Vaught's Conjecture for Differentially Closed Fields

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http://www.math.uic.edu/∼marker/vcdcf-slides.pdf

Vaught's Conjecture for ω-stable Theories

Let $I(T, \kappa)$ be the number of nonisomorphic models of T of cardinality κ.

Theorem 1 (Shelah 1981) *If T is an ω-stable theory in a countable language and $I(T, \aleph_0) > \aleph_0$, then $I(T, \aleph_0) = 2^{\aleph_0}$.*

Theorem 2 (Hrushovski–Sokolović 1992) *There are $2^{\aleph_0}$ countable differentially closed fields of characteristic zero.*

Outline for Tutorial

- Simple examples using dimensions to code graphs into ω-stable theories

- Survey of the model theory of differentially closed fields

- Proof of Hrushovski–Sokolović Theorem

Coding Graphs with Dimensions

Example 1 Let T_1 be the following theory in the language $\{V, X, +, \pi\}$.

- V and X are disjoint sorts
- $(V, +)$ is a torsion free divisible abelian group (i.e. V is a $\mathbb{Q}$-vector space)
- $\pi : X \to V$ is onto
- each fiber $\pi^{-1}(v)$ is infinite.

Countable models are determined by dim(V).

Uncountable Models of T_1

Let G be a graph of cardinality $\kappa \geq \aleph_1$ such that every vertex has valance at least 2.

Let $\mathcal{M}_0$ be the prime model of T_1 over $A \subset V$ a linearly independent set of size κ.

In $\mathcal{M}_0$, for $v \in V$, $\pi^{-1}(v)$ is countable.

We assume that A is the set of verticies of G. $B = \{a + b : a, b \in A, (a, b) \in G\}$.

Lemma 3 *There is $\mathcal{M}(G) \models T_1$ such that $|\pi^{-1}(a)| = \aleph_0$ if $a \in A \cup B$ and $|\pi^{-1}(a)| = \kappa$ for $a \in V \setminus (A \cup B)$.*

Recovering the Graph from $\mathcal{M}(G)$

Let $S = \{a \in V : |\pi^{-1}(a)| = \aleph_0\} = A \cup B$.

We say that $\{x, y, z\} \subseteq S$ is a *triangle* if x, y, z are pairwise independent but not independent.

Lemma 4 *Every triangle is of the form $\{a, b, a+b\}$ for some $a, b \in A$.*

Proof (sketch) Any three elements of A are independent.

Any three elements of B are independent. The hardest case $a+b, b+c$ and $a+c$ are interdefinable with a, b, c (as $(a+b)+(b+c)-(a+c) = 2b$).

If $x \in A$ and $y, z \in B$ they are independent.

If $a, b, c \in A$, then $a, b, a+c$ are independent.

Since every vertex has valance at least 2,

$$A = \{a \in S : a \text{ is in at least two triangles}\}$$

and (a, b) is an edge if and only if there is a $c \in S$, $\{a, b, c\}$ is a triangle.

Thus we can recover G from $\mathcal{M}(G)$. If $G \ncong G'$, then $\mathcal{M}(G) \ncong \mathcal{M}(G')$.

Proposition 5 $I(T_1, \aleph_0) = \aleph_0$, $I(T_1, \kappa) = 2^\kappa$ *for all* $\kappa \geq \aleph_1$.

Observation In countable models of T_1 we don't have enough choices to do coding.

Example 2 $\mathcal{L} = \{V, X, +, \pi, f\}$ let $T_2 \supset T_1$ so that each $(\pi^{-1}(v), f) \equiv (\mathbb{Z}, s)$ [where $s(x) = x + 1$].

For each v, $\dim(\pi^{-1}(v)) \geq 1$ is the number of copies of $\mathbb{Z}$ in $\pi^{-1}(v)$.

Let G be a graph as above with vertex set A of cardinality $\kappa \geq \aleph_0$.

Lemma 6 *There is $\mathcal{M}(G) \models T_2$ of cardinality κ with $A \subseteq V$ independent such that for $a \in V$*

$\dim(\pi^{-1}(a)) = 1$ *if $a \in A$ or $a = b + c$ where $(b, c) \in G$ and*

$\dim(\pi^{-1}(a)) = \kappa$ *otherwise.*

Corollary 7 $I(T_2, \kappa) = 2^\kappa$ *for all $\kappa \geq \aleph_0$.*

Homework

- Work out the details for T_1 and T_2.

Example 3 Change Example 1 by making V a vector space over $\mathbb{F}_2$. Show that T_3 is $\aleph_0$-categorical with $I(T_3, \kappa) = 2^\kappa$ for all uncountable κ. (Hint: Use triangle free graphs)

Example 4 Change Example 2 by making V a set with no additional structure. Show that

$$I(T_4, \aleph_\alpha) \leq (\alpha + \aleph_0)^{(\alpha+\aleph_0)}$$

Example 5 Change Example 2 by making $V \equiv (\mathbb{Z}, s)$. Show that

$$I(T_5, \aleph_\alpha) \leq (\alpha + \aleph_0)^{(\alpha+\aleph_0)^{\aleph_0}}$$

Observations

For this method of coding graphs using dimensions to work, we seem to need:

- large family of types $(p_a : a \in A)$, $p_a \in S(a)$, to which we can assign dimensions (for Vaught's Conjecture we would like to be able to assign different countable dimensions).

- the ability to realize one type in the family while omitting others (orthogonality)

- good notion of independence in A with lots of elements $a, b, c \in A$, pairwise independent but not independent (non-triviality)

Differential Fields

A *differential field* (K, δ) is a field K with a derivation $\delta : K \to K$ such that

$$\delta(x + y) = \delta(x) + \delta(y)$$

$$\delta(xy) = x\delta(y) + y\delta(x).$$

We will assume all fields have characteristic 0.

Examples i) $\mathbb{R}(t)$ where $\delta(t) = 1$

ii) Mer(U) the field of meromorphic functions on $U \subseteq \mathbb{C}$

Differential Polynomials

If (K, δ) is a differential field, we form $K\{X_1, \ldots, X_n\}$ the ring of *differential polynomials* in n-variables.

$$K[X_1, \ldots, X_n, X_1', \ldots, X_n', \ldots, X_1^{(m)}, \ldots, X_n^{(m)}, \ldots]$$

and extend the derivation by $\delta(X_i^{(j)}) = X_i^{(j+1)}$.

For example

$$X' - aX$$

$$(X'')^2 - X^3 - aX - b$$

The *order* of f is the largest n such that some $X_i^{(n)}$ occurs in f.

Differentially Closed Fields

We say that (K, δ) is *differentially closed* (DCF) if whenever $f_1, \ldots, f_m \in K\{X_1, \ldots, X_n\}$ and there is $L \supseteq K$ where

$$L \models \exists \overline{v}\ f_1(\overline{v}) = \ldots = f_m(\overline{v}) = 0,$$

then

$$K \models \exists \overline{v}\ f_1(\overline{v}) = \ldots = f_m(\overline{v}) = 0.$$

Differentially closed fields are the existentially closed differential fields.

Most Embarrasing Question: What's an example of a differentially closed field?

There are no natural examples.

Theorem 8 (Seidenberg) *Every countable differential field is isomorphic to a field of germs of meromorphic functions.*

If there are no natural models, why do we study differentially closed fields?

Reason 1: They provide useful universal domains for studying algebraic differential equations. The model theory of DCF has proved useful in studying:

- Differential Galois Theory
- Differential Algebraic Groups
- Diophantine Geometry

Reason 2: As Gerald Sacks said in **Saturated Model Theory**, DCF is the "least misleading example" of an ω-stable theory.

Many interesting phenomena from all over model theory are witnessed in DCF, including:

- Robinson Style: Quantifier Elimination, Model Completeness

- Morley Style: ω-stability, prime model extensions

- Shelah Style: forking, orthogonality, DOP, ENI-DOP

- Zilber Style: geometric stability, ω-stable groups

Quantifier Elimination

The first results on DCF are due to Robinson, with improvements by Blum.

Theorem 9 DCF *is axiomatizable.*

Blum Axioms: If $f, g \in K\{X\}$ and order$(f) >$ order g, there is $x \in K$ with $f(x) = 0$ and $g(x) \neq 0$.

Theorem 10 DCF *has quantifier elimination and hence is model complete.*

Differential Nullstelensatz

We say that an ideal $I \subseteq k\{X_1, \ldots, X_n\}$ is a *differential ideal* if whenever $f \in I$, then $f' \in I$.

Theorem 11 *Let* $K \models \mathrm{DCF}$*. Suppose* $P \subseteq K\{X_1, \ldots, X_n\}$ *is a prime differential ideal,* $f_1, \ldots, f_m \in P$ *and* $g \notin P$*. Then there is* $\overline{x} \in K^n$ *such that*

$$f_1(\overline{x}) = \ldots = f_m(\overline{x}) = 0 \wedge g(\overline{x}) \neq 0.$$

Proof Let $L \supseteq K$ be a DCF containing the differential domain $K\{\overline{X}\}/P$. In L, $X_1/P, \ldots, X_n/P$ are a solution to $f_1 = \ldots = f_m = 0 \wedge g \neq 0$. By model completeness, there is a solution in K.

The Kolchin Topology

A *Kolchin closed* $V \subseteq K^n$ is a finite union of sets of the form

$$\{\overline{x} \in K^n : f_1(\overline{x}) = \ldots = f_m(\overline{x}) = 0\}$$

where $f_1, \ldots, f_m \in K\{\overline{X}\}$.

Proposition 12 *$X \subseteq K^n$ is definable if and only if it is a finite Boolean combination of Kolchin closed sets.*

Types and Ideals

We say that an ideal $I \subseteq k\{X_1, \ldots, X_n\}$ is a *differential ideal* if whenever $f \in I$, then $f' \in I$.

If $k \subseteq K \models$ DCF and $\overline{a} \in K$, then, by quantifier elimination, tp$(\overline{a}/k)$ is deterimined by

$$I_{\overline{a}} = \{f \in k\{\overline{X}\} : f(\overline{a}) = 0\}$$

a prime differential ideal.

Proposition 13 *There is a bijection between $S_n(k)$ and prime differential ideals in $k\{X_1, \ldots, X_n\}$*

Proof If P is a prime differential ideal, then $R = k\{X_1, \ldots, X_n\}/P$ is a differential domain. Let K be the differential closure of the fraction field of R and let $\overline{a} \in K$ be $(X_1/P, \ldots, X_n/P)$. Then $I_{\overline{a}} = P$.

Differential Basis Theorem

Theorem 14 *If k is a differential field, then there are no infinite ascending chains of radical differential ideals in $k\{\overline{X}\}$. Every prime differential ideals are finitely generated.*

Corollary 15 *An arbitrary intersection of Kolchin closed sets is Kolchin closed.*

Corollary 16 *If $k \subseteq K$ and $\overline{a} \in K$, there is V a Kolchin closed set defined over k such that $\overline{a} \in V$ and if $W \subset V$ is defined over k, then $\overline{a} \notin W$. We say* tp$(\overline{a}, k)$ *is the generic type of V.*

Proof Let V be the intersection of all Kolchin closed W defined over k with $\overline{a} \in W$.

Every type is the generic type of some Kolchin closed set.

ω-stability

Corollary 17 DCF *is ω-stable.*

Proof We know $|S_n(k)|$ is the number of prime differential ideals in $k\{X_1, \ldots, X_n\}$. Since prime differential ideals are finitely generated there are only $|k|$ differential prime ideals in $k\{\overline{X}\}$.

Differential Closures

Definition 18 Let k be a differential field. We say that $K \models$ DCF is a *differential closure* of k if $k \subseteq K$ and whenever $L \models$ DCF and $k \subseteq L$, there is a differential field embedding $\eta : K \to L$ fixing k pointwise.

Differential closures are prime model extensions.

Theorem 19 *i) Differential closures exist*

ii) Differential closures are unique up to isomorphism.

iii) Every element of the differential closure of k realizes an isolated type in $S(k)$.

iv) Differential closures need not be minimal

By Morley i) and iii) are always true of prime model extensions in ω-stable theories.

By Shelah ii) is always true of prime model extensions in stable theories.

iv) was proved independently by Rosenlicht, Kolchin and Shelah.

The Field of Constants

Let $C = \{x : \delta(x) = 0\}$. C is an algebraically closed field.

Proposition 20 *If $X \subseteq K^n$ is definable, then $X \cap C^n$ is definable in $(C, +, \cdot)$.*

Proof By quantifier elimination and the triviality of δ on C, $X = V \cap C^n$ where $V \subseteq K^n$ is definable in $(K, +, \cdot)$.

By stability of ACF, X is definable in $(C, +, \cdot)$.

Corollary 21 *C is strongly minimal.*

One invariant of $K \models$ DCF is the transcendence degree of the field of constants.

Differential Transcendentals

Let $k \subseteq K$. We say $a_1, \ldots, a_n \in K$ are *differentially independent* over k, if $I_{\overline{a}} = \{0\}$.

The *differential transcendence degree* of K/k, $\mathrm{td}_\delta(K/k)$, is the maximal cardinality of a differential independent set.

The differential transcendence degree over $\mathbb{Q}$ is a second invariant of K.

At one point it was conjectured that $(\mathrm{td}_\delta(K/\mathbb{Q}), \mathrm{td}(C))$ determined K up to isomorphism.

Linear Equations

Let $K \models$ DCF, $a_0, \ldots, a_n \in K$ and let

$$f(X) = a_n X^{(n)} + \ldots + a_1 X' + a_0 X.$$

Using the usual theory of linear ODEs we prove:

Proposition 22 *The solution set to $f(X) = 0$ is an n-dimensional vector space over C.*

Corollary 23 *The formula $f(x) = 0$ has Morley rank n.*

Corollary 24 *The type of a differential transcendental has Morley rank at least ω.*

Rank and Order

Proposition 25 *If $g \in K\{X\}$ has order n, then the formula $g(x) = 0$ has Morley rank at most n.*

Corollary 26 *The type of a differential transcendental has Morley rank exactly ω.*

The equation $XX'' = X'$ has order 2 but Morley rank 1.

Strongly Minimal Sets

Recall that a definable set $X \subseteq \mathbb{K}^n$ is *strongly minimal* if is infinite, but has no infinite coinfinite definable subset.

- What are the strongly minimal sets in DCF?

The first natural example is the constant field C. Are there any others?

Recall that strongly minimal sets come equiped with a pre-gometry given by algebraic closure.

The Zilber Trichotemy

- A strongly minimal set X is *trivial* if

$$\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(a)$$

for all $A \subseteq X$.

For example, a set with no structure and $(\mathbb{Z}, s)$ are trivial.

Modular Strongly Minimal Sets

• A strongly minimal set is *modular* if whenever $a \in \mathrm{cl}(B, c)$ there is $b \in \mathrm{cl}B$ such that $a \in \mathrm{cl}(b, c)$.

For example, $(V, +)$ a $\mathbb{Q}$-vector space. $\mathrm{cl}(A) = \mathrm{span}(A)$. If

$$a = \sum m_i b_i + nc$$

let $b = \sum m_i b_i$.

Theorem 27 (Hrushovski) *Nontrivial modular strongly minimal sets are nonorthogonal to an interpretable strongly minimal group.*

In modular groups every definable subset of G^n is a Boolean combination of cosets of definable subgroups.

Nonmodular strongly minimal sets

Algebraically closed fields are nonmodular strongly minimal sets. If $a_0, \ldots, a_{n-1}$ are algebraically independent and x is a solution to

$$x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$

then x is not algebraic over any subfield of $\mathbb{Q}(a_0, \ldots, a_{n-1})$ of transcendence degree less than n.

Zilber conjectured that algebraically closed fields were the only nonmodular strongly minimal sets. Hrushovski showed this is false in general.

Zilber's Principle In natural settings the only nonmodular strongly minimal sets "are" algebraically closed fields.

Strongly Minimal Sets in DCF

Theorem 28 (Hrushovski–Sokolović) *In* DCF *if* X *is a nonmodular strongly minimal set, there is a definable finite-to-one* $f : X \to C$.

In particular X is nonorthogonal to C (we will define this later)

Trivial Strongly Minimal Sets

There are trivial strongly minimal sets in DCF.

Theorem 29 (Rosenlicht,Kolchin,Shelah) *The equations*

$$X' = X^3 - X^2 \text{ and } X' = \frac{X}{X+1}$$

define trivial strongly minimal sets.

Indeed these equations define infinite sets of indiscernibles ($\pm$ finitely many points).

Are these sets useful for many model constructions? Yes, for $\kappa \geq \aleph_1$. But in the countable case they always have dimension $\aleph_0$.

Conjecture 30 *In* DCF *any trivial strongly minimal set is* $\aleph_0$*-categorical.*

Are there nontrivial modular strongly minimal sets?