# Turán problems and shadows III: expansions of graphs

Alexandr Kostochka* Dhruv Mubayi† Jacques Verstraëte‡

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**Abstract**

The expansion $G^+$ of a graph $G$ is the 3-uniform hypergraph obtained from $G$ by enlarging each edge of $G$ with a new vertex disjoint from $V(G)$ such that distinct edges are enlarged by distinct vertices. Let $\mathrm{ex}_3(n,F)$ denote the maximum number of edges in a 3-uniform hypergraph with $n$ vertices not containing any copy of a 3-uniform hypergraph $F$. The study of $\mathrm{ex}_3(n,G^+)$ includes some well-researched problems, including the case that $F$ consists of $k$ disjoint edges [5], $G$ is a triangle [4, 8, 17], $G$ is a path or cycle [11, 12], and $G$ is a tree [6, 7, 9, 10, 13]. In this paper we initiate a broader study of the behavior of $\mathrm{ex}_3(n,G^+)$. Specifically, we show

$$\mathrm{ex}_3(n,K_{s,t}^+) = \Theta(n^{3-3/s})$$

whenever $t > (s-1)!$ and $s \geq 3$. One of the main open problems is to determine for which graphs $G$ the quantity $\mathrm{ex}_3(n,G^+)$ is quadratic in $n$. We show that this occurs when $G$ is any bipartite graph with Turán number $o(n^{\varphi})$ where $\varphi = \frac{1+\sqrt{5}}{2}$, and in particular, this shows $\mathrm{ex}_3(n,Q^+) = \Theta(n^2)$ where $Q$ is the three-dimensional cube graph.

## 1 Introduction

An $r$-uniform hypergraph $F$, or simply *$r$-graph*, is a family of $r$-element subsets of a finite set. We associate an $r$-graph $F$ with its edge set and call its vertex set $V(F)$. Given an $r$-graph $F$, let $\mathrm{ex}_r(n,F)$ denote the maximum number of edges in an $r$-graph on $n$ vertices that does not contain $F$. The *expansion* of a graph $G$ is the 3-graph $G^+$ with edge set $\{e \cup \{v_e\} : e \in G\}$ where $v_e$ are distinct vertices not in $V(G)$. By definition, the expansion of $G$ has exactly $|G|$ edges. Note that Füredi and Jiang [9, 10] used a notion of expansion to $r$-graphs for general $r$, but this paper considers only 3-graphs.

Expansions include many important hypergraphs whose extremal functions have been investigated. For instance, the celebrated Erdős-Ko-Rado Theorem [5] for 3-graphs is the case of expansion of a matching. A well-known result is that $\mathrm{ex}_3(n,K_3^+) = \binom{n-1}{2}$ [4, 8, 17]. If a graph

*University of Illinois at Urbana–Champaign, Urbana, IL 61801 and Sobolev Institute of Mathematics, Novosibirsk 630090, Russia. E-mail: kostochk@math.uiuc.edu. Research of this author is supported in part by NSF grant DMS-1266016 and by grants 12-01-00631 and 12-01-00448 of the Russian Foundation for Basic Research.

†Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607. E-mail: mubayi@uic.edu. Research partially supported by NSF grants DMS-0969092 and DMS-1300138.

‡Department of Mathematics, University of California at San Diego, 9500 Gilman Drive, La Jolla, California 92093-0112, USA. E-mail: jverstra@math.ucsd.edu. Research supported by NSF Grant DMS-1101489.

is not 3-colorable then its expansion has positive Turán density and this case is fairly well understood [15, 18], so we focus on the case of expansions of 3-colorable graphs. It is easy to see that $\mathrm{ex}_3(n, G^+) = \Omega(n^2)$ unless $G$ is a star (the case that $G$ is a star is interesting in itself, and for $G = P_2$ determining $\mathrm{ex}_3(n, G^+)$ constituted a conjecture of Erdős and Sós [6] which was solved by Frankl [7]). The authors [12] had previously determined $\mathrm{ex}_3(n, G^+)$ exactly (for large $n$) when $G$ is a path or cycle of fixed length $k \geq 3$, thereby answering questions of Füredi-Jiang-Siever [11] and Füredi-Jiang [10]. The case when $G$ is a forest is solved asymptotically in [13], thus settling a conjecture of Füredi [9]. The following straightforward result provides general bounds for $\mathrm{ex}_3(n, G^+)$ in terms of the number of edges of $G$.

**Proposition 1.1.** *If $G$ is any graph with $v$ vertices and $f \geq 4$ edges, then for some $a > 0$,*

$$an^{3-\frac{3v-9}{f-3}} \leq \mathrm{ex}_3(n, G^+) \leq (n-1)\mathrm{ex}_2(n, G) + (f + v - 1)\binom{n}{2}.$$

The proof of Proposition 1.1 is given in Section 3. Some key remarks are that $\mathrm{ex}_3(n, G^+)$ is not quadratic in $n$ if $f > 3v - 6$, and if $G$ is not bipartite then the upper bound in Proposition 1.1 is cubic in $n$. This suggests the question of identifying the graphs $G$ for which $\mathrm{ex}_3(n, G^+) = O(n^2)$, and in particular evaluation of $\mathrm{ex}_3(n, G^+)$ for planar $G$.

## 1.1 Expansions of planar graphs

We give a straightforward proof of the following proposition, which is a special case of a more general result of Füredi [9] for a larger class of triple systems.

**Proposition 1.2.** *Let $G$ be a graph with treewidth at most two. Then $\mathrm{ex}_3(n, G^+) = O(n^2)$.*

On the other hand, there are 3-colorable planar graphs $G$ for which $\mathrm{ex}_3(n, G^+)$ is not quadratic in $n$. To state this result, we need a definition. A proper $k$-coloring $\chi : V(G) \to \{1, \ldots, k\}$ is *acyclic* if every pair of color classes induces a forest in $G$. We pose the following question:

**Question 1.** *Does every planar graph $G$ with an acyclic 3-coloring have $\mathrm{ex}_3(n, G^+) = O(n^2)$?*

Let $g(n, k)$ denote the maximum number of edges in an $n$-vertex graph of girth larger than $k$.

**Proposition 1.3.** *Let $G$ be a planar graph such that in every proper 3-coloring of $G$, every pair of color classes induces a subgraph containing a cycle of length at most $k$. Then $\mathrm{ex}_3(n, G^+) = \Omega(ng(n, k)) = \Omega(n^{2+\Theta(\frac{1}{k})})$.*

The last statement follows from the known fact that $g(n, k) \geq n^{1+\Theta(\frac{1}{k})}$. The octahedron graph $O$ is an example of a planar graph where in every proper 3-coloring, each pair of color classes induces a cycle of length four, and so $\mathrm{ex}_3(n, O^+) = \Omega(n^{5/2})$. Even wheels do not have acyclic 3-colorings, and we do not know whether their expansions have quadratic Turán numbers.

**Question 2.** *Does every even wheel $G$ have $\mathrm{ex}_3(n, G^+) = O(n^2)$?*

### 1.2 Expansions of bipartite graphs

The behavior of $\mathrm{ex}_3(n, G^+)$ when $G$ is a dense bipartite graph is somewhat related to the behavior of $\mathrm{ex}_2(n, G)$ according to Proposition 1.1. In particular, Proposition 1.1 shows that for $t \geq s \geq 2$ and some constants $a, c > 0$,

$$an^{3-\frac{3s+3t-9}{st-3}} \leq \mathrm{ex}_3(n, K_{s,t}^+) \leq cn^{3-\frac{1}{s}}.$$

We show that both the upper and lower bound can be improved to determine the order of magnitude of $\mathrm{ex}_3(n, K_{s,t}^+)$ when good constructions of $K_{s,t}$-free graphs are available (see Alon, Rónyai and Szabo [2]):

**Theorem 1.4.** *Fix* $3 \leq s \leq t$. *Then* $\mathrm{ex}_3(n, K_{s,t}^+) = O(n^{3-\frac{3}{s}})$ *and, if* $t > (s-1)! \geq 2$, *then* $\mathrm{ex}_3(n, K_{s,t}^+) = \Theta(n^{3-\frac{3}{s}})$.

The case of $K_{3,t}$ is interesting since $\mathrm{ex}_3(n, K_{3,t}^+) = O(n^2)$, and perhaps it is possible to determine a constant $c$ such that $\mathrm{ex}_3(n, K_{3,3}^+) \sim cn^2$, since the asymptotic behavior of $\mathrm{ex}_2(n, K_{3,3})$ is known, due to a construction of Brown [3] and the upper bounds of Füredi [9]. In general, the following bounds hold for expansions of $K_{3,t}$:

**Theorem 1.5.** *For fixed* $r \geq 1$ *and* $t = 2r^2+1$, *we have* $(1-o(1))\frac{t-1}{12}n^2 \leq \mathrm{ex}_3(n, K_{3,t}^+) = O(n^2)$.

The upper bound in this theorem is a special case of a general upper bound for all graphs $G$ with $\sigma(G^+) = 3$ (see Theorem 1.7). Finally, we prove a general result that applies to expansions of a large class of bipartite graphs.

**Theorem 1.6.** *Let* $G$ *be a graph with* $\mathrm{ex}_2(n, G) = o(n^{\varphi})$, *where* $\varphi = (1+\sqrt{5})/2$ *is the golden ratio. Then* $\mathrm{ex}_3(n, G^+) = O(n^2)$.

Let $\mathbb{Q}$ be the graph of the 3-dimensional cube (with 8 vertices and 12 edges). Erdős and Simonovits [6] proved $\mathrm{ex}_2(n, \mathbb{Q}) = O(n^{1.6}) = o(n^{\varphi})$, so a corollary to Theorem 1.6 is that

$$\mathrm{ex}_3(n, \mathbb{Q}^+) = \Theta(n^2).$$

Determining the growth rate of $\mathrm{ex}_2(n, \mathbb{Q})$ is a longstanding open problem. Since it is known that for any graph $G$ the 1-subdivision of $G$ has Turán Number $O(n^{3/2})$ – see Alon, Krivelevich and Sudakov [1] – Theorem 1.6 also shows that for such graphs $G$, $\mathrm{ex}_3(n, G^+) = \Theta(n^2)$. Erdős conjectured that $\mathrm{ex}_2(n, G) = O(n^{3/2})$ for each 2-degenerate bipartite graph $G$. If this conjecture is true, then by Theorem 1.6, $\mathrm{ex}_3(n, G^+) = O(n^2)$ for any 2-degenerate bipartite graph $G$.

### 1.3 Crosscuts

A set of vertices in a hypergraph containing exactly one vertex from every edge of a hypergraph is called a *crosscut* of the hypergraph, following Frankl and Füredi [8]. For a 3-uniform hypergraph $F$, let $\sigma(F)$ be the minimum size of a crosscut of $F$ if it exists, i.e.,

$$\sigma(F) := \min\{|X| : \forall e \in F, |e \cap X| = 1\}$$

if such an $X$ exists. Since the triple system consisting of all edges containing exactly one vertex from a set of size $\sigma(F)-1$ does not contain $F$, we have

$$\mathrm{ex}_3(n,F) \geq (\sigma(F)-1+o(1))\binom{n}{2}. \tag{1}$$

An intriguing open question is: For which $F$ an asymptotic equality is attained in (1)? Recall that a graph has tree-width at most two if and only if it has no subdivision of $K_4$. Informally, these are subgraphs of a planar graph obtained by starting with a triangle, and then picking some edge $uv$ of the current graph, adding a new vertex $w$, and then adding the edges $uw$ and $vw$.

**Question 3.** *Is it true that*

$$\mathrm{ex}_3(n,G^+) \sim (\sigma(G^+)-1)\binom{n}{2} \tag{2}$$

*for every graph $G$ with tree-width two?*

If $G$ is a forest or a cycle, then (2) holds [12, 13] (corresponding results for $r>3$ were given by Füredi [9]). If $G$ is any graph with $\sigma(G^+)=2$, then again (2) holds [13]. Proposition 1.1 and Theorem 1.4 give examples of graphs $G$ with $\sigma(G^+)=4$ and $\mathrm{ex}_3(n,G^+)$ superquadratic in $n$. This leaves the case $\sigma(G^+)=3$, and in this case, Theorem 1.5 shows that $\mathrm{ex}_3(n,K_{3,t}^+)/n^2 \to \infty$ as $t\to\infty$, even though $\sigma(K_{3,t}^+)=3$ for all $t\geq 3$. A quadratic upper bound for $\mathrm{ex}_3(n,K_{3,t}^+)$ in Theorem 1.5 is a special case of the following theorem:

**Theorem 1.7.** *For every $G$ with $\sigma(G^+)=3$, $\mathrm{ex}_3(n,G^+)=O(n^2)$.*

## 2 Preliminaries

**Notation and terminology.** A 3-graph is called a *triple system*. The edges will be written as unordered lists, for instance, $xyz$ represents $\{x,y,z\}$. For a set $X$ of vertices of a hypergraph $H$, let $H-X=\{e\in H : e\cap X=\emptyset\}$. If $X=\{x\}$, then we write $H-x$ instead of $H-X$. The *codegree* of a pair $\{x,y\}$ of vertices in $H$ is $d_H(x,y)=|\{e\in H : S\subset e\}|$ and for a set $S$ of vertices, $N_H(S)=\{x\in V(H) : S\cup\{x\}\in H\}$ so that $|N_H(S)|=d_H(S)$ when $|S|=2$. The *shadow* of $H$ is the graph $\partial H=\{xy : \exists\, e\in H, \{x,y\}\subset e\}$. The edges of $\partial H$ will be called the *sub-edges* of $H$.

An $r$-graph $H$ is *$d$-full* if every sub-edge of $H$ has codegree at least $d$.

Thus $H$ is $d$-full is equivalent to the fact that the minimum non-zero codegree in $H$ is at least $d$. The following lemma from [13] extends the well-known fact that each graph $G$ has a subgraph of minimum degree at least $d$ with at least $|G|-(d-1)|V(G)|$ edges.

**Lemma 2.1.** *For $r\geq 2, d\geq 1$, every $n$-vertex $r$-graph $H$ has a $(d+1)$-full subgraph $F$ with*

$$|F| \geq |H| - d\,|\partial H|.$$

*Proof.* A *d-sparse sequence* is a maximal sequence $e_1, e_2, \ldots, e_m \in \partial H$ such that $d_H(e_1) \leq d$, and for all $i > 1$, $e_i$ is contained in at most $d$ edges of $H$ which contain none of $e_1, e_2, \ldots, e_{i-1}$. The $r$-graph $F$ obtained by deleting all edges of $H$ containing at least one of the $e_i$ is $(d+1)$-full. Since a $d$-sparse sequence has length at most $|\partial H|$, we have $|F| \geq |H| - d|\partial H|$. □

## 3 Proofs of Propositions

**Proof of Proposition 1.1.** The proof of the lower bound in Proposition 1.1 is via a random triple system. The idea is to take a random graph not containing a particular graph $G$, and then observe that the triple system of triangles in the random graph does not contain $G^+$. Consider the random graph on $n$ vertices, whose edges are placed independently with probability $p$, to be chosen later. If $X$ is the number of triangles and $Y$ is the number of copies of $G$ in the random graph, then

$$\mathbb{E}(X) = p^3\binom{n}{3} \qquad \mathbb{E}(Y) \leq p^f n^v.$$

Therefore choosing $p = 0.1n^{-(v-3)/(f-3)}$, since $f \geq 4$, we find

$$\mathbb{E}(X - Y) \geq 0.0001n^{3-3(v-3)/(f-3)}.$$

Now let $H$ be the triple system of vertex-sets of triangles in the graph obtained by removing one edge from each copy of $G$ in the random graph. Then $\mathbb{E}(|H|) \geq \mathbb{E}(X - Y)$, and $G^+ \not\subset H$. Select an $H$ so that $|H| \geq 0.0001n^{3-3(v-3)/(f-3)}$. This proves the lower bound in Proposition 1.1 with $a = 0.0001$.

Now suppose $G$ is a bipartite graph with $f$ edges $e_1, e_2, \ldots, e_f$ and $v$ vertices. If a triple system $H$ on $n$ vertices has more than $(n-1)\mathrm{ex}_2(n, G) + (f + v - 1)\binom{n}{2}$ triples, then by deleting at most $(f + v - 1)\binom{n}{2}$ triples we arrive at a triple system $H' \subset H$ which is $(f + v)$-full, by Lemma 2.1 and $|H'| > (n - 1)\mathrm{ex}_2(n, G)$. There exists $x \in V(H')$ such that more than $\mathrm{ex}_2(n, G)$ triples of $H'$ contain $x$. So the graph of all pairs $\{x, y\}$ such that $\{w, x, y\} \in H'$ contains $G$. Since every pair $\{w, y\}$ has codegree at least $f + v$, we find vertices $z_1, z_2, \ldots, z_f \notin V(G)$ such $e_i \cup \{z_i\} \in H'$ for all $i = 1, 2, \ldots, f$, and this forms a copy of $G^+$ in $H'$. □

**Proof of Proposition 1.2.** Let $G$ be a graph of tree-width two. Then $G \subset F$, where $F$ is a graph obtained from a triangle by repeatedly adding a new vertex and joining it to two adjacent vertices of the current graph. It is enough to show $\mathrm{ex}_3(n, F^+) = O(n^2)$. Suppose $F$ has $v$ vertices and $f$ edges. By definition, $F$ has a vertex $x$ of degree two such that the neighbors $x'$ and $x''$ of $x$ are adjacent. Then $F' := F - x$ has $v - 1$ vertices and $f - 2$ edges. Let $H$ be an $n$-vertex triple system with more than $(v + f - 1)\binom{n}{2}$ edges. By Lemma 2.1, $H$ has a $(v + f)$-full subgraph $H'$. We claim $H'$ contains $F^+$. Inductively, $H'$ contains a copy $H''$ of the expansion of $F'$. By the definition of $H'$, $\{x', x''\}$ has codegree at least $v + f$ in $H'$. Therefore we may select a new vertex $z$ that is not in $H''$ such that $\{z, x', x''\}$ is an edge of $H'$, and now $F$ is embedded in $H'$ by mapping $x$ to $z$. $\square$

**Proof of Proposition 1.3.** Let $G$ be a 3-colorable planar graph with the given conditions. To show $\mathrm{ex}_3(n, G^+) = \Omega(ng(n,k))$, form a triple system $H$ on $n$ vertices as follows. Let $F$ be a bipartite $\lfloor \frac{n}{2} \rfloor$-vertex graph of girth $k+1$ with at least $\frac{1}{2} g(\lfloor \frac{n}{2} \rfloor, k)$ edges. Let $U$ and $V$ be the partite sets of $F$. Let $X$ be a set of $\lceil \frac{n}{2} \rceil$ vertices disjoint from $U \cup V$. Then set $V(H) = U \cup V \cup X$ and let the edges of $H$ consist of all triples $e \cup \{x\}$ such that $e \in F$ and $x \in X$. Then

$$|H| \geq |X| \cdot g(\lfloor \frac{n}{2} \rfloor, k) = \Omega(ng(n,k)).$$

Now $\partial H$ has a natural 3-coloring given by $U, V, X$. If $G^+ \subset H$, then $G \subset \partial H$ and therefore $G$ is properly colored, with color classes $V(G) \cap U$, $V(G) \cap V$ and $V(G) \cap X$. By the assumptions on $G$, $V(G) \cap (U \cup V)$ induces a subgraph of $G$ which contains a cycle of length at most $k$. However, that cycle is then a subgraph of $F$, by the definition of $H$, which is a contradiction. Therefore $G^+ \not\subset H$. $\square$

# 4 Proof of Theorem 1.6

**Proof of Theorem 1.6.** Suppose $\mathrm{ex}_2(n, G) = o(n^{\varphi})$ and $|G| = k$, and $H$ is an $G^+$-free 3-graph with $|H| \geq (k+1)\binom{n}{2}$. By Lemma 2.1, $H$ has a $k$-full-subgraph $H_1$ with at least $n^2/3$ edges. If $G \subset \partial H_1$, then we can expand $G$ to $G^+ \subset H_1$ using that $H_1$ is $k$-full. Therefore $|\partial H_1| \leq \mathrm{ex}_2(n, G) = o(n^{\varphi})$. By Lemma 2.1, and since $|H_1| \geq \delta n^2$, $H_1$ has a non-empty $n^{2-\varphi}$-full subgraph $H_2$ if $n$ is large enough. Let $H_3$ be obtained by removing all isolated vertices of $H_2$ and let $m = |V(H_3)|$. Since $H_3$ is $n^{2-\varphi}$-full, $m > n^{2-\varphi}$. Since $H_1$ is $G^+$-free, $H_3 \subset H_1$ is also $G^+$-free, and therefore if $F = \partial H_3$, $|V(F)| = |V(H_3)| = m$ and $|F| \leq \mathrm{ex}_2(m, G) = o(m^{\varphi})$. So some vertex $v$ of the graph $F = \partial H_3$ has degree $o(m^{\varphi-1})$. Now the number of edges of $F$ between vertices of $N_F(v)$ is at least the number of edges of $H_3$ containing $v$. Since $H_3$ is $n^{2-\varphi}$-full, there are at least $\frac{1}{2} n^{2-\varphi} |N_F(v)|$ such edges. On the other hand, since the subgraph of $F$ induced by $N_F(v)$ does not contain $G$, the number of such edges is $o(|N_F(v)|^{\varphi})$. It follows that $n^{2-\varphi} = o(|N_F(v)|^{\varphi-1})$. Since $|N_F(v)| = o(m^{\varphi-1}) = o(n^{\varphi-1})$, we get $2 - \varphi < (\varphi - 1)^2$, contradicting the fact that $\varphi$ is the golden ratio. $\square$

# 5 Proof of Theorem 1.4

**Proof of Theorem 1.4.** For the upper bound, we repeat the proof of Theorem 1.6 when $F = K_{s,t}$, using the bounds $\mathrm{ex}_2(n, K_{s,t}) = O(n^{2-1/s})$ provided by the Kövari-Sós-Turán Theorem [14], except at the stage of the proof where we use the bound on $\mathrm{ex}_2(|N_G(v)|, F)$, we may now use

$$\mathrm{ex}_2(|N_G(v)|, K_{s-1,t}) = O(|N_G(v)|^{2-1/(s-1)})$$

for if the subgraph of $G$ of edges between $N_G(v)$ contains $K_{s-1,t}$, then by adding $v$ we see $G$ contains $K_{s,t}$. A calculation gives $|H| = O(n^{3-3/s})$.

For the lower bound we must show that $\mathrm{ex}_3(n, K_{s,t}^+) = \Omega(n^{3-3/s})$ if $t > (s-1)!$. We will use the *projective norm graphs* defined by Alon, Rónyai and Szabo [2]. Given a finite field $\mathbb{F}_q$ and an integer $s \geq 2$, the norm is the map $N : \mathbb{F}_{q^{s-1}}^* \to \mathbb{F}_q^*$ given by $N(X) = X^{1+q+\cdots+q^{s-2}}$. The norm is a (multiplicative) group homomorphism and is the identity map on elements of $\mathbb{F}_q^*$. This implies that for each $x \in \mathbb{F}_q^*$, the number of preimages of $x$ is exactly

$$\frac{q^{s-1}-1}{q-1} = 1 + q + \cdots + q^{s-2}. \tag{3}$$

**Definition 5.1.** *Let $q$ be a prime power and $s \geq 2$ be an integer. The projective norm graph $PG(q,s)$ has vertex set $V = \mathbb{F}_{q^{s-1}} \times \mathbb{F}_q^*$ and edge set*

$$\{(A,b)(B,b) : N(A+B) = ab\}.$$

**Lemma 5.2.** *Fix an integer $s \geq 3$ and a prime power $q$. Let $x \in \mathbb{F}_q^*$, and $A, B \in \mathbb{F}_{q^{s-1}}$ with $A \neq B$. Then the number of $C \in \mathbb{F}_{q^{s-1}}$ with*

$$N\left(\frac{A+C}{B+C}\right) = x \tag{4}$$

*is at least $q^{s-2}$.*

*Proof.* By (3) there exist distinct $X_1, \ldots, X_{q^{s-2}+1} \in \mathbb{F}_{q^{s-1}}^*$ such that $N(X_i) = x$ for each $i$. As long as $X_i \neq 1$, define

$$C_i = \frac{BX_i - A}{1 - X_i}.$$

Then $(A + C_i)/(B + C_i) = X_i$, and $C_i \neq C_j$ for $i \neq j$ since $A \neq B$. □

**Lemma 5.3.** *Fix an integer $s \geq 3$ and a prime power $q$. The number of triangles in $PG(q,s)$ is at least $(1 - o(1))q^{3s-3}/6$ as $q \to \infty$.*

*Proof.* Pick a vertex $(A,a)$ and then one of its neighbors $(B,b)$. The number of ways to do this is at least $q^{s-1}(q-1)(q^{s-1}-1)$. Let $x = a/b$ and apply Lemma 5.2 to obtain at least $q^{s-2} - 2$ distinct $C \notin \{-A, -B\}$ satisfying (4). For each such $C$, define

$$c = \frac{N(A+C)}{a} = \frac{N(B+C)}{b}.$$

Then $(C,c)$ is adjacent to both $(A,a)$ and $(B,b)$. Each triangle is counted six times in this way and the result follows. □

For appropriate $n$ the $n$-vertex norm graphs $PG(q,s)$ (for fixed $s$ and large $q$) have $\Theta(n^{2-1/s})$ edges and no $K_{s,t}$. By Lemma 5.3 the number of triangles in $PG(q,s)$ is $\Theta(n^{3-3/s})$. The hypergraph $H$ whose edges are the vertex sets of triangles in $PG(q,s)$ is a 3-graph with $\Theta(n^{3-3/s})$ edges and no $K_{s,t}^+$. This completes the proof of Theorem 1.4. □

## 6 Proof of Theorems 1.5 and 1.7

We need the following result.

**Theorem 6.1.** *Let $F$ be a 3-uniform hypergraph with $v$ vertices and* $\mathrm{ex}_3(n, F) < c\binom{n}{2}$. *Then* $ex_3(n, (\partial F)^+) < (c + v + |F|)\binom{n}{2}$.

*Proof.* Suppose we have an $n$ vertex 3-uniform hypergraph $H$ with $|H| > (c + v + |H|)\binom{n}{2}$. Apply Lemma 2.1 to obtain a subhypergraph $H' \subset H$ that is $(v + |F|)$-full with $|H'| > c\binom{n}{2}$. By definition, we may find a copy of $F \subset H'$ and hence a copy of $\partial F \subset \partial H'$. Because $H'$ is $(v + |F|)$-full, we may expand this copy of $\partial F$ to a copy of $(\partial F)^+ \subset H' \subset H$ as desired. □

Define $H_t$ to be the 3-uniform hypergraph with vertex set $\{a, b, x_1, y_1, \ldots, x_t, y_t\}$ and $2t$ edges $x_i y_i a$ and $x_i y_i b$ for all $i \in [t]$. It is convenient (though not necessary) for us to use the following theorem of the authors [16].

**Theorem 6.2. ([16])** *For each $t \geq 2$, we have* $\mathrm{ex}_3(n, H_t) < t^4\binom{n}{2}$.

**Proof of Theorems 1.5 and 1.7.** First we prove the upper bound in Theorem 1.7. Suppose $\sigma(G^+) \leq 3$. This means that $G$ has an independent set $I$ and set $R$ of edges such that $I$ intersects each edge in $G - R$, and $|I| + |R| \leq 3$. It follows that $G$ is a subgraph of one of the following graphs (Cases (i) and (ii) correspond to $|I| = 1$, Case (iii) corresponds to $|I| = 2$, and Case (iv) corresponds to $|I| = 3$):

(i) $K_4 - e$ together with a star centered at one of the degree 3 (in $K_4 - e$) vertices,

(ii) two triangles sharing a vertex $x$ and a star centered at $x$,

(iii) the graph obtained from $K_{2,t}$ by adding an edge joining two vertices in the part of size $t$,

(iv) $K_{3,t}$.

Now suppose we have a 3-uniform $H$ with $|H| > cn^2$ for some $c > |G| + |V(G)|$. Applying Lemma 2.1, we find a $c$-full $H' \subset H$ with $|H'| > c\binom{n}{2}$. As in the proof of Theorem 6.1, it is enough to find $G$ in $\partial H'$. Since $|H'| > c\binom{n}{2}$, the codegree of some pair $\{x, y\}$ is at least $c + 1$. Then the shadow of the set of triples in $H'$ containing $\{x, y\}$ contains the graph of the form i). Similarly, $H'$ contains two edges sharing exactly one vertex, say $x$, and the shadow of the set of triples in $H'$ containing $x$ contains the graph of the form ii). If $G$ is of the form in iii), we apply Theorems 6.1 and 6.2 and observe that $\partial H_t \supset G$. Finally, if $G \subseteq K_{3,t}$ then we apply Theorem 1.4.

For the lower bound in Theorem 1.5, we use a slight modification of the construction in Theorem 1.4. Set $s = 3$ and let $r | q - 1$. Let $Q_r$ denote a subgroup of $\mathbb{F}_q^*$ of order $r$. Define the graph $H = H_r(q)$ with $V(H) = \mathbb{F}_{q^2} \times \mathbb{F}_q^*/Q_r$ and two vertices $(A, aQ_r)$ and $(B, bQ_r)$ are adjacent in $H$ if $N(A + B) \in abQ_r$. Then $H$ has $n = (q^3 - q^2)/r$ vertices and each vertex has degree $q^2 - 1$. It also follows from [2] that $H$ has no $K_{3,t}$ where $t = 2r^2 + 1$. Now we construct a

3-uniform hypergraph $H'$ with $V(H') = V(H)$ and whose edges are the triangles of $H$. We must count the number of triangles in $H$ to determine $|H'|$. For every choice of $(A, a), (B, b)$ in $\mathbb{F}_{q^2} \times \mathbb{F}_q^*$, the number of $(C, c) \in \mathbb{F}_{q^2} \times \mathbb{F}_q^*$ with $C \neq A, B$, $N(A + C) = ac$ and $N(B + C) = bc$ is at least $q - 2$ by (the proof of) Lemma 5.3. Consequently, the number of $(C, c)$ such that $N(A+C) \in acQ_r$ and $N(B+C) \in bcQ_r$ is at least $r^2(q-2)$. Since $(C, c)$ satisfies these equations iff $(C, cq)$ satisfies these equations for all $q \in Q_r$ (i.e. the solutions come in equivalence classes of size $r$), the number of common neighbors of $(A, aQ_r)$ and $(B, bQ_r)$ is at least $r(q - 2)$. The number of edges in $H$ is at least $(1 - o(1))q^5/2r$, so the number of triangles in $H$ is at least $(1 - o(1))q^6/6 = (1 - o(1))(r^2/6)n^2$. $\square$

## 7 Concluding remarks

• In this paper we studied $\mathrm{ex}_3(n, G^+)$ where $G$ is a 3-colorable graph. If $G$ has treewidth two, then we believe $\mathrm{ex}_3(n, G^+) \sim (\sigma(G^+) - 1)\binom{n}{2}$ (Question 3), and if $G$ has an acyclic 3-coloring, then we believe $\mathrm{ex}_3(n, G^+) = O(n^2)$ (Question 1). We are also not able to prove or disprove $\mathrm{ex}_3(n, G^+) = O(n^2)$ when $G$ is an even wheel (Question 2). This is equivalent to showing that if $F$ is an $n$-vertex graph with a superquadratic number of triangles, then $F$ contains every even wheel with a bounded number of vertices.

• A number of examples of 3-colorable $G$ with superquadratic $\mathrm{ex}_3(n, G^+)$ were given. In particular we determined the order of magnitude of $\mathrm{ex}_3(n, K_{s,t}^+)$ when near-extremal constructions of $K_{s,t}$-free bipartite graphs are known. One may ask for the asymptotic behavior of $\mathrm{ex}_3(n, K_{3,t}^+)$ for each $t \geq 3$, since in that case we have shown $\mathrm{ex}_3(n, K_{3,t}^+) = \Theta(n^2)$. Finally, we gave a general upper bound on $\mathrm{ex}_3(n, G^+)$ when $G$ is a bipartite graph, and showed that if $G$ has Turán number much smaller than $n^{\varphi}$ where $\varphi$ is the golden ratio, then $\mathrm{ex}_3(n, G^+) = O(n^2)$. Determining exactly when $\mathrm{ex}_3(n, G^+)$ is quadratic in $n$ remains an open problem for further research.

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