Sparse hypergraphs with low independence number

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Abstract

Let $K_4^{(3)}$ denote the complete 3-uniform hypergraph on 4 vertices. Ajtai, Erdős, Komlós, and Szemerédi (1981) asked if there is a function $\omega(d) \to \infty$ such that every 3-uniform, $K_4^{(3)}$-free hypergraph H with N vertices and average degree d has independence number at least $\frac{N}{d^{1/2}}\omega(d)$. We answer this question by constructing a 3-uniform, $K_4^{(3)}$-free hypergraph with independence number at most $2\frac{N}{d^{1/2}}$. We also provide counterexamples to several related conjectures and improve the lower bound of some hypergraph Ramsey numbers.

1 Introduction

A k-uniform hypergraph H is a pair $H = (V, E)$, where V is the vertex set and $E \subset \binom{V}{k}$ is the edge set. We refer to the edge set of the hypergraph by H and the vertex set by $V(H)$. The degree of a vertex in $V(H)$ is the number of edges containing that vertex. An independent set in a hypergraph is a subset of $V(H)$ which contains no edge of H. The independence number of H, denoted $\alpha(H)$, is the maximum size of an independent set in H. Turán [18] showed that $\alpha(G) \geq \frac{N}{d+1}$ for any graph G with N vertices and average degree d. Spencer [17] extended Turán's result to hypergraphs by showing that for all $k \geq 1$ there is a c_k so that every $(k+1)$-uniform hypergraph H with average degree d satisfies $\alpha(H) \geq c_k \frac{N}{d^{1/k}}$.

When G is a graph, Turán's bound can be improved if G is forbidden from containing a fixed subgraph. Ajtai, Komlós, and Szemerédi [3] showed that if G is triangle-free,

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then

$$\alpha(G) \geq \frac{1}{100}\frac{N}{d}\log d. \tag{1}$$

Ajtai, Erdős, Komlós, and Szemerédi [1] subsequently showed that if $t \geq 4$ and G is K_t-free, then $\alpha(G) \geq c_t \frac{N}{d} \log\log d$.

Let H be a $(k+1)$-uniform hypergraph with N vertices and average degree d. Ajtai, Komlós, Pintz, Spencer, and Szemerédi [2] showed that there exists a positive constant c_k such that if H contains no 2, 3, or 4 cycles, then

$$\alpha(H) \geq c_k \frac{N}{d^{1/k}} \log^{1/k} d. \tag{2}$$

Applications of (2) have been found in number theory [3], discrete geometry [12], coding theory [14], and Ramsey theory [4]. Ajtai, Erdős, Komlós and Szemerédi asked if, like in the graph case, (2) could also be extended to other families of hypergraphs.

Question 1 (Ajtai-Erdős-Komlós-Szemerédi [1])**.** *Is there a function $\omega(d) \to \infty$ such that if a 3-uniform hypergraph H contains no $K_4^{(3)}$ (or even $K_4^{-(3)}$), then $\alpha(H) \geq \frac{N}{d^{1/2}}\omega(d)$?*

We construct hypergraphs which negatively answer this question, even in the $K_4^{-(3)}$ case. The construction is presented in Section 2. In Section 3, we generalize this construction to k-uniform hypergraphs and disprove several conjectures related to Question 1. We also discuss an application to hypergraph Ramsey numbers.

2 3-uniform construction

In this section, we answer Question 1 by constructing a $K_4^{-(3)}$-free, 3-uniform hypergraph H with independence number at most $2N/d^{1/2}$. The hypergraph H is constructed from the complete bipartite graph $K_{n,n}$ with vertex classes $[n]$ and $[n]$. The vertices of H correspond to edges in the graph, while the edges of H correspond to 3-edge paths which open in the increasing direction:

$$V(H) = [n] \times [n]$$
$$E(H) = \{\{ab, ac, db \in [n] \times [n] : c > b, d > a\}.$$

H is clearly 3-uniform, contains $N = n^2$ vertices, and has average degree $d = 3(n-1)^2/4$. For $v \in V(H)$, consider the link graph $L_v = \{uw : uvw \in E(H)\}$. The components of L_v are either stars (when v is in the role of ac) or a bipartite graph (when v is in the role of ab). $K_4^{-(3)}$, on the other hand, contains a vertex whose link graph contains a triangle, so H must be $K_4^{-(3)}$-free.

Let $S \subset V(H)$. If $|S| \geq 2n$, then the edges in $K_{n,n}$ corresponding to the vertices in S contain a cycle on at least four vertices. The smallest vertex on this cycle is contained in a 3-edge path which opens in the increasing direction, and this path corresponds to an edge in H. Therefore, $\alpha(H) < 2n < 2N/d^{1/2}$.

3 Related problems and conjectures

3.1 Ramsey numbers for 3-uniform tight paths

Our construction also provides the correct order of magnitude for some new 3-uniform Ramsey numbers. Let F be a 3-uniform hypergraph. Recall that the Ramsey number $r(F,t)$ is the smallest n so that every red-blue coloring of the edges of $K_n^{(3)}$ contains a red F or a blue complete $K_t^{(3)}$. Let P_s denote the 3-uniform hypergraph with vertex set $[s+2]$ and edge set $\{\{i,i+1,i+2\} : i \in [s]\}$. P_s is called the 3-uniform tight path. Results of Phelps and Rödl [16] imply that the Ramsey number of P_2 satisfies

$$r(P_2,t) = \Theta(t^2/\log t).$$

It is easy to prove that for fixed s, we have $\mathrm{ex}(n,P_s) = O(n^2)$ and this immediately implies that

$$r(P_s,t) = O(t^2).$$

Indeed, if we have a P_s-free 3-uniform hypergraph on $c_s n$ vertices (c_s large), then its average degree is at most $c'_s n$, so it has an independent set of size at least $t = c''_s n^{1/2}$.

We now show that the construction in Section 2 contains no P_4, which improves the lower bound of $r(P_s,t)$ for $s \geq 4$. The order of magnitude of $r(P_3,t)$ remains open.

Theorem 2. *Fix $s \geq 4$. Then $r(P_s,t) = \Theta(t^2)$.*

Proof. We only need to prove the lower bound, which follows by observing that the hypergraph H in Section 2 contains no P_4. Recall that every link graph of H has one component that is a complete bipartite graph and all of its other components are stars; further, the pairs of vertices which form edges in the bipartite component appear in exactly one edge of H. On the other hand, the link graph of each of the degree 3 vertices in P_4 contains a 3-edge path and one of the pairs of vertices which form an edge in this path is contained in two edges of P_4. □

3.2 Generalization to k-uniform hypergraphs

The construction in Section 2 starts with a bipartite graph and builds a hypergraph whose edges correspond to 3-edge paths in the graph. In this section, we generalize this method by starting with a multipartite hypergraph and building a new hypergraph whose edges correspond to some fixed hypergraph. The resulting hypergraphs provide counterexamples to various conjectures concerning k-uniform hypergraphs.

3.2.1 Chromatic number of k-uniform hypergraphs

A proper coloring of a hypergraph H is a partition of $V(H)$ into independent sets. The chromatic number of H, denoted $\chi(H)$, is the minimum number of parts needed in a proper coloring of H. Erdős and Lovász [9] showed that every $(k+1)$-uniform hypergraph with maximum degree Δ has $\chi(H) \le c_k \Delta^{1/k}$. Strengthening (2), Frieze and the second author [11] showed that every $(k+1)$-uniform linear hypergraph with maximum degree Δ satisfies $\chi(H) \le c'_k (\frac{\Delta}{\log \Delta})^{1/k}$. In [10, 11], the same authors conjectured a stronger positive answer to the question of Ajtai, Erdős, Komlós, and Szemerédi.

Conjecture 3 (Frieze-Mubayi [10, 11])**.** *If F is a $(k+1)$-uniform hypergraph and H is an F-free $(k+1)$-uniform hypergraph with maximum degree Δ, then $\chi(H) \le c_F(\Delta/\log \Delta)^{1/k}$.*

Let T_k be the k-uniform hypergraph with $k+1$ edges $e_1, \ldots, e_k, f$ where for all $i \neq j$ we have $e_i \cap e_j = S$ and $f \supset e_i - S$ for some S with $|S| = k-1$. In other words, k edges share the same set of $k-1$ points and the last edge contains the remaining vertex from each of the k edges. A k-uniform hypergraph has independent neighborhoods if it contains no copy of T_k. Bohman, Frieze, and the second author [5] conjectured a weaker version of Conjecture 3: if H is a 3-uniform hypergraph with maximum degree Δ and independent neighborhoods, then $\chi(H) = o(\Delta^{1/2})$.

The construction in Section 2 shows that both of these conjectures are false for 3-uniform hypergraphs. We now generalize that construction to disprove these conjectures for k-uniform hypergraphs.

3.2.2 Construction from positive strong k-simplices

Fix $k \ge 2$. A *k-simplex* is a collection of $k+1$ sets with empty intersection, every k of which have nonempty intersection. A *strong k-simplex* S_k, introduced in [15], is the k-uniform hypergraph with vertex set $\{v_1, v'_1, \ldots, v_k, v'_k\}$ and edge set $\{e, e_1, \ldots, e_k\}$

where $e = \{v_1, \dots, v_k\}$ and $e_i = e \cup \{v_i'\} - v_i$ (e is called the central edge). Given disjoint sets $X_1, \dots, X_k$ with each $X_i \cong [n]$, a *positive strong k-simplex* S_k^+ is a k-partite strong simplex satisfying $v_i, v_i' \in X_i$ and $v_i' > v_i$ for each $i = 1, \dots, k$.

Let $X_1, \dots, X_k$ be disjoint sets each isomorphic to $[n]$. Define the $(k+1)$-uniform hypergraph H_k with vertex set $X_1 \times \cdots \times X_k$ and edge set

$$H_k = \{A \subset X_1 \times \cdots \times X_k : A \cong S_k^+\}.$$

For example, H_2 corresponds to the construction in Section 2.

Fix a k-uniform hypergraph F_k. The Zarankiewicz number $z(n, F_k)$ is the maximum number of edges in a k-partite k-uniform hypergraph with parts of size n that contains no copy of F_k. Since copies of S_k^+ correspond to edges of H_k,

$$\alpha(H_k) \le z(n, S_k^+).$$

We may thus use the following lemma below to bound $\alpha(H_k)$.

Lemma 4. *Fix $k \ge 2$. Then $z(n, S_k^+) \le 2kn^{k-1}$.*

Proof. We proceed by induction on k. The base case $k = 2$ follows from Section 2. For the induction step, suppose we are given a k-partite $H \subset X_1 \times \cdots \times X_k$ with $|H| > 2kn^{k-1}$, where each $X_i \cong [n]$. For a vertex v in a k-uniform hypergraph H, define its link to be the $(k-1)$-uniform hypergraph $L_v = \{S \subset V(H) : v \notin S, S \cup \{v\} \in H\}$. For a set of vertices T, let $d_H(T)$ denote the number of edges containing T.

For each $v \in X_1$, let L_v be the link $(k-1)$-uniform hypergraph of v. Let $A_v \subset L_v$ comprise those $(k-1)$-sets T with $d_H(T) = 1$ and $B_v = L_v - A_v$.

Let B_v^+ be the set of all $S \in B_v$ such that there exists $v' > v$ with $S \in L_{v'}$. We will find $x \in X_1$ with $|B_x^+| > 2(k-1)n^{k-2}$ and then apply induction. Now

$$\sum_{v \in X_1} |B_v^+| = \sum_{\substack{S \in X_2 \times \cdots \times X_k: \\ d_H(S) \ge 2}} (d_H(S) - 1) \ge \sum_{\substack{S \in X_2 \times \cdots \times X_k: \\ d_H(S) \ge 2}} d_H(S) - n^{k-1} = \sum_{v \in X_1} |B_v| - n^{k-1}.$$

Thus

$$\begin{aligned} 2kn^{k-1} < |H| = \sum_{v \in X_1} |L_v| = \sum_{v \in X_1} |A_v| + \sum_{v \in X_1} |B_v| &\le n^{k-1} + \sum_{v \in X_1} |B_v| \\ &\le 2n^{k-1} + \sum_{v \in X_1} |B_v^+|. \end{aligned}$$

Consequently, there exists $x \in X_1$ with $|B_x^+| > 2(k-1)n^{k-2}$. Apply induction to B_x to obtain a copy of S_{k-1}^+ in $X_2 \times \cdots \times X_k$. To form S_k^+, begin by enlarging each edge of S_{k-1}^+ with x. Add another edge by enlarging the central edge e by some other vertex $y \in X_1$ with $y > x$. Note that y exists since $e \in B_x^+$ and $d_H(e) > 1$. We have thus obtained a copy of S_k^+, where $e \cup \{x\}$ is the central edge. $\square$

Notice H_k has $N = n^k$ vertices and maximum degree $\Delta \leq (k+1)n^k$. By Lemma 4,

$$\alpha(H_k) \leq 2kn^{k-1} = 2k(k+1)^{1/k} \frac{n^k}{((k+1)n^k)^{1/k}} \leq 2k(k+1)^{1/k} \frac{N}{\Delta^{1/k}},$$

and

$$\chi(H_k) \geq \frac{\Delta^{1/k}}{2k(k+1)^{1/k}}.$$

Recall that T_{k+1} is the $(k+1)$-uniform hypergraph with $k+2$ edges $e_1, \ldots, e_{k+1}, f$ where for all $i \neq j$, $e_i \cap e_j = S$ and $f \supset e_i - S$ for some S with $|S| = k$. Suppose $S_{k,1}^+, \ldots, S_{k,k+1}^+$ satisfy $S_{k,i}^+ \cap S_{k,j}^+ = S$, for $i \neq j$ and $|S| = k$. Since each strong k-simplex is positive, they must share a single central edge. Thus the edges in $(S_{k,1}^+ \cup \cdots \cup S_{k,k+1}^+) - S$ share a single vertex and so do not form a positive strong k-simplex. Therefore H_k does not contain any copy of T_{k+1}, disproving Conjecture 3 and the weaker conjecture of [5].

3.2.3 c-sparse hypergraphs

A hypergraph is *c-sparse* if every vertex subset S spans at most $c|S|^2$ edges. By Spencer's extension of Turán's bound, every c-sparse hypergraph H with N vertices satisfies $\alpha(H) \geq c'_k \sqrt{N}$. Phelps and Rödl [16] improved this to $\alpha(H) \geq c'_k \sqrt{N \log N}$ for linear 3-uniform hypergraphs. In 1986, de Caen (see [7]) conjectured that a similar improvement holds even for c-sparse hypergraphs (observe that linear implies $\frac{1}{2}$-sparse).

Conjecture 5 (De Caen [7])**.** *For every positive c, there is a function $\omega(N) \to \infty$ such that every c-sparse 3-uniform hypergraph H with N vertices satisfies $\alpha(H) \geq \omega(N)\sqrt{N}$.*

Recently, Kostochka, the second author, and Verstraëte [13] posed a stronger version of de Caen's conjecture: for every positive c, there is a function $\omega(N) \to \infty$ such that every c-sparse 3-uniform hypergraph H with N vertices and average degree d satisfies $\alpha(H) \geq \omega(N) \frac{N}{d^{1/2}}$.

Observe that the construction in Section 2 is 1-sparse, so S_2 immediately provides a counterexample to Conjecture 5 and the conjecture of [13]. However, for $k \geq 3$, H_k is not c-sparse for any constant c, so one may ask whether or not for $k \geq 3$ and every positive c there is a function $\omega(N) \to \infty$ such that every c-sparse $(k+1)$-uniform hypergraph H with N vertices satisfies $\alpha(H) \geq \omega(N)N^{1/k}$. The next section provides a counterexample to this generalization of de Caen's conjecture.

3.2.4 Construction from special k-clusters

A *k-cluster*, introduced in [15], is a collection of $k+1$ sets with empty intersection whose union has size at most $2k$. The family of *special k-clusters* $\mathcal{D}_k$ is the k-uniform hypergraph family that is defined inductively as follows: $\mathcal{D}_2 = \{D_2\}$, where D_2 is the path with three edges. For $k \geq 3$, $\mathcal{D}_k$ is the family of k-uniform hypergraphs which can be constructed as follows: begin with any $D_{k-1} \in \mathcal{D}_{k-1}$, which is assumed inductively to have $2(k-1)$ vertices and two disjoint edges a and b. Then D_k is a member of $\mathcal{D}_k$ if it can be formed by adding two new vertices x, y to D_{k-1}, enlarging all edges of D_{k-1} by including x, and enlarging a by including y. Thus D_k has $2k$ vertices and $k+1$ edges, two of which are disjoint. We will use D_k to denote an arbitrarily chosen member of $\mathcal{D}_k$.

Following the construction from Section 3.2.2, define the $(k+1)$-uniform hypergraph J_k with vertex set $X_1 \times \cdots \times X_k$ and edge set

$$J_k = \{A \subset X_1 \times \cdots \times X_k : A \cong D_k\}.$$

Lemma 6. *Fix $k \geq 2$ and $D_k \in \mathcal{D}_k$. Then $z(n, D_k) \leq kn^{k-1}$.*

Proof. We proceed by induction on k. For the base case $k = 2$, observe that D_2 is the path with 3 edges. If H is a bipartite graph with more than $2n$ edges, then H contains a cycle with at least four edges, which contains a copy of D_2. For the induction step, suppose we are given k-partite H with $|H| > kn^{k-1}$ with parts $X_1, \ldots, X_k$ each of size n. For each $v \in X_1$, define L_v, A_v, and B_v as in the proof of Lemma 4. Then

$$kn^{k-1} < |H| = \sum_{v \in X_1} |L_v| = \sum_{v \in X_1} |A_v| + \sum_{v \in X_1} |B_v| \leq n^{k-1} + \sum_{v \in X_1} |B_v|. \tag{3}$$

Consequently, there exists $x \in X_1$ with $|B_x| > (k-1)n^{k-2}$. Let D_{k-1} be the member of $\mathcal{D}_{k-1}$ that gives rise to D_k in the inductive construction of D_k. Apply induction to B_x to obtain a copy of D_{k-1} in B_x. To form D_k, begin by enlarging each edge of D_{k-1} with x. Add another edge by enlarging one of the two disjoint edges a, b of D_{k-1} (say a) by some other vertex $y \in X_1$. Note that y exists since $a \in B_x$. We have thus obtained a copy of D_k, where $a \cup \{y\}$ and $b \cup \{x\}$ are the disjoint edges. $\square$

By Lemma 6, $\alpha(J_k) \leq kn^{k-1}$, so it suffices to show that J_k is 2^{2k^2-2k-1}-sparse. Let $S \subset V(J_k)$. The vertex set of a copy of D_k is determined by the two disjoint edges in D_k. There are at most $\binom{2k}{k}^{k-1}$ possibilities for the remaining $k-1$ edges. Therefore we may associate every pair of vertices in S to at most $\binom{2k}{k}^{k-1}$ edges in the subgraph induced by S. Since every edge corresponds to at least one pair of vertices, the number of edges in S is at most

$$\binom{2k}{k}^{k-1}\binom{|S|}{2} < 2^{2k^2-2k-1}|S|^2.$$

This disproves the generalization of de Caen's conjecture to k-uniform hypergraphs.

4 Concluding remarks

- H_k is a counterexample to Conjecture 3 with N vertices and maximum degree $\Theta(N)$. Sparser counterexamples with fN vertices and maximum degree N can be constructed by taking the disjoint union of f copies of H_k.

- Benny Sudakov suggested the following generalization of H_2 to $(k+1)$-uniform hypergraphs, which provides a denser counterexample to Conjecture 3 for $k \geq 3$. Let G be the $(k+1)$-uniform hypergraph with vertex set $[n] \times [n]$ and edge set

  $$\{(x_1, y_1), (x_1, y_2), (x_2, y_2), \ldots, (x_k, y_2) : y_2 < y_1, x_i < x_{i+1} \text{ for } i \in [k-1]\}.$$

  In other words, each edge corresponds to an L with k points on its base. It is not hard to see that G has maximum degree $\Theta(n^k)$, independence number $\Theta(n)$, and contains no copy of T_{k+1}.

- For $1 < r < k+1$, say that a $(k+1)$-uniform hypergraph is (c, r)-sparse if every vertex subset S spans at most $c|S|^r$ edges. A partial Steiner $(k+1, k)$-system is a $(k+1)$-uniform hypergraph with every k vertices in at most one edge. Such a system has average degree at most n^{k-1} and, by [13], has independence number at least $c'(n \log n)^{1/k}$ for some positive c'. This result cannot be extended to the larger class of (c, k)-sparse $(k+1)$-uniform hypergraphs, as shown by the following $(c, 3)$-sparse 4-uniform hypergraphs with independence number $O(n^{1/3})$.

  Let F be the set of 3-partite 3-uniform hypergraphs with four edges such that one of the edges is contained in the union of the other three. Then it is an easy exercise to show (by induction on n for example) that $z(n, F) = O(n)$, so our general construction provides a 4-uniform, $(c, 3)$-sparse hypergraph $H(F)$ on n^3 vertices with $\alpha(H(F)) = O(n)$ (for $(c, 3)$-sparse, use the argument in Section 3.2.4).

  We remark that, in addition, $H(F)$ contains no $K_{163}^{(4)}$ for the vertex set of a copy of $K_{163}^{(4)}$ would correspond to a set of $163 = 1 + 3!(4-1)^3$ 3-uniform edges, and by the Erdős-Rado sunflower lemma, these edges would contain a sunflower C of size 4. But the 4 vertices in $H(F)$ corresponding to the edges of C cannot form an edge in $H(F)$ since not one of them is contained in the union of the other three.

- Define the 3-uniform hypergraphs $F_5 = \{abc, abd, cde\}$ and $C_3 = \{abc, cde, efa\}$. The authors [6] recently answered Question 1 positively if, in addition to $K_4^{-(3)}$, F_5 and C_3 are also forbidden. It would be interesting to answer Question 1 if only $K_4^{-(3)}$ and C_3 are forbidden.

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