.

ERGODIC THEORY OF TRANSLATION SURFACES

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1. Three definitions of translation surface or flat surface and examples

In this survey article we describe the ergodic theory of flows on translation surfaces. We relate this theory to the dynamics of the $\mathrm{SL}(2,\mathbb{R})$-action on the moduli space of translation surfaces. We describe recent results on the diagonal subgroup also known as the Teichmüller geodesic flow and results on the unipotent flow.

There is considerable overlap of material here with the survey article [22] as well as with the survey article of T.Schmidt and P.Hubert in this volume. ([11]).

We are going to give three (equivalent) definitions of translation surface. Equivalently, these will be called flat surfaces with trivial linear holonomy or just flat surfaces. The first definition is via charts. The second definition is the most geometric and is by glued polygons. The third definition is complex analytic. They arise as the flat structure associated to a holomorphic 1-form on a Riemann surface. We will indicate (but not provide a complete proof) their equivalence.

Let M be a closed topological surface, of genus $g \geq 1$.

Definition 1. A translation surface consists of a a finite set of points (the singularity set) $\Sigma = \{x_1, x_2, \ldots, x_m\}$ and an open cover of $M - \Sigma$ by sets $\{U_\alpha\}$ together with charts $\phi_\alpha : U_\alpha \to \mathbb{R}^2$ such that for all α, β, with $U_\alpha \cap U_\beta \neq \emptyset$,

$$\phi_\alpha \phi_\beta^{-1}(v) = v + c.$$

At each singular point the surface has a $2\pi c$ cone singularity.

Specifically, since the Euclidean metric on the plane is preserved by translations, the notion of direction and parallel lines makes sense on the complement of the singularity set. In fact we get a metric ds, by pulling back the Euclidean metric on the plane via these coordinate charts. In this metric geodesics that do not go through singularities

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are straight lines in a fixed direction, and such geodesics never intersect themselves, except possibly to close up.

Definition 2. For each direction θ and each non-singular point p define the flow $\phi_t(p)$ to be the point obtained after moving in the direction θ for time t, starting at p.

The flow $\phi_t : X \to X$ preserves the natural Euclidean measure (normalized to have total area one) on the surface. It is defined for all time only on the set of full measure of points that do not run into a singularity either in forwards or backwards time. A major part of these notes will be devoted to describing ergodic properties of this flow.

At each singular point we write $ds^2 = dr^2 + (crd\theta)^2$, a conical singularity written in polar co-ordinates. We require c to be a positive integer. For $c = 1$, we simply recover the Euclidean metric. If $c > 1$, we have a $2\pi c$ cone angle. We can think of a point with a $2\pi c$ cone angle as $2c$ Euclidean half discs glued together along half lines — For the case $c = 2$ see figure 1.

The total angle around each vertex is required to be $2\pi c$, c a positive integer.

Geodesics can change direction if they go through a singular point. A pair of straight lines through the singular point form a geodesic if the angle between them is at least 2π.

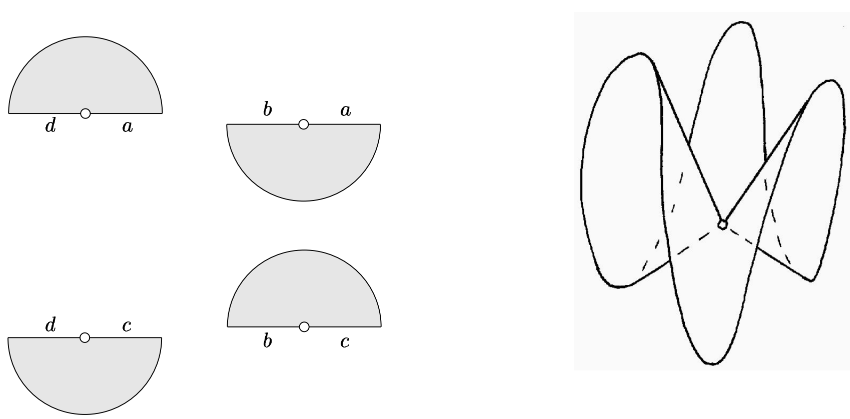


FIGURE 1. Flat surface near a singularity

Metrically we can also describe these as flat surfaces with conical singularities of the above type and trivial linear holonomy. The latter means that parallel transport of a vector around a path missing the singularities comes back to the same vector. This explains why these surfaces are also called *flat surfaces with trivial linear holonomy.*

Definition 3. A saddle connection is a geodesic joining two of the singularities with no singularities in its interior.

In each coordinate chart it is a straight line in the Euclidean metric. An oriented saddle conection determines a vector called the *holonomy vector* of the saddle connection.

It is a standard fact (see [26]) that between any two points there is a unique geodesic in any homotopy class. In particular, there is a unique geodesic joining any two singularities in each homotopy class. The geodesic is a union of saddle connections. We sketch an argument which says that the set of holonomy vectors of saddle connections is a discrete subset of $\mathbb{R}^2$. This fact is used in the proof of Veech dichotomy and is implicit in any discussion of counting problems. (See the articles of Hubert-Schmidt and Eskin in this volume). Another sketch is given in [11].

Lift the metric to the universal cover, to give a complete metric on the hyperbolic plane. Fix a Dirichlet fundamental domain $\mathcal{F}$ for the action of the covering group. Let D be its diameter. A ball of radius $R+D$ about a base point in $\mathcal{F}$ intersects only a finite number of translates of $\mathcal{F}$. Any saddle connection of length at most R must lift to a saddle connection joining a singularity in $\mathcal{F}$ to a singularity in a ball of radius $R+D$. There are only finitely many such points and hence only finitely many such saddle connections.

As mentioned in the article of P.Hubert and T.Schmidt ([11]), the $\mathrm{SL}(2,\mathbb{R}^)$ action on flat surfaces can be defined as postcomposition with charts- we discuss this action in more detail later.

Our next definition of a flat surface is the most geometric and often useful when we need to visualize these objects.

Definition 4. A *translation surface* is a finite union of Euclidean polygons $\{\Delta_1, \Delta_2, \ldots, \Delta_n\}$ such that

- the boundary of every polygon is oriented so that the polygon lies to the left
- for every $1 \le j \le n$, for every oriented side s_j of Δ_j there is a $1 \le k \le n$ and an oriented side s_k of Δ_k so that s_j and s_k are parallel and of the same length. They are glued together in the opposite orientation by a parallel translation. (Note that this means that as one moves along a glued edge, one polygon appears to the left, the other to the right.

It follows that the total angle around each vertex is $2\pi c$, c a positive integer. Note that when we speak of Euclidean polygons we fix their embedding into a standard Euclidean plane up to a parallel translation.

In particular we distinguish two polygons obtained one from the other by a nontrivial rotation. Another way to say the same thing is that we equip a translation surface with a choice of vertical direction.

The rational billiard table examples (see [11]) yield surfaces of this form. However, note that in general we do not require the angles of the polygons to be rational, as is the case for the billiards. The best way to see this definition is by considering a few examples:

The first example is a regular octagon with opposite sides identified. This gives rise to a surface of genus two with one singularity of angle 6π (all the vertices collapse to one point, yielding an angle $8(3\pi/4)$). This is an example of a Veech surface which satisfies the Veech dichotomy (see [11]). Namely for any direction, either all the orbits in that direction are closed or equally distributed.

Another example also gives a surface in genus two but which turns out to have very different ergodic properties.

Consider a $1 \times 1/2$ rectangle with a barrier of length $\alpha/2$ hanging down from the top of the rectangle at its midpoint: that is, a vertical line segment from $(1/2, 1/2)$ to $(1/2, 1/2 - \alpha/2)$. Billiards in this polygon gives rise ([11]) to a surface with opposite sides identified(of side length two), with two slits of length α inside it (see Figure 2). The left side of the left slit is identified with the right side of the right slit, and the right side of the left slit is identified with the left side of the right one. The rectangle with slits yields a torus with two holes, when opposite sides are identified, and when the slits are glued, the result is a genus two surface. There are two singularities, each with a 4π cone angle coming from the endpoints of the glued slit.

In fact this example illustrates the definition by polygons. There are four generalized 7-gons, each of which has six vertex angles of $\pi/2$ and one angle 2π.

When α is rational, the surface is a Veech surface and the Veech group is a finite index subgroup of $\mathrm{SL}(2, \mathbb{Z})$. These are particular examples of arithmetic Veech surfaces.

Now in general, since the gluings of the polygons are realized by parallel translations, it is clear that a surface satisfying the definition by polygons satisfies the definition by charts. Conversely, one can show that a translation surface has a triangulation by geodesic triangles so a surface satisfying the first definition satisfies the second.

The third definition is complex-analytic.

Definition 5. A *translation surface* is given by a pair (X, ω) where X is a Riemann surface and ω is a holomorphic 1-form (Abelian differential) on X.

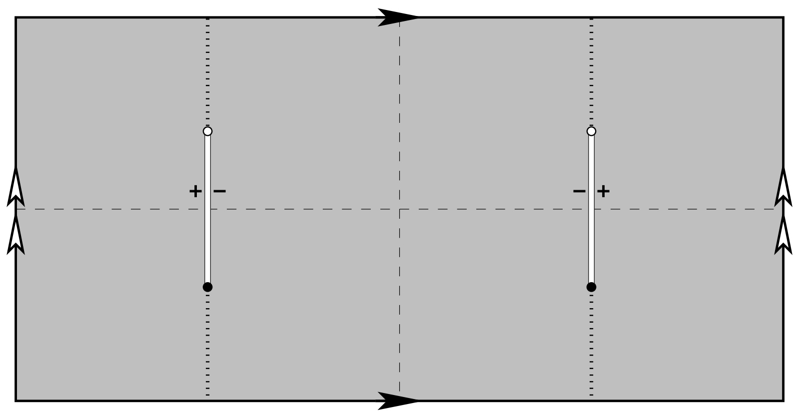


FIGURE 2. Slit torus example

Recall that this means that to each holomorphic chart z is assigned a holomorphic function $f(z)$ such that in an overlapping chart ζ with function $g(\zeta)$, the relation is

$$g(\zeta)\frac{d\zeta}{dz} = f(z).$$

In the article of P.Hubert and T.Schmidt ([11]) they show how to go from a pair (X, ω) to a collection of charts where the transition maps are translations, i.e., our first definition (zeroes of the 1-form correspond to singularities, etc.). Specifically in a neighborhood of a point p_0 which is not a zero of ω there are holomorphic coordinates z defined by

$$z(p) = \int_{p_0}^{p} \omega$$

which give $\omega = dz$. In an overlapping neighborhood similarly defined coordinates z' will satisfy

$$z' = z + c$$

so that the change of coordinates is a translation. At a zero of order k in appropriate coordinates

$$\omega = z^k dz = d(\frac{z^{k+1}}{k+1})$$

and so the surface is locally a $k+1$ fold cover over the complex plane. This means that the zero of order k gives rise to a singularity with cone angle $2\pi(k+1)$.

In this language the (affine) holonomy of a saddle connection β coincides with $\int_\beta \omega = \int_\beta dz$, where we have identified the complex numbers with $\mathbb{R}^2$, and so we can consider the holonomy to be a complex number (with real and imaginary parts) or as a vector.

To get from the first definition to this one, simply pull back the natural 1-form dz via the charts. This defines a holomorphic 1-form ω on the surface. The cone singularity gives rise to a zero of ω.

2. Spaces of translations surfaces and Riemann surfaces

For the rest of this article we will use the notation (X, ω) to refer to a translation surface.

A translation surface has three pieces of topological data: the genus, the set of zeros, and the multiplicity of the singularities. We can represent the topological data by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, where α_i denotes the order of the ith zero. It is classical and in any case follows from either an Euler characteristic argument or from the Gauss-Bonnet theorem that

$$\sum_{i=1}^{k} \alpha_i = 2g - 2.$$

For example, given the data $\alpha = (2)$, the surface has genus two with one singularity with cone angle 6π. Given $\alpha = (1, 1)$, the genus is still two, but with two singularities, each of cone angle 4π.

We want to consider the space of all translation surfaces with fixed topological data. For this, we need to define an equivalence relation on such surfaces.

We say that two surfaces are equivalent, if there is an orientation preserving isometry from one to the other preserving the given preferred direction. This definition distinguishes between polygons that differ by rotations. In the complex analytic definition, it distinguishes between (X, ω) and $(X, e^{i\theta}\omega)$.

Definition 6. Given topological data α, we define the moduli space $\mathcal{H}(\alpha)$ as the space of translation surfaces with topological data α together with a choice of direction under the above equivalence relation. If we add the condition that the surfaces have area 1 we denote the resulting space by $\mathcal{H}_1(\alpha)$. These moduli spaces are also called strata.

On the other hand for any genus g we may define the Riemann moduli space $\mathcal{M}_g$ as the space of Riemann surfaces of genus g up to conformal equivalence. Every closed Riemann surface of genus $g > 1$ carries a metric of constant curvature -1 in its conformal class, so $\mathcal{M}_g$ is also the space of hyperbolic metrics on a surface up to equivalence by isometries.

For each $\alpha = (\alpha_1, \dots, \alpha_k)$, define g by $2g - 2 = \sum_{i=1}^{k} \alpha_i$. There is then a map

$$\pi : \mathcal{H}(\alpha) \to \mathcal{M}_g$$

which sends (X, ω) to X. The map only remembers the complex structure on the surface defined by the Abelian differential.

As a main motivating example, let us consider the space of tori with specified directions, i.e., $\mathcal{H}(\emptyset)$. Recall that while two tori differing by a rotation are identical as metric spaces, the vertical direction on each torus is distinct, so we do not consider them the same point— as opposed to the moduli space of Riemann surfaces $\mathcal{M}_1$ where these are the same point.

The space of tori $\mathcal{H}(\emptyset)$ can also be viewed as the space of unit volume lattices in $\mathbb{R}^2$ (together with a specified direction), which is identified with the symmetric space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ (if one ignores the direction, we get instead $\mathbb{H}^2/\mathrm{SL}(2,\mathbb{Z})$, the moduli space of tori $\mathcal{M}_1$).

In general the moduli spaces $\mathcal{H}(\alpha)$ are not necessarily connected, though each has no more than three connected components. The components have been classified by Kontsevich and Zorich [16].

If we allow reflections as well as translations in gluings (or equivalently, allow transitions to be of the form $z \mapsto \pm z + c$, we get quadratic differentials and the classification is different.

3. SL(2, ℝ)-action and invariant measures

Recall from the survey paper of Hubert and Schmidt ([11]) the $\mathrm{SL}(2,\mathbb{R})$-action. In the language of polygons, we can define the action as follows. Given a translation surface (X, ω) (i.e., a finite collection of polygons $\{\Delta_i\}$) and a matrix $A \in \mathrm{SL}(2,\mathbb{R})$, we define the translation surface $A \cdot (X, \omega)$ by the collection of polygons $\{A\Delta_i\}$. The gluing pattern is preserved since linear maps preserve parallel lines. One can check that the definition does not depend on how one represents the surface as a union of polygons.

In the language of complex analysis, the action of the rotation

$$r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is the same as multiplying the Abelian differential ω by $e^{i\theta}$.

The action of a matrix in $\mathrm{SL}(2,\mathbb{R})$ does not change the topological data of a flat surface. Thus, for each stratum $\mathcal{H}(\alpha)$, we have an $\mathrm{SL}(2,\mathbb{R})$-action. We are interested in defining a measure μ on $\mathcal{H}_1(\alpha)$ which is invariant under this action.

We do this by defining co-ordinates for this space, and then pulling back natural Lebesgue measure on the co-ordinate space. Our first co-ordinates will arise from our "visual" definition of the moduli space using polygons.

Suppose $\{\Delta_i\}$, a collection of polygons, represents a point in $\mathcal{H}(\alpha)$. It is obvious that there is some finite collection of sides $v_1, v_2, \ldots, v_N$ which determine the surface. For example, for a flat torus, the surface is determined by two sides v_1, v_2 of a parallelogram. For the surface to be in $\mathcal{H}_1(\emptyset)$ one has the further condition that the area determined by the polygon is *one*, which we denote by $v_1 \wedge v_2 = 1$. Another example is the octagon, for which we need four vectors (once a side is determined, so is its opposite).

These v_i yield local co-ordinates for $\mathcal{H}(\alpha)$ giving a map $\phi : \mathcal{H}(\alpha) \to (\mathbb{R}^2)^N$. We consider Lebesgue λ measure on $(\mathbb{R}^2)^N$, restricted to the hypersurface corresponding to the area 1 surfaces and define $\mu = \phi^*\lambda$. This measure is independent of the choice of co-ordinates and the way the surface is cut into polygons (in particular the number of polygons may change, but the number of sides neccessary to determine the surface does not).

A more formal way to see this definition, is by starting with the surface (X, ω), and its set of singularities Σ. Consider the relative homology group $H_1(X, \Sigma; \mathbb{Z})$. This is an $N = 2g + n - 1$ dimensional space, where n is the number of singularities. Fix a basis $\{\beta_1, \beta_2, \ldots, \beta_N\}$. Define coordinates for (X, ω) by $\{\int_{\beta_i} \omega\} \in \mathbb{R}^{2N}$. Once again consider Lebesgue measure on the image of this map, and pull it back to get a measure on $\mathcal{H}(\alpha)$. This is more easily seen to be invariant of choices— in particular, any change of basis is a determinant one matrix. For the same reason it is invariant under the $\mathrm{SL}(2, \mathbb{R})$-action

Returning to the torus, recall that the space of tori

$$\mathcal{H}_1(\emptyset) = \mathrm{SL}(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{Z}).$$

It has finite volume because $\mathrm{SL}(2, \mathbb{Z})$ is a lattice in $\mathrm{SL}(2, \mathbb{R})$. We can see this directly because the space of tori (without normalization) is simply the set of all pairs of non-colinear vectors $v_1, v_2 \in \mathbb{R}^2$. This space clearly has infinite Lebesgue measure. When restricting to the area one tori, the space is non-compact, since the vector v_1 can be arbitrarily short. However the space $\mathcal{H}_1(\emptyset)$ has finite volume because of the easily proven fact:

$$\mu\{(v_1, v_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : |v_1 \wedge v_2| \leq 1\} < \infty.$$

A similar computation explains the finite measure in general. We will sketch this explanation. In A.Eskin's survey article ([8]) he explains how to actually compute the measures of these spaces.

On any flat surface (X, ω), consider a closed geodesic in some direction which doesn't hit any singularities. Then there is a cylinder, containing this curve, which is filled with closed curves, parallel of the same length. If we make it as large as possible, it is called a metric cylinder. If $g > 1$, the boundary of the metric cylinder is a union of saddle connections. It turns out that for each genus g, there is a universal constant $C(g)$ such that if $\operatorname{diam}(X, \omega) \geq C(g)$, there is a metric cylinder on the surface such that the distance h across the cylinder satisfies $h \sim \operatorname{diam}(X, \omega)$; that is, they are comparable up to a definite factor.

Since the measure is defined by the holonomy along saddle connections, the measure of the part of moduli space corresponding to surfaces of diameter at most $C(g)$ is finite. On the part of the moduli space consisting of surfaces with large diameter, the above consideration says that these (area 1) surfaces have cylinders with small circumference and large distance across them. We can take as part of the basis for the homology a curve parallel to the cylinder with holonomy v_1 and a curve across the cylinder with holonomy vector v_2. But recalling that

$$\mu\{(v_1, v_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : |v_1 \wedge v_2| \leq 1\} < \infty,$$

we have that the measures of these "cusps" are finite, and thus we have the following theorem. Complete proofs can be found in [28] and [21].

Theorem 1. *For each stratum $\mathcal{H}_1(\alpha)$, $\mu(\mathcal{H}_1(\alpha)) < \infty$.*

4. Ergodicity of flows defined by translation surfaces

In this section we begin the discussion of the properties of the flow ϕ_t defined for each direction θ.

To avoid the problem of measures which are concentrated on the singular set, we consider only measures supported on the *punctured surface* $X - \Sigma$.

The first notion is purely topological. We will say that the flow in direction θ is minimal if there are no closed curves in direction θ. Equivalently, other than a finite number of saddle connections, every orbit that does not run into a singularity is dense, and if an orbit runs into a singularity in forward (resp. backwards) time then it is dense in backward (resp. forward) time.

Recall that a flow is ergodic if any invariant set has measure zero or measure one. Let ν be surface area. In this case the Birkhoff ergodic theorem states that for $f \in L^1(X,\nu)$, and for almost all p,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(\phi_t(p))dt = \int_X f d\nu.$$

If this convergence holds for every point p, and every continuous function f, the flow is said to be uniquely ergodic. This is equivalent to saying that the measure ν is the unique normalized flow-invariant measure on $X \setminus \Sigma$.

For motivation, we once again turn to the case of the torus $\mathbb{R}^2/(\mathbb{Z}\oplus\mathbb{Z})$. If the direction θ has rational slope, then every orbit is closed. On the other hand, if θ has irrational slope, then the flow is minimal, and moreover by the classical theorem of Weyl, the flow is uniquely ergodic.

However, even in the case of the torus there is a flow constructed by Furstenberg [10], which is minimal, but not uniquely ergodic. For a general treatment of the subject of nonunique ergodicity, see Section 14.5 of the book [13] and Sections 12.3 and 12.4 of [12].

We now want to exhibit a minimal non-uniquely ergodic example on a translation surface of genus 2. Veech [27] considered the following dynamical system. Take a pair of unit circles and mark off a segment of length β on each circle in the counterclockwise direction with one endpoint at $(1,0)$. Start on one circle and rotate counterclockwise by angle θ until the point lands in the segment. Then switch to the corresponding point on the other circle, rotate by θ until the orbit lands in the segment again, switch back to the first circle and so forth. Veech showed that for any irrational θ with unbounded partial quotients in its continued fraction expansion, there are irrational β so that the dynamical system is minimal, but not uniquely ergodic. What happens is that sets of orbits of positive measure spend asymptotically more than half their time on one circle and less than half the time on the other.

This dynamical system can be seen to be equivalent to the billiard flow on the billiard table with a slit described in Figure 2. Recall, it was given by a rectangular $1\times 1/2$ table with a slit of length $\alpha/2 = (1-\beta)/2$ hanging down from the midpoint of the top side. The surface (X,ω) associated to has genus two, with two singular points, each with angle 4π. It is formed from a 2×1 rectanagle with a pair of slits and appropriate identifications. Now take two circles in the vertical direction. The first follows one side of the slit and then a vertical segment of length β joining the two singularities and which passes through the point $(1/2,1)\sim(1/2,0)$. The second follows the other slit and passes through $(3/2,1)\sim(3/2,0)$. The first return map to those circles of a flow in direction θ, gives the dynamical system described by Veech.

In this section we show how to build these minimal nonergodic examples geometrically. Additional details can be found in [22].

Theorem 2. *When β is irrational there are uncountably many directions θ such that the flow in direction θ is minimal and not ergodic.*

In order to prove the theorem we will view the surface (X, ω) differently. Cut the surface along the pair of dotted vertical lines that go from P to Q in Figure 2. The result is a pair of tori each with a hole consisting of the pair of vertical lines. Each torus then can be thought of as a standard square tori T slit along a segment w_0 going from $p_1 = (0,0)$ to $p_2 = (0, \alpha)$. The surface (X, ω) is reformed by gluing the tori together pairwise along w_0. The union of this pair of slits partitions the surface (X, ω) into two pieces A_{w_0} and B_{w_0} of equal area.

We will look for other slits w' defining (X, ω). That is, we want another pair of saddle connections w' joining p_1 to p_2 so that their union also splits (X, ω) into two pieces of equal area. The new slit w' will cut the original slit, and so the new partition $A_{w'} \cup B_{w'}$ of (X, ω) will differ from the original.

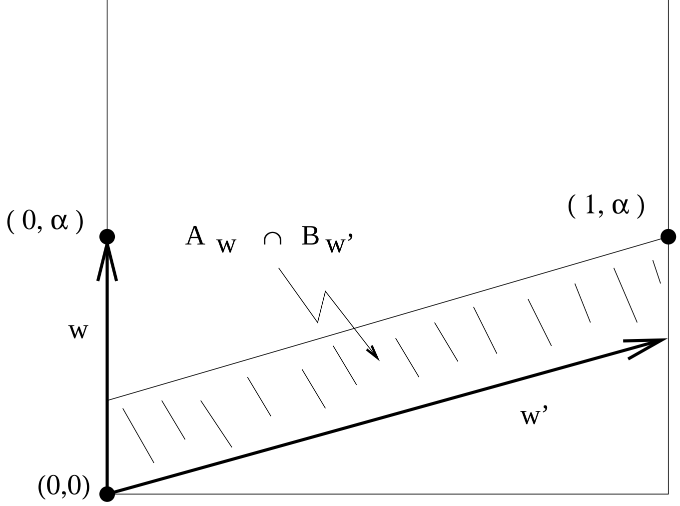
On the universal cover $\mathbb{R}^2$ of the torus T, the new slit w' is a line from $(0,0)$ to $(m, \alpha + n)$ for some integers m, n. The condition that the pair of slits w' divide (X, ω) is equivalent to the condition that m and n are both even. Equivalently, on T, w' intersects w an odd number of times in its interior. It is also equivalent to saying that w and w' are homologous $mod(2)$ on T. Then the change in partition on (X, ω) measured by $c = (A_w \cap B_{w'}) \cup (B_w \cap A_{w'})$ is a union of an even number of parallelograms with sides on w and w' (here thought of as vectors). Thus the area of c is bounded by $2|w \times w'|$. (See Figure 3).

Figure 3. Sheet Interchange.

The main step in the proof is to find uncountably many sequences $\{w_n\}$ of vectors determining partitions $\{A_n, B_n\}$ such that

$$\sum_{n=1}^{\infty} \nu(A_{n+1} \triangle A_n) < \infty.$$

Here ν is area on the surface. The directions of any sequence of these vectors will converge to a limiting direction θ. Assuming that such sequences can be found, we show first that the flow in direction θ is not ergodic.



Let

$$A_\infty = \liminf A_n = \{x : \exists N : x \in A_n, \forall n \geq N\}$$

and let B_∞ be defined similarly. The condition $\sum \nu(A_{n+1} \Delta A_n) < \infty$ and the Borel-Cantelli lemma imply

$$\nu\{x : x \in A_n \triangle A_{n+1} \text{ infinitely many } n\} = 0$$

so $\nu((X, \omega)\backslash(A_\infty \cup B_\infty) = 0$. By symmetry, we get $\nu(A_\infty) = \nu(B_\infty) = 1$.

Now we claim that A_∞ is a.e. invariant under the flow $\{\phi_t\}$ in direction θ, i.e.,

$$\nu(\phi_t(A_\infty) \triangle A_\infty) = 0$$

for all times t. Assume that the claim is false so that there is some $\delta > 0$ and t_0 such that

$$\nu(\phi_{t_0}(A_\infty) \triangle A_\infty) \geq \delta > 0. \tag{1}$$

Without loss of generality we may assume that the limiting direction is vertical. It follows from the summability condition on the areas that

$$h_n \to 0,$$

where h_n is the horizontal component of the holonomy of w_n. (Recall the holonomy is a vector). Pick n such that

$$\nu(A_n \triangle A_\infty) < \delta/8 \tag{2}$$

and

$$t_0 h_n < \delta/8 \tag{3}$$

The flow invariance of the measure, (2), (1) and the triangle inequality imply

$$\nu(\phi_{t_0}(A_n) \triangle A_n) > \delta - 2\delta/8 = 3\delta/4.$$

Thus at time t_0 at least $3\delta/8$ of the measure of A_n flows to its complement. However if a point crosses w_n, the boundary of A_n at time t_0 of the flow, its vertical distance to w_n must be at most t_0. The set of points whose vertical distance to w_n is at most t_0 lie in a parallelogram whose sides are w_n and a vertical segment of length t_0. The area of a such a parallelogram is $h_n t_0 < \delta/8$ by (3). We have a contradiction, proving the claim.

From the claim there is an argument that says there is a set A' with $\nu(A' \triangle A_\infty) = 0$ such that A' is ϕ_t invariant. This implies that the flow in direction θ is not ergodic completing the first step.

Let us return to finding an uncountable number of sequences of w_n satisfying the condition $\sum_n \nu(A_{n+1} \triangle A_n) < \infty$. We wish to show that the limiting directions are distinct for then we will have constructed an uncountable number of nonergodic directions. This will guarantee an uncountable number of *minimal* nonergodic directions, since in a nonminimal direction there is a saddle connection, and there are only countably many saddle connections.

Fix any sequence ρ_n such that $\sum \rho_n < \infty$. We will build an infinite directed tree with each "parent" vertex w_j leading to a pair of "child" vertices w_{j+1}. At level j there will be 2^j vertices. Each vertex will correspond to a pair (p_j, q_j) which will yield a slit joining $(0,0)$ to $(p_j, q_j + \alpha)$.

Let $w_0 = (0, \alpha)$ and suppose inductively we have found 2^j vectors $w_j = (p_j, q_j + \alpha)$ at stage j. For any pair (p_j, q_j) form the ratio $\frac{q_j+\alpha}{p_j}$, the slope of the slit. Define δ_j to be the minimum distance between the slopes of any pair of distinct w_j at level j. For any (p_j, q_j) we will look for integer solutions r, s of

$$2|p_j s - (q_j + \alpha) r| < \rho_j.$$

Since α is irrational, so is each $\frac{p_j}{\alpha + q_j}$, and so there are infinitely many coprime solutions (r, s) of the above inequality. Choose any two sets of solutions (r_j, s_j) so that

$$\frac{\rho_j}{(q_j + \alpha)(q_j + \alpha + 2s_j)} \leq \delta_j/4$$

and for each, set $p_{j+1} = p_j + 2r_j; q_{j+1} = q_j + 2s_j$ and then

$$w_{j+1} = (p_{j+1}, q_{j+1} + \alpha).$$

A direct calculation also shows that

$$\nu(A_{j+1} \triangle A_j) < 2|w_{j+1} \times w_j| \leq 4|p_j s_j - (q_j + \alpha) r_j| < 2\rho_j,$$

giving the desired summability condition.

The proof will be complete when we show that the directions of a sequence of w_j converge and distinct sequences give distinct limiting directions. A calculation shows

$$|\frac{p_j}{q_j+\alpha}-\frac{p_{j+1}}{q_{j+1}+\alpha}|\leq \delta_j/4;$$

that is, the distance between the slopes of a parent and child is at most $\delta_j/4$. The triangle inequality says that the distance between slopes of children of the same parent is at most $\delta_j/2$ and so $\delta_{j+1}<\delta_j/2$.

Since the distance between the slopes of a parent and a child goes to 0, the slopes of the vertices w_j of any geodesic in the tree converges.

We finally show that limits of slopes of w_j along distinct geodesics are different. For suppose two geodesics l_1, l_2 are different for the first time at level j with vertices w_j^1, w_j^2 (thought of as parents). Let θ_1, θ_2 be the limiting slopes of the vertices along l_i. Since the slope of each child at level $m+1$ is within $\delta_m/4$ of the slope of the parent at level m, summing the geometric series says that the difference of the slope of θ_i and the slope of w_j^i is smaller than $\delta_j/2$. Since the slopes of w_j^1, w_j^2 are at least δ_j apart, we must have $\theta_1\neq\theta_2$.

5. Further results on unique ergodicity

The above construction was generalized recently [7] to show that on any translation surface in genus 2 which is not a Veech surface there is some direction for which the flow is minimal and not ergodic. A natural question is which translation surfaces have minimal nonergodic directions. Veech surfaces do not, due to the Veech Dichotomy (which was proved in [11]).

The existence of minimal nonergodic directions led to work about their prevalence. For each (X,ω), define $NE(X,\omega)$ to be the set of $\theta\in[0,2\pi)$ such that the flow ϕ_t in the θ direction is not ergodic. Equivalently, $NE(X,\omega)$ is the set of θ such that the flow ϕ_t in the vertical direction of $e^{i\theta}\omega$, is not ergodic. In [14] it was shown that the Lebesgue measure of $NE(X,\omega)$ is 0. The idea of the proof is the following. Let

$$r_\theta=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

be the rotation group in $SL(2,\mathbb{R})$ and let

$$g_t=\begin{pmatrix}e^t & 0\\ 0 & e^{-t}\end{pmatrix}$$

be the diagonal group.

The action of the diagonal subgroup is known as Teichmüller geodesic flow, since images of these orbits under the projection π to the Riemann moduli space $\mathcal{M}_g$ are geodesics in the Teichmüller metric on $\mathcal{M}_g$.

One shows that for large t most points on the circle $g_t r_\theta(X,\omega)$ are not near the cusp in moduli space. This is combined with the following Theorem ([19]) (whose proof is sketched in the next section).

Theorem 3. *Suppose (X,ω) is a translation surface. Suppose the flow in the vertical direction is not uniquely ergodic. Then $X_t = \pi g_t(X,\omega)$ eventually leaves every compact set in $\mathcal{M}_g$. That is, the Teichmüller geodesic associated to (X,ω) is divergent.*

We note that this theorem is also one of the ingredients in the proof of the Veech dichotomy ([11]).

We continue with some remarks about Theorem 3. It is a basic fact that the moduli space $\mathcal{M}_g$ is non-compact. The reason is that one may have a sequence of surfaces and curves on those surfaces whose lengths in the hyperbolic metric (assume $g > 1$) go to zero. Such surfaces cannot converge to a compact surface. On the other hand it is a basic fact [24] that this is the only way to leave compact sets in $\mathcal{M}_g$. Namely, if X_n is a sequence that eventually leaves every compact set, then there is a sequence of curves γ_n such that the length of γ_n (in the hyperbolic metric on X_n) goes to zero.

It is easy to see that if there is a closed leaf in the vertical direction (in particular, the flow is not minimal) of ω, then X_t eventually leaves every compact set of $\mathcal{M}_g$. Namely, since g_t shrinks lengths in the vertical direction by a factor of e^t, the length in the flat metric of $g_t(X,\omega)$ of any closed vertical leaf goes to 0. If there were a subsequence of X_t converging to a compact surface X_0, there would be a further subsequence of $g_t(X,\omega)$ converging to some (X_0,ω_0). This (X_0,ω_0) would assign 0 length to a closed curve, which is impossible.

In the minimal case there are no closed vertical leaves. This means that under the flow g_t the length of any *fixed* curve γ must go to infinity in the flat metric of $g_t(X,\omega)$ as $t \to \infty$. What the Theorem 3 says is that there is a sequence of distinct simple closed curves γ_n such that for any $\varepsilon > 0$, for sufficiently large t there is a curve $\gamma_n = \gamma_n(t)$ such that the length of γ_n in the flat metric of $g_t(X,\omega)$ is smaller than ε.

The measure 0 result was generalized by Veech [29] to Borel probability measures on $[0,2\pi)$ that satisfy certain growth conditions. Normalized Cantor-Lebesgue measure on the Cantor middle third set is an example of such a measure.

Further work concerns the Hausdorff dimension of $NE(X,\omega)$. In [21] it was shown that for each component of each stratum (other than several low dimensional exceptional cases covered by the Weyl theorem) there is a $\delta > 0$ such that for μ a.e. (X,ω) in the component, $NE(X,\omega)$ has Hausdorff dimension δ. The construction of these non-ergodic directions on a generic surface uses a method similar to that described in the Veech example.

In [19] it was shown that the Hausdorff dimension of $NE(X,\omega)$ is always bounded by $1/2$. The proof of this result is also based on Theorem 3 and estimates on counting saddle connections. As discussed above, Theorem 3 says that for $\theta \in NE(X,\omega)$, for all large times there is a short saddle connection. Typically there may be many such short intersecting saddle connections at any given time, and this collection of short saddle connections change with time. However one can make a choice of a short saddle connection in this collection at any time so that successive choices are *disjoint.* Thus the proof amounts to estimating the size of the set of angles θ such that along the orbit $g_t r_\theta(X,\omega)$ there is a sequence of saddle connections that become successively short, and such that each is disjoint from its predecessor. This problem can be reduced to counting problems for saddle connections. There is an estimate [20], [9] which says that the number of saddle connections of length T grows at most quadratically in T, and another estimate which says for fixed saddle connection of length l, the number of *disjoint* saddle connections of length at most L grows roughly linearly in $\frac{L}{l}$. The comparison of linear growth and quadratic growth accounts for the dimension $1/2$.

Y. Cheung [4] has shown that this estimate is sharp. Specifically, suppose an irrational α satisfies a Diophantine condition that for some $s > 0$ there are no fractions p/q that satisfy

$$|\alpha - p/q| < \frac{1}{q^s} \tag{4}$$

Then for the rectangular table with a slit of length $\alpha/2$ described in section 4, $\dim NE(X,\omega) = 1/2$.

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6. Boshernitzan's Theorem and sketch of proof of Theorem 3

Before we turn to the proof of the Theorem 3, we state an alternative criterion for divergence, given by Boshernitzan [3], formulated in terms of interval exchange maps.

If we consider the flow in the vertical direction, then by considering the first return to a piece of horizontal transversal I, we obtain an interval exchange map T, whose discontinuity points correspond to leaves that run into singular points before returning to I. Suppose T exchanges k intervals. Let $T^{(n)}$ be the nth iterate of T. It will be an interval exchange on approximately kn intervals. Let m_n denote the length of the shortest of these intervals.
Theorem 4. [3] *If T is not uniquely ergodic, then $nm_n \to 0$.*

This is slightly weaker than Theorem 3 as it only guarantees divergence of the geodesic in a stratum, whereas Theorem 3 guarantees divergence in moduli space.

To explain the difference, in a stratum $\mathcal{H}(\alpha_1, \dots, \alpha_k)$ where $k > 1$, one may leave compact sets by a sequence of translation surfaces such that a pair of singularities come close together. If no closed curve becomes short then one stays in a compact set of $\mathcal{M}_g$.

Now the reason that the criterion $nm_n \to 0$ implies divergence in the stratum is as follows. The discontinuity points on I of the interval exchange $T^{(n)}$ are points of the form $T^{(-l)}(x)$ for $l \leq n$, where x is a discontinuity point of T. Hence each interval yields a saddle connection crossing it such that the vertical component of its holonomy has length $O(n)$. However, the short interval yields a saddle connection γ_n such that in addition, the horizontal component of its holonomy is $O(m_n)$. Since $nm_n \to 0$, for some interval of times t, the length of both the vertical and horizontal component of γ_n in the metric of $g_t(X, \omega)$ are small. One can show that for any time t there is such a γ_n.

We are now ready to sketch the proof of Theorem 3. Let $\{\phi_t\}$ denote the vertical flow of ω. As explained above, we can assume it is minimal but not uniquely ergodic. The set of invariant probability measures for ϕ_t is a finite dimensional convex set and the extreme points are mutually singular ergodic measures. For sake of argument assume there are exactly two ν_1, ν_2. (The general case is almost the same). Since ν_i is invariant under the vertical flow, for any horizontal interval I, the measure ν_i decomposes into

$$\nu_i = \mu_i \times dy$$

where μ_i is an ergodic measure on I invariant under the first return map ψ and y is the coordinate in the vertical direction. Since the ν_i are mutually singular, so are the μ_i and I can be chosen so that

$$\mu_1(I) \neq \mu_2(I).$$

Let χ_I be the indicator function of I. We say x is generic for μ_i if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\chi_I(\psi^n(x))=\mu_i(I).$$

Thus if x_i is a generic point of μ_i then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\chi_I(\psi^n(x_1))\neq\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\chi_I(\psi^n(x_2)) \tag{5}$$

It is a fact that μ_i almost all points of I are generic for μ_i.

We argue by contradiction. If the theorem is false, there is a sequence of times $t_n\to\infty$ and X_0 such that $X_{t_n}\to X_0\in\mathcal{M}_g$. Since the part of $\mathcal{H}_1(\alpha)$ that lies over a compact set of $\mathcal{M}_g$ is also compact, by passing to further subsequences, we can assume there is ω_0 an Abelian differential on X_0 such that $g_{t_n}(X,\omega)\to(X_0,\omega_0)$.

Let $x_i\in I$ be generic for $\mu_i, i=1,2$. We follow the image of x_i under the flow g_{t_n} and denote its image by $g_{t_n}(x_i)$. Note that each term in the sequence $g_{t_n}(x_i)$ is a point on a different Riemann surface X_i that evolves over time. Since the surfaces in question are compact, by passing to further subsequences we can assume that there exists $y_i\in X_0$ such that $g_{t_n}(x_i)\to y_i$. Since the surface X_0 is connected, and the set of generic points is of full measure for each μ_i and each of these sets is invariant under ϕ_t, it is not hard to show that we can pick x_i generic for μ_i such that y_1,y_2 lie on the same horizontal line h_1 of the limiting translation surface (X_0,ω_0). We will show that this contradicts (5).

Let l_1,l_2 short vertical lines of (X_0,ω_0) through y_1,y_2 and let R be the Euclidean rectangular box with vertical sides l_1 and l_2 and one horizontal side h_1. If l_1,l_2 are chosen small enough, R will have no singularities in its interior. Then the number of intersections of every connected horizontal line of (X_0,ω_0) with l_1 will differ with the number of its intersections with l_2 by at most 1.

For $i=1,2$, let $l_{i,n}$ denote bounded segments of the vertical leaf of $g_{t_n}(X,\omega)$ through $g_{t_n}(x_i)$ of equal length such that $l_{i,n}\to l_i$. Thus for n sufficiently large, with small error, every horizontal segment of $g_{t_n}(X,\omega)$ intersecting $l_{1,n}$ will intersect $l_{2,n}$. In particular, this is true for the long horizontal segment $g_{t_n}(I)$. Pulling back by g_{t_n}, we see that $g_{t_n}^{-1}(l_{i,n})$ are very long vertical leaves of the same length with respect to the original (X,ω) through x_1 and x_2, such that the ratio of the number of their intersections with I is approximately 1. But this is a contradiction to (5).

We describe how the Veech nonergodic example described earlier fits into the above theorem. The theorem says that for all large enough time t there is a curve $\gamma(t)$ on X_t with small hyperbolic length; the curve depends on the time. There is a sequence of dividing curves formed from slits $w_j = (p_j, q_j + \alpha)$ such that each w_j becomes short in hyperbolic length for a *finite* interval of time before it becomes long. It is a standard fact in hyperbolic geometry [2] that intersecting curves are never simultaneously short, so the intervals of times that different slits are short in hyperbolic length are disjoint. The slit curves therefore cannot account for all the short curves in the family X_t. What happens is that each slit divides the surface into a pair of tori, and before that slit curve becomes long, a (r_j, s_j) curve on each torus becomes short, where recall from the construction, $p_{j+1} = p_j + 2r_j, q_{j+1} = q_j + 2s_j$. This also occurs for a finite interval of time before becoming long. It stays short until the next slit becomes short, defining a new pair of tori and the process repeats.

One way to think about why such examples are impossible in genus one, is that on a torus there are no disjoint nonhomotopic curves.

7. Further results on dynamics of actions of subgroups of $\mathrm{SL}(2, \mathbb{R})$

The first set of results have to do with the Teichmuller flow. The converse to Theorem 3 is not true. It is possible to construct examples of divergent geodesics such that the flow ϕ_t in the vertical direction is uniquely ergodic ([6]). Another interesting line of work has concerned the rate of divergence of geodesics $g_t(X, \omega)$. Cheung [5] has recently shown that one can find geodesics with arbitrarily slow rates of divergence. Let $(X, \omega) \in \mathcal{H}(1,1)$ be a surface which is a double cover over the torus and which is not a Veech surface. (An example is given by the slit torus considered in Section 4 with irrational α). Then given any function $R(t) \to \infty$ there is a direction θ so that $\pi g_t(X, e^{i\theta}\omega)$ diverges in moduli space in $\mathcal{M}_g$, and such that

$$\tau(\pi g_t(X, e^{i\theta}\omega), \pi(X, e^{i\theta}\omega)) < R(t)$$

for all large t. Here $\tau(\cdot, \cdot)$ is the Teichmuller metric on $\mathcal{M}_g$.

He also showed that if α satisfies (4) and $e_0 > \max(2, s)$, then there is a Hausdorff dimension $1/2$ of directions $\theta \in NE(X, \omega)$ such that the sublinear rate of divergence

$$r_+(\theta) = \limsup_{t \to \infty} \frac{\log \tau(\pi g_t(X, e^{i\theta}\omega), \pi(X, e^{i\theta}\omega))}{\log t} \le 1 - \frac{1}{e_0}$$

holds. It would be interesting to know if slow rates of divergence in general implies unique ergodicity.

It is known [18], [28] that the flow g_t is ergodic with respect to the natural "Lebesgue" measure μ on each component of each stratum. For the principal stratum (all simple zeroes) this implies in particular that the projection to the Riemann moduli space of almost every geodesic is dense. Thus the set of cobounded geodesics $g_t(X,\omega)$; those geodesics whose projection to the moduli space remain in some compact set (depending on the geodesic) has measure 0. One can then ask about the Hausdorff dimension of the set of cobounded geodesics.

The case of $g = 1$ is classical. Suppose X is the standard square torus and ω is the 1-form $e^{i(\pi/2-\alpha)}dz$. The lines in direction α are vertical with respect to ω. The behavior of $g_t(X,\omega)$ in the moduli space $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ is determined by the continued fraction expansion of α. In particular, the orbit is cobounded iff α has bounded partial quotients. The set of these irrational numbers has measure 0 and Hausdorff dimension 1.

The result for general (X,ω) is recent work of Kleinbock and Weiss ([17]). They show that for any (X,ω), the set of $\theta \in [0,2\pi)$ such that $g_t(X,e^{i\theta}\omega)$ is cobounded has Hausdorff dimension 1.

Other intersting and important recent work in the dynamics in moduli space has been inspired by the dynamics of flows of subgroups of G acting on G/Γ, where G is a Lie group and Γ is a lattice subgroup. (See [15] for a survey).

The most important analogy is with the horocycle flow

$$h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

The Kleinbock-Weiss theorem is in turn based on work of Minsky and Weiss [23] on the horocycle flow. Let H denote this subgroup. It is a basic principle that g_t orbits can be quite wild. For example, the closure can be a Cantor set. On the other hand H orbits are constrained. In $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ every horocycle orbit is either closed or dense. It is a basic question in the subject to find all H orbit closures of points (X,ω) and all measures invariant under the action of H. (See the article by Eskin in these proceedings for more on this problem, which one can call the Ratner problem in moduli space).

Veech [29] showed that horocycle orbits do not diverge in the stratum. Minsky and Weiss gave a quantitative version of this result which shows that horocycle orbits spend most of their time in a compact set. To explain their result, introduce the terminology of $l(\gamma,(X,\omega))$ to represent the length of the saddle connection γ with respect to the metric of (X,ω), and K_ε the set of (X,ω) such that for every saddle connection γ, $l(\gamma,(X,\omega)) \geq \varepsilon$.

Theorem 5. *There are positive constants C, α, ρ_0 depending only on the topology of the surface such that if (X,ω), an interval $I \subset \mathbb{R}$ and $0 \leq \rho \leq \rho_0$ satisfy the condition*

for any saddle connection γ there is $s \in I$ such that $l(\gamma, h_s(X,\omega)) \geq \rho$,

then for any $\varepsilon > 0$

$$|\{s \in I : h_s(X,\omega) \notin K_\varepsilon\}| \leq C(\frac{\varepsilon}{\rho})^\alpha |I|.$$

Another recent result of Smillie and Weiss [25] classifies *minimal* sets for the horocycle flow. A set is minimal if it is invariant, closed, and there is no proper invariant closed subset. The authors first describe examples of minimal sets and then show that *every* minimal set is given by such an example. To describe the examples suppose in the horizontal direction all leaves of (X,ω) are closed so that (X,ω) decomposes into a union of cylinders each of which is swept out by closed horizontal leaves. Let $\mathcal{O} = \overline{H(X,\omega)}$. Then

- every $(Y,\sigma) \in \mathcal{O}$ admits a cylinder decomposition $C_1 \cup \ldots \cup C_r$ where each C_i is swept out by closed horizontal leaves.
- There is an isomorphism between $\mathcal{O}$ and a d dimensional torus where d is the dimension of the $\mathbb{Q}$ linear subspace of $\mathbb{R}$ spanned by the moduli of $C_1, \ldots, C_r$. The isomorphism conjugates the H-action on $\mathcal{O}$ with a one parameter translational flow.
- The restriction of the H-action to $\mathcal{O}$ is minimal.

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