \models \psi \Leftrightarrow \mathbb{F}_p \models \psi \Leftrightarrow L \models \psi,$$

where $\mathbb{F}_p$ is the p-element field if $p > 0$ and the rationals if $p = 0$. Thus,

$$K \models \phi \Leftrightarrow K \models \psi \Leftrightarrow L \models \psi \Leftrightarrow L \models \phi.$$

Thus $K \equiv L$ and ACF_p is complete.

Definable Sets and Constructible Sets

Quantifier elimination has a geometric interpretation. We begin by looking at the sets defined by quantifier free formulas.

Lemma 6.3 *Let* K *be a field. The subsets of* K^n *defined by atomic formulas are exactly those of the form* $V(p) = \{x$ *for some* $p \in K[\overline{X}]$*. A subset of* K^n *is quantifier-free definable if and only if it is a Boolean combination of Zariski closed subsets.*

Proof If $\phi(\overline{x}, \overline{y})$ is an atomic $\mathcal{L}_\mathrm{r}$-formula, then there is $q(\overline{X}, \overline{Y}) \in \mathbb{Z}[\overline{X}, \overline{Y}]$ such that $\phi(\overline{x}, \overline{y})$ is equivalent to $q(\overline{x}, \overline{y}) = 0$. If $X = \{\overline{x} : \phi(\overline{x}, \overline{a})\}$, then $X = V(q(\overline{X}, \overline{a}))$ and $q(\overline{X}, \overline{a}) \in K[\overline{X}]$. On the other hand, if $p \in K[\overline{X}]$, there is $q \in \mathbb{Z}[\overline{X}, \overline{Y}]$ and $\overline{a} \in K^m$ such that $p(\overline{X}) = q(\overline{X}, \overline{a})$. Then, $V(p)$ is defined by the quantifier-free formula $q(\overline{X}, \overline{a}) = 0$.

If $X \subseteq K^n$ is a finite Boolean combination of Zariski closed sets we call X *constructible*. If K is algebraically closed, the constructible sets have much stronger closure properties.

Corollary 6.4 *Let* K *be an algebraically closed field.*

i) $X \subseteq K^n$ *is constructible if and only if it is definable.*

ii) **(Chevalley's Theorem)** *The image of a constructible set under a polynomial map is constructible.*

Proof i) By Lemma 6.3, the constructible sets are exactly the quantifier-free definable sets, but by quantifier elimination every definable set is quantifier-free definable.

ii) Let $X \subseteq K^n$ be constructible and $p : K^n \to K^m$ be a polynomial map. Then, the image of $X = \{y \in K^m : \exists x \in K^n \; p(x) = y\}$. This set is definable and hence constructible.

Quantifier elimination has very strong consequences for definable subsets of K.

Corollary 6.5 *If K is an algebraically closed field and $X \subseteq K$ is definable, then either X or $K \setminus X$ is finite.*

Proof By quantifier elimination X is a finite Boolean combination of sets of the form $V(p)$, where $p \in K[X]$. But $V(p)$ is either finite or (if $p = 0$) all of K.

We say that a theory T is *strongly minimal* if for any $\mathcal{M} \models T$ and any definable $X \subseteq M$ either X or $M \setminus X$ is finite. This is a very powerful assumption. For example, it can be shown that any strongly minimal theory in a countable language is κ-categorical for every uncountable κ.

The model-completeness of algebraically closed fields can be used to give a proof of the Nullstellensatz.

Theorem 6.6 (Hilbert's Nullstellensatz) *Let K be an algebraically closed field. Suppose thatI and J are radical ideals in $K[X_1, \ldots, X_n]$ and $I \subset J$. Then $V(J) \subset V(I)$. Thus $X \mapsto I(X)$ is a bijective correspondence between Zariski closed sets and radical ideals.*

Proof Let $p \in J \setminus I$. By Primary Decomposition, there is a prime ideal $P \supseteq I$ such that $p \notin P$. We will show that there is $x \in V(P) \subseteq V(I)$ such that $p(x) \neq 0$. Thus $V(I) \neq V(J)$. Because P is prime, $K[\overline{X}]/P$ is a domain and we can take F, the algebraic closure of its fraction field.

Let $q_1, \ldots, q_m \in K[X_1, \ldots, X_n]$ generate I. Let a_i be the element X_i/P in F. Because each $q_i \in P$ and $p \notin P$,

$$F \models \bigwedge_{i=1}^{m} q_i(\overline{a}) = 0 \wedge p(\overline{a}) \neq 0.$$

Thus

$$F \models \exists \overline{w} \bigwedge_{i=1}^{m} q_i(\overline{w}) = 0 \wedge p(\overline{w}) \neq 0$$

and by model-completeness

$$K \models \exists \overline{w} \bigwedge_{i=1}^{m} q_i(\overline{w}) = 0 \wedge p(\overline{w}) \neq 0.$$

Thus there is $\overline{b} \in K^n$ such that $q_1(\overline{b}) = \ldots = q_m(\overline{b}) = 0$ and $p(\overline{b}) \neq 0$. But then $\overline{b} \in V(P) \setminus V(J)$.

The next corollary is a simple consequence of model completeness.

Corollary 6.7 *Suppose $K \subseteq L$ are algebraically closed fields, V and W are varieties defined over K and $f : V \to W$ is a polynomial isomorphism defined over L. Then there is an isomorphism defined over K.*

Proof Suppose $f : V \to W$ is a polynomial isomorphism defined over L and f and f^{-1} both have degree at most d. As in the proof of Ax's Theorem we can write down an $\mathcal{L}$-formula Ψ with parameters from K saying that for some choice of coefficients there is a polynomial bijection from V between V and W where the polynomials have degree at most d. Since $L \models \Psi$, by model completeness, $K \models \Psi$. Thus we can choose an isomorphism defined over K.

Quantifier elimination gives us a powerful tool for analyzing definability in algebraically closed fields. For example, we will give the following characterization of definable functions.

Definition 6.8 Let $X \subseteq K^n$. We say that $f : X \to K$ is *quasirational* if either

i) K has characteristic zero and for some rational function $q(\overline{X}) \in K(X_1, \ldots, X_n)$, $f(\overline{x}) = q(\overline{x})$ on X, or

ii) K has characteristic $p > 0$ and for some rational function $q(\overline{X}) \in K(\overline{X})$, $f(\overline{x}) = q(\overline{x})^{\frac{1}{p^n}}$.

Rational functions are easily seen to be definable. In algebraically closed fields of characteristic p, the formula $x = y^p$ defines the function $x \mapsto x^{\frac{1}{p}}$, because every element has a unique p^{th}-root. Thus, every quasirational function is definable.

Proposition 6.9 *If $X \subseteq K^n$ is constructible and $f : X \to K$ is definable, then there are constructible sets $X_1, \ldots, X_m$ and quasirational functions $\rho_1, \ldots, \rho_m$ such that $\bigcup X_i = X$ and $f|X_i = \rho_i|X_i$.*

Proof Let $\Gamma(v_1, \ldots, v_n) = \{f(\overline{x}) \neq \rho(\overline{x}) : \rho \text{ a quasirational function}\} \ \cup\{\overline{v} \in X\} \cup$ ACF $\cup$ Diag(K).

Claim Γ is not satisfiable.

Suppose that Γ is consistent. Let $L \models$ ACF $+$Diag(K) with $b_1, \ldots, b_n \in L$ such that for all $\gamma(\overline{v}) \in \Gamma$, $L \models \gamma(\overline{b})$.

Let K_0 be the subfield of L generated by K and $\overline{b}$. Then, K_0 is the closure of $B = \{b_1, \ldots, b_n\}$ under the rational functions of K. Let K_1 be the closure of B under all quasirational functions. If K has characteristic 0, then $K_0 = K_1$. If K has characteristic $p > 0$, $K_1 = \bigcup K_0^{\frac{1}{p^n}}$, the perfect closure of K_0.

By model-completeness, $K \prec L$, thus f^L, the interpretation of f in L, is a function from X^L to L, extending f. Because $L \models \Gamma(\overline{b})$, $f(\overline{b})$ is not in K_1. Because K_1 is perfect there is an automorphism α of L fixing K_1 pointwise such that $\alpha(f^L(\overline{b})) \neq f^L(\overline{b})$. But f^L is definable with parameters from K; thus, any automorphism of L which fixes K and fixes $\overline{a}$ must fix $f(\overline{a})$, a contradiction. Thus Γ is unsatisfiable.

Thus, by compactness, there are quasirational functions $\rho_1, \ldots, \rho_m$ such that

$$K \models \forall x \in X \bigwedge f(\overline{x}) = \rho_i(\overline{x}).$$

Let $X_i = \{\overline{x} \in X : f(\overline{x}) = \rho_i(\overline{x})\}$. Each X_i is definable.

We end by stating two more far reaching definability results for algebraically closed. They are a bit more involved–and ideally best understood using the model theoretic tool of *ω-stability* that we will not discuss in these lectures.

Let K be algebraically closed.

Theorem 6.10 (Elimination of Imaginaries) *Suppose $X \subseteq K^n$ is definable and E is a deifnable equivalence relation on X. There is a definable $f : X \to K^m$ for some m such that xEy if and only if $f(x) = f(y)$.*

This is related to the existence of fields of definitions. It is a useful tool for viewing projective, quasiprojective or abstract varieties (at least in the style of Weil) as constructible objects.

Theorem 6.11 *Let $G \subseteq K^n$ be a definable group. Then G is definably isomorphic to an algebraic group.*

Combining these we could conclude that if G is an algebraic group and H is a normal algebraic subgroup, then G/H is an algebraic group.