Coloring sparse hypergraphs

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Abstract

Fix $k \geq 3$, and let G be a k-uniform hypergraph with maximum degree Δ. Suppose that for each $l = 2, \ldots, k-1$, every set of l vertices of G is in at most $\Delta^{\frac{k-l}{k-1}}/f$ edges. Then the chromatic number of G is $O((\frac{\Delta}{\log f})^{\frac{1}{k-1}})$. This extends results of Frieze and the second author [10] and Bennett and Bohman [2].

A similar result is proved for 3-uniform hypergraphs where every vertex lies in few triangles. This generalizes a result of Alon, Krivelevich, and Sudakov [1], who proved the result for graphs.

Our main new technical contribution is a deviation inequality for positive random variables with expectation less than 1. This may be of independent interest and have further applications.

1   Introduction

A hypergraph G is a tuple consisting of a set of vertices V and a set of edges E, which are subsets of V; we will often associate G with its edge set E. A hypergraph has rank k if every edge contains between 2 and k vertices and is k-uniform if every edge contains exactly k vertices. A proper coloring of G is an assignment of colors to the vertices so that no edge is monochromatic. The chromatic number, $\chi(G)$, is the minimum number of colors in a proper coloring of G.

A hypergraph is *linear* if every pair of vertices is contained in at most one edge. A *triangle* in a hypergraph G is a set of three distinct edges $e, f, g \in G$ and three distinct

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vertices $u, v, w \in V(G)$ such that $u, v \in e$, $v, w \in f$, $w, u \in g$, and $\{u, v, w\} \cap e \cap f \cap g = \emptyset$. For example, the three triangles in a 3-uniform hypergraph are $C_3 = \{abc, cde, efa\}$, $F_5 = \{abc, abd, ced\}$, and $K_4^- = \{abc, bcd, abd\}$.

The *degree* of a vertex $u \in V(G)$ is the number of edges containing that vertex. The *maximum degree* of a hypergraph G is the maximum degree of a vertex $v \in V(G)$. Improving on results of Catlin [6], Lawrence [14], Borodin and Kostochka [5], and Kim [12], Johansson [11] showed that if G is a triangle-free graph with maximum degree Δ, then

$$\chi(G) = O(\frac{\Delta}{\log \Delta}). \tag{1}$$

Random graphs show that the $\log \Delta$ factor in (1) is optimal. Recently, Frieze and the second author [10] generalized (1) to linear k-uniform hypergraphs, and the current authors [7] proved slightly stronger results for $k = 3$.

Theorem 1 ([10]). *Fix $k \geq 3$. If G is a k-uniform, linear hypergraph with maximum degree Δ, then*

$$\chi(G) = O(\frac{\Delta}{\log \Delta})^{\frac{1}{k-1}}.$$

Theorem 2 ([7]). *If G is a 3-uniform, triangle-free hypergraph with maximum degree Δ, then*

$$\chi(G) = O((\frac{\Delta}{\log \Delta})^{1/2}).$$

Alon, Krivelevich, and Sudakov [1] extended (1) by showing that if every vertex $u \in V(G)$ is in at most Δ^2/f triangles, then $\chi(G) = O(\frac{\Delta}{\log f})$, where $\Delta \to \infty$. They used this to show that if G contains no copy of H, where H is a fixed graph such that $H - u$ is biparite for some $u \in V(H)$, then $\chi(G) = O(\Delta/\log \Delta)$. In this paper, we give similar improvements to the results of [10] and [7].

Given two hypergraphs H and A, a map $\phi : V(H) \to V(A)$ is an isomorphism if for all $E \subset V(H)$, $\phi(E) \in A$ if and only if $E \in H$. If there exists an isomorphism $\phi : V(H) \to V(A)$, we say H is isomorphic to A and denote this by $H \cong_\phi A$. Given a hypergraph H and $v \in V(H)$, let

$$\Delta_{H,v}(G) = \max_{u \in V(G)} |\{A \subset G : A \cong_\phi H \text{ and } \phi(u) = v\}|$$

and

$$\Delta_H(G) = \min_{v \in V(H)} \Delta_{H,v}(G).$$

For example, suppose G is a d-regular graph, and H is the path with edges xy and yz. Then $\Delta_{H,x}(G) = \Delta_{H,z}(G) = d(d-1)$, while $\Delta_{H,y}(G) = \binom{d}{2}$. Thus $\Delta_H(G) = \binom{d}{2}$. Another example appears after the statement of Theorem 5.

We improve the result of [7] with the following theorem. For a hypergraph H, let $v(H) = |V(H)|$.

Theorem 3. *Let G be a 3-uniform hypergraph with maximum degree Δ. Let $\mathcal{T}$ denote the set of 3-uniform triangles. If*

$$\Delta_H(G) \le \Delta^{(v(H)-1)/2}/f$$

for all $H \in \mathcal{T}$, then

$$\chi(G) = O((\frac{\Delta}{\log f})^{1/2}).$$

Notice that the hypotheses of Theorem 3 are satisfied when G is linear and $f = \Delta^{1/2}$, so Theorem 3 implies Theorem 1 for 3-uniform hypergraphs.

Given a rank k hypergraph G and $A \subset V(G)$, let

$$d_j(A) = |\{B \in G : |B| = j, A \subset B\}|$$

for each $1 \le j \le k$. When the hypergraph is k-uniform, we write $d(A)$ instead of $d_k(A)$. For $1 \le l \le j \le k$, define the maximum (j,l)-degree of G, denoted $\Delta_{j,l}(G)$, to be $\max_{A \subset V(G):|A|=l} d_j(A)$ and the maximum j-degree of G to be $\Delta_{j,1}(G)$.

Consider the random greedy algorithm for forming an independent set I in a hypergraph G: at each step of the algorithm, a vertex v is chosen at random from the set of vertices $V(G) - I$ such that $I \cup \{v\}$ contains no edge of G. The algorithm terminates when no such v exists. Notice that when the vertex set of G consists of the edges of a complete graph, and the edges of G correspond to triangles in this graph, the random greedy independent set algorithm reduces to the triangle-free process (see [3], [4], [9]). Bennett and Bohman [2] have recently shown that this algorithm terminates with a large independent set I with high probability, unifying many of the previous results on H-free processes. Define the *b-codegree* of a pair of distinct vertices v, v' to be the number of edges $e, e' \in G$ such that $v \in e$, $v' \in e'$, and $|e \cap e'| = b$. Let $\Gamma_b(G)$ be the maximum b-codegree of G.

Theorem 4 (Bennett, Bohman [2]). *Let $k \ge 2$ and $\epsilon > 0$ be fixed. Let G be a k-uniform, D-regular hypergraph on N vertices such that $D > N^\epsilon$. If*

$$\Delta_{k,l}(G) < D^{\frac{k-l}{k-1}-\epsilon} \ \text{for } l = 2, \ldots, k-1$$

and $\Gamma_{k-1}(G) < D^{1-\epsilon}$ then the random greedy independent set algorithm produces an independent set I in G with

$$|I| = \Omega(N(\frac{\log N}{D})^{\frac{1}{k-1}})$$

with probability $1 - \exp\{-N^{\Omega(1)}\}$.

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Our next theorem improves and extends Bennett and Bohman's result on the independence number to chromatic number when $k \geq 3$. We also weaken the hypothesis by not requiring any condition on $\Gamma_{k-1}(G)$. Note that an important aspect of [2] is that the random greedy procedure results in a large independent set. We do not make any such claims in our result below.

Theorem 5. *Fix $k \geq 3$. Let G be a k-uniform hypergraph with maximum degree Δ. If*

$$\Delta_{k,l}(G) \leq \Delta^{\frac{k-l}{k-1}}/f \ \ for \ l = 2, \ldots, k-1,$$

then

$$\chi(G) = O((\frac{\Delta}{\log f})^{\frac{1}{k-1}}).$$

The following example shows that Theorem 3 implies Theorem 5 when $k = 3$ and also motivates our definition of Δ_H: Suppose G is a 3-uniform hypergraph with maximum 3-degree Δ and maximum $(3, 2)$-degree at most $\Delta^{1/2}/f$. Recall that $F_5 = \{abc, abd, ced\}$. Notice $\Delta_{F_5,e} \leq \Delta^2$, while $\Delta_{F_5,a} \leq \Delta^2/f^2$; thus $\Delta_{F_5} \leq \Delta^2/f^2$. We also have $\Delta_{C_3} \leq \Delta^{5/2}/f$ and $\Delta_{K_4^-} \leq 4\Delta^{3/2}/f$. Theorem 3 therefore implies $\chi(G) = O((\Delta/\log f)^{1/2})$.

Theorem 3 and Theorem 5 both follow from a general partitioning lemma, which is a generalization of the main result of [1] to hypergraphs.

Definition 6. Let G be a rank k hypergraph. G is $(\Delta, \omega_2, \ldots, \omega_k)$-*sparse* if G has maximum k-degree at most Δ, and for all $1 \leq l < j \leq k$, G has maximum (j, l)-degree at most $\Delta^{\frac{j-l}{k-1}}\omega_j$.

Recall that a hypergraph H is *connected* if for all $u, v \in V(H)$, there exists a sequence of edges $e_1, \ldots, e_n \in H$ such that $u \in e_1$, $v \in e_n$, and $e_i \cap e_{i+1} \neq \emptyset$ for $1 \leq i < n$.

Lemma 7. *Fix $k \geq 2$. Let G be a rank k hypergraph, and let $\mathcal{H}$ be a finite family of fixed, connected hypergraphs. Let $f = \Delta^{O(1)}$, where f is sufficiently large. Suppose that*

- *G is $(\Delta, \omega_2, \ldots, \omega_k)$-sparse, where $\omega_j = \omega_j(\Delta) = f^{o(1)}$ for all $2 \leq j \leq k$.*

- *For all $H \in \mathcal{H}$, $\Delta_H(G) \leq \Delta^{\frac{v(H)-1}{k-1}}/f^{v(H)}$.*

Then $V(G)$ can be partitioned into $O(\Delta^{\frac{1}{k-1}}/f)$ parts such that the hypergraph induced by each part is $\mathcal{H}$-free and has maximum j-degree at most $2^{2k}f^{j-1}\omega_j$, for each $1 \leq j \leq k$.

Our main new contribution is a technical lemma (Lemma 13 in Section 4) that allows us to prove a large deviation inequality for positive random variables with mean less than 1. This may be of independent interest.

4

1.1 Organization

In Section 2, we use Lemma 7 to prove Theorems 3 and 5. The remaining sections are devoted to Lemma 7. In Section 3, we state two concentration theorems that we will use in Section 5. In Section 4, we prove our main probabilistic lemma, which is also used in Section 5. Section 5 contains the proof of Lemma 7.

1.2 Notation

All of our O and o notation is with respect to the parameter $\Delta \to \infty$. In particular, when we write $f = O(g)$, we mean that there is a positive constant c such that $f < cg$.

Given a hypergraph G and $A \subset V(G)$, we define the $(j - |A|)$-uniform hypergraph

$$L_{G,j}(A) = \{B - A : B \in G \text{ with } A \subset B \text{ and } |B| = j\}.$$

Let $N_{G,j}(A) = V(L_{G,j}(A))$. When G is k-uniform, we drop the subscript j. When G is clear from the context, we drop the subscript G.

2 Proof of Theorem 3 and 5

We first prove a simple bound on the chromatic number of general rank k hypergraphs. The proof is based on a proof of Erdős and Lovász [8] for k-uniform hypergraphs. This bound will allow us to assume that f is sufficiently large in the proofs of Theorems 3 and 5.

2.1 Chromatic number of rank k hypergraphs

We will need the following version of the local lemma.

Asymmetric Local Lemma ([16]). *Consider a set $\mathcal{E} = \{A_1, \ldots, A_n\}$ of (typically bad) events that such each A_i is mutually independent of $\mathcal{E} - (\mathcal{D}_i \cup A_i)$, for some $\mathcal{D}_i \subset \mathcal{E}$. If for each $1 \leq i \leq n$*

- $\Pr[A_i] \leq 1/4$, *and*

- $\sum_{A_j \in \mathcal{D}_i} \Pr[A_j] \leq 1/4$,

then with positive probability, none of the events in $\mathcal{E}$ occur.

Lemma 8. *Fix $k \geq 2$. Let G be a rank k hypergraph with maximum j-degree Δ_j, $j = 2, \ldots, k$. Then*

$$\chi(G) \leq \max_{j=2}^{k} (4k(k-1)\Delta_j)^{1/(j-1)}.$$

Proof. Set

$$r = \max_{j=2}^{k} (4k(k-1)\Delta_j)^{1/(j-1)}.$$

Then $\Delta_j \leq \frac{r^{j-1}}{4k(k-1)}$ for each $j = 2, \ldots, k$. Assign each vertex $u \in V(G)$ a color chosen uniformy at random from $[r]$. For each edge $e \in G$, let B_e denote the event that each vertex in e received the same color. Then $\Pr[B_e] = r^{1-|e|}$. The event B_e depends only on events B_f such that $B_e \cap B_f \neq \emptyset$. Since

$$\sum_{f \in H: e \cap f \neq \emptyset} \Pr[B_f] = \sum_{j=2}^{k} \sum_{\substack{f \in H: e \cap f \neq \emptyset, \\ |f| = j}} \Pr[B_f] \leq \sum_{j=2}^{k} |e| \Delta_j r^{1-j} \leq 1/4,$$

the Asymmetric Local Lemma implies that there exists a coloring where no event B_e occurs. $\qquad\square$

2.2 Proof of Theorem 3

We will use the following stronger version of Theorem 2.

Theorem 9 ([7])**.** *Suppose G is a rank 3, triangle-free hypergraph with maximum 3-degree Δ and maximum 2-degree Δ_2. Then*

$$\chi(G) = O(\max\{(\frac{\Delta}{\log \Delta})^{1/2}, \frac{\Delta_2}{\log \Delta_2}\}).$$

In fact, we will prove the following stronger result.

Theorem 10. *Let G be a rank 3 hypergraph with maximum 3-degree at most Δ and maximum 2-degree at most Δ_2. Let $\mathcal{H}$ denote the family of rank 3 triangles. If*

$$\Delta_H(G) \leq \max\{\Delta^{1/2}, \Delta_2\}^{v(H)-1}/f$$

for all $H \in \mathcal{H}$, then

$$\chi(G) = O(\max\{\frac{\Delta}{\log f})^{1/2}, \frac{\Delta_2}{\log f}\}).$$

Proof. By Lemma 8, we may assume that f is sufficiently large. We consider two cases.

Case 1: $\Delta_2 \leq (\Delta \log f)^{1/2}$.

Set $g = \frac{f}{\log^{5/2} f}$. Since $\Delta_2 \le (\Delta \log f)^{1/2}$ and $v(H) \le 6$ for all $H \in \mathcal{H}$,

$$\Delta_H(G) \le \max\{\Delta^{1/2}, \Delta_2\}^{v(H)-1}/f \le (\Delta \log f)^{\frac{v(H)-1}{2}}/f \le \Delta^{\frac{v(H)-1}{2}}/g \qquad (2)$$

for each $H \in \mathcal{H}$.

Let

$$K = \{A \subset V(G) : |A| = 2 \text{ and } d_3(A) \ge \Delta^{1/2}\}.$$

Define the rank 3 hypergraph

$$G' = G - \{E \in G : A \subset E \text{ for some } A \in K\} \cup K.$$

In other words, if a pair of vertices has high codegree, then we replace all 3-edges containing the pair with a 2-edge between the pair. Hence any proper coloring of G' is also a proper coloring of G.

We will now apply Lemma 7 to G' with the function $g' := g^{1/6}$. Set $\omega_2 = \log^{1/2} g$ and $\omega_3 = 1$. We first check that G' is $(2\Delta, \omega_2, \omega_3)$-sparse. Note first that $\omega_2 = g'^{o(1)}$, $\omega_3 = g'^{o(1)}$, and G' has maximum 3-degree at most 2Δ. Also, by definition of K, G' has maximum $(3, 2)$-degree at most $\Delta^{1/2} < (2\Delta)^{1/2}\omega_3$. Let $u \in V(G)$. Since $|L_{G,3}(u)| \le \Delta$, u is in at most $2\Delta^{1/2}$ sets in K. Thus, G' has maximum $(2, 1)$-degree at most

$$(\Delta \log f)^{1/2} + 2\Delta^{1/2} < (2\Delta \log g)^{1/2} = (2\Delta)^{1/2}\omega_2,$$

which shows that G' is $(2\Delta, \omega_2, \omega_3)$-sparse.

Let $H' \in \mathcal{H}$. Suppose $T' \subset G'$ is a copy of H' which is not in G. Since each $e \in T' - G$ corresponds to at least $\Delta^{1/2}$ edges of size 3 in G, T' corresponds to at least $(\Delta^{1/2} - 5)^{|T'-G|}$ copies of H in G, where H is the triangle obtained by replacing each $e \in T' - G$ with a distinct size 3 edge containing e and some vertex outside of T'. Thus

$$\Delta_{H'}(G')(\Delta^{1/2} - 5)^{v(H)-v(H')} \le \Delta_H(G).$$

Using (2), this implies

$$\Delta_{H'}(G') \le 2\Delta_H(G)\Delta^{(v(H')-v(H))/2} \le 2\Delta^{(v(H')-1)/2}/g \le 2\Delta^{(v(H')-1)/2}/(g^{1/6})^{v(H')}.$$

Lemma 7 therefore implies that there exists a partition of $V(G') = V(G)$ into $O(\Delta^{1/2}/g')$ parts such that each part is $\mathcal{H}$-free and has maximum j-degree at most $O(g'^{j-1}\omega_j)$. By Theorem 9, we may properly color each part with

$$O(\max\{\frac{g'}{\log^{1/2} g'}, \frac{g'\omega_2}{\log(g'\omega_2)}\}) = O(\frac{g'}{\log^{1/2} g'})$$

different colors, resulting in a total of $O((\frac{\Delta}{\log g'})^{1/2}) = O((\frac{\Delta}{\log f})^{1/2})$ colors.

Case 2: $\Delta_2 > (\Delta \log f)^{1/2}$.

Set $\Delta' = \frac{\Delta_2^2}{\log f}$. Then $\Delta < \Delta'$. Thus G has maximum 3-degree at most Δ', and

$$\Delta_H(G) \le \max\{\Delta^{1/2}, \Delta_2\}^{v(H)-1}/f < \max\{\Delta'^{1/2}, \Delta_2\}^{v(H)-1}/f$$

for all $H \in \mathcal{H}$. We may therefore apply case 1 with Δ' in the role of Δ to obtain a coloring with at most $O((\frac{\Delta'}{\log f})^{1/2}) = O(\frac{\Delta_2}{\log f})$ colors. $\qquad\square$

2.3 Proof of Theorem 5

Let G be a k-uniform hypergraph with maximum degree Δ such that $\Delta_{k,l}(G) \le \Delta^{\frac{k-l}{k-1}}/f$ for $l = 2, \dots, k-1$. As in the proof of Theorem 9, we may assume that f is sufficiently large. Set $\omega_j = 1$ for $2 \le j \le k$. Observe that G is $(\Delta, \omega_2, \dots, \omega_k)$-sparse. For each $l = 2, \dots, k-1$, let H_l be the k-uniform hypergraph consisting of two edges which share exactly l vertices. Set $\mathcal{H} = \{H_2, \dots, H_{k-1}\}$ and $f' = f^{1/(2k-2)}$. Then for $a \in V(H_l)$ with $d_{H_l}(a) = 2$,

$$\Delta_{H_l} \le \Delta_{H_l,a} \le \Delta\binom{k-1}{l-1}\Delta_{k,l} \le \binom{k-1}{l-1}\Delta^{\frac{2k-1-l}{k-1}}/f \le \binom{k-1}{l-1}\Delta^{\frac{2k-1-l}{k-1}}/f'^{v(H_l)},$$

so Lemma 7 implies that there exists a partition of $V(G)$ into $O(\Delta^{\frac{1}{k-1}}/f')$ parts such that each part is $\mathcal{H}$-free and has maximum degree at most $O(f'^{k-1})$. Since each part is $\mathcal{H}$-free, each part is linear and can be colored with $O(\frac{f'}{\log^{1/(k-1)} f'})$ colors by Theorem 1. Using a different set of colors for each part gives a proper coloring of G with $O((\frac{\Delta}{\log f})^{\frac{1}{k-1}})$ colors.

3 Concentration results

Theorem 11 (McDiarmid [15]). *For $i = 1, \dots, n$, let Ω_i be a probability space, and let $\Omega = \prod_{i=1}^{n} \Omega_i$. For each i, let $X_i : \Omega_i \to \mathcal{A}_i$ be a random variable, and let $f : \prod \mathcal{A}_i \to \mathbb{R}$. Suppose that f satisfies $|f(x) - f(x')| \le c_i$ whenever the vectors x and x' differ only in the ith coordinate. Let $Y : \Omega \to \mathbb{R}$ be the random variable $f(X_1, \dots, X_n)$. Then for any $t > 0$,*

$$\Pr[|Y - \mathbf{E}[Y]| > t) \le 2e^{-2t^2/\sum_{i=1}^{n} c_i^2}$$

Theorem 12 (Kim-Vu [13]). *Suppose F is a hypergraph such that $|f| \le s$ for all $f \in F$. Let*

$$F' = \{f \in F : z_i = 1 \text{ for all } i \in f\},$$

where the z_i, $i \in V(F)$ are independent random variables taking values in $[0, 1]$. For $A \subset W$ with $|A| \le s$, let

$$Z_A = \sum_{f \in F : f \supset A} \prod_{i \in f - A} z_i.$$

Let $M_A = \mathbf{E}[Z_A]$ and $M_j = \max_{A : |A| \ge j} M_A$ for $j \ge 0$. Then there exist positive constants $a = a(s)$ and $b = b(s)$ such that for any $\lambda > 0$,

$$\Pr\left[\big| |F'| - \mathbf{E}[|F'|] \big| \ge a\lambda^s \sqrt{M_0 M_1} \right] \le b|V(F)|^{s-1} e^{-\lambda}.$$

4 Deviation inequality

To motivate our next lemma, which is the main novel ingredient in this work, we outline the proof of Lemma 7, which appears in Section 5. We are given a hypergraph G and a fixed hypergraph H, and we know that each $u \in V(G)$ is in very few copies of H. Our goal is to produce a coloring of G such that the subhypergraph induced by each color is H-free. We randomly color the vertices of G, hoping to remove all copies of H in the induced subgraph of each color. Consider some $u \in V(G)$. Since u is in few copies of H, the expected number of these which remain in u's color is much less than 1. We would like to conclude that with very low probability, u is in more than c copies of H in the subgraph induced by u's color, for some constant c. We could then use the Local Lemma to find a coloring such that u is in at most c copies of H in the subgraph induced by u's color, which would bring us close to our goal of removing all copies of H from the subgraph. To draw this conclusion, we could try to let F_u be the set of copies of H which contain u, and let F'_u be the set of copies of H which contain u and whose vertices all receive u's color. Then we could apply Theorem 12 to bound F'_u and make the above conclusion. However, in our case, "very low probability" requires us to set $\lambda = \Omega(\log |V(F_u)|)$. Since M_0 and M_1 are always at least 1 (the empty product is taken to be 1), this only allows us to conclude that u is in $\Omega(\log |V(F_u)|)$ copies of H. Since $|V(F_u)|$ will be tending to ∞, this is not small enough.

Frieze and the second author [10] overcame this by using a two-step random process. However, their proof used the linearity of G and the assumption that H is a triangle, and we were not able to easily duplicate their method. Our approach, which can be viewed as a generalization of the method in [1], is to instead bound the *transversal number* of the hypergraph F'_u. A *transversal* of F_u is a subset $K \subset V(F_u)$ such that $A \cap K \ne \emptyset$ for all $A \in F_u$. In other words, the subhypergraph of F_u induced by $V(F_u) - K$ has no edges. The *transversal number* of F_u, denoted $\tau(F_u)$, is the minimum size of a transversal of F_u.

Returning to our outline, we will use Lemma 13 below to show that the probability that $\tau(F'_u)$ is large is very low.

Lemma 13. *Suppose F is an s-uniform hypergraph, and z_i, $i \in V(F)$ are independent random indicator variables with $\Pr[z_i = 1] = p$, for all $i \in V(F)$. Let*

$$F' = \{A \in F : \forall i \in A, z_i = 1\}.$$

Suppose there exists $\alpha > 0$ such that $|F|p^{(1-\alpha)s} < 1$. Then for any $c \geq e2^s s\alpha$,

$$\Pr[\tau(F') > s^2(c/\alpha)^{s+1}] \leq s^2|V(F)|^{s-1}p^c.$$

We then repeat this for all $v \in V(G)$ in the same color class as u. Let K_v be the set of vertices in a transversal of size at most $\tau(F'_v)$. Following [1], we create a 2-graph on these vertices, with an edge from v to each vertex in K_v. Since $\tau(F'_v)$ is bounded by a constant, this graph has constant out-degree and can be properly colored with a constant number of colors. Since the neighborhood of v in this graph is a transversal and all of the vertices in the neighborhood of v received a different color than v, none of the edges of F_v (which correspond to copies of H containing v) could survive in v's new color class. Thus the subgraph induced by each new color contains no copies of H.

4.1 Proof of Lemma 13

We will need the following simple proposition to prove Lemma 13. It is a straightforward generalization of the well-known fact that a graph with many edges has either a large matching or a large star. Let F be a k-uniform hypergraph. Recall that $M \subset F$ is a *matching* if $A \cap B = \emptyset$ for any $A, B \in M$ with $A \neq B$. When F is k-uniform, recall that for all $A \subset V(F)$,

$$L_F(A) = \{B - A : B \in F \text{ with } A \subset B\}.$$

Proposition 14. *If F is a k-uniform hypergraph, then there exists $A \subset V(F)$ such that $L_F(A)$ contains a matching of size at least $\frac{|F|^{1/k}}{k-|A|}$.*

Proof. We induct on k. If $k = 1$, the claim holds with $A = \emptyset$. Assume the result for k, and let F be a $k+1$-uniform hypergraph with maximum degree Δ. By the greedy coloring algorithm, F can be partitioned into $k\Delta$ matchings, so F contains a matching with $\frac{|F|}{k\Delta}$ edges. Thus $\Delta > |F|^{k/(k+1)}$ or F contains a matching with at least $\frac{|F|}{k|F|^{k/(k+1)}} = \frac{|F|^{1/(k+1)}}{k}$ edges. In the second case, we are done (with $A = \emptyset$), so assume there exists $u \in V(F)$ with $d(u) > |F|^{k/(k+1)}$. Set $A = \{u\}$ and consider the k-uniform hypergraph $L_F(A)$. By

induction, there exists $B \subset V(L_F(A))$ such that $L_{L(A)}(B)$ contains a matching of size at least

$$\frac{|L_F(A)|^{1/k}}{(k-|B|)} = \frac{|L_F(A)|^{1/k}}{k+1-|A \cup B|} > \frac{(|F|^{k/(k+1)})^{1/k}}{k+1-|A \cup B|} = \frac{|F|^{1/(k+1)}}{k+1-|A \cup B|}.$$

Since $B \subset V(L_F(A)) \subset V(F)$ and $L_{L(A)}(B) = L_F(A \cup B)$, this completes the proof. $\square$

Proof of Lemma 13. For $k = 0, 1, \ldots, s$, set $\tau_k = |F|p^{(1-\alpha)k}$. For $k = 1, \ldots, s$, let

$$H_k = \{A \subset V(F) : |A| = k, d(A) > \tau_k, \text{ and } \forall B \subsetneq A, d(B) \leq \tau_{|B|}\}.$$

Fix k, where $1 \leq k \leq s$. Let $B \in \binom{V(F)}{b}$, where $b < k$. Suppose there exists $A \in H_k$ with $B \subsetneq A$. By definition of H_k, $d(B) \leq \tau_b$. Since each such A corresponds to at least τ_k sets in F, and each of these τ_k sets is counted at most $\binom{s}{k}$ times under this correspondence, this implies

$$\tau_k |\{A \in H_k : B \in \binom{A}{b}\}| \leq \binom{s}{k} d(B) < 2^s \tau_b. \tag{3}$$

Consider the k-uniform hypergraph $Z_k = \{A \in H_k : z_i = 1 \ \forall i \in A\}$. Suppose $|Z_k| \geq (c/\alpha)^k$. Then by Proposition 14, there exists $B \in \binom{V(Z_k)}{b}$, where $b < k$, such that the $(k-b)$-uniform hypergraph $L_{Z_k}(B)$ contains a matching of size at least $x_b := \frac{c/\alpha}{(k-b)}$. Each edge in this matching corresponds to an edge $A \in H_k$ with $B \subset A$. By (3), there are at most $2^s \tau_b / \tau_k$ edges. Thus the probability that such a B exists is at most

$$\sum_{b=1}^{k-1} \sum_{B \in \binom{V(H_k)}{b}} \binom{2^s \tau_b / \tau_k}{x_b} p^{(k-b)x_b} \leq \sum_{b=1}^{k-1} \sum_{B \in \binom{V(H_k)}{b}} (\frac{e2^s \tau_b}{\tau_k x_b})^{x_b} p^{(k-b)x_b}$$

$$< \sum_{b=1}^{k-1} \sum_{B \in \binom{V(H_k)}{b}} (\frac{\tau_b p^{k-b}}{\tau_k})^{x_b}$$

$$= \sum_{b=1}^{k-1} \sum_{B \in \binom{V(H_k)}{b}} p^{((1-\alpha)b+k-b-(1-\alpha)k)x_b}$$

$$= \sum_{b=1}^{k-1} \sum_{B \in \binom{V(H_k)}{b}} p^{\alpha(c/\alpha)}$$

$$< k|V(F)|^{k-1} p^c.$$

Therefore, for each $k = 1, \ldots, s$,

$$\Pr[|Z_k| > (c/\alpha)^k] < k|V(F)|^{k-1} p^c.$$

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Hence

$$\Pr[|\bigcup_{k=1}^{s} Z_k| \ge s(c/\alpha)^s]] < \Pr[|\bigcup_{k=1}^{s} Z_k| \ge \sum_{k=1}^{s}(c/\alpha)^k]$$

$$< \sum_{k=1}^{s} k|V(F)|^{k-1}p^c$$

$$< s^2|V(F)|^{s-1}p^c.$$

Since each edge in $\bigcup_{k=1}^{s} Z_k$ contains at most s vertices, this implies

$$\Pr[|\bigcup_{k=1}^{s} V(Z_k)| \ge s^2(c/\alpha)^s] < s^2|V(F)|^{s-1}p^c.$$

We now claim that $\bigcup_{k=1}^{s} V(Z_k)$ is a transversal of F'. Suppose $A \in F$ and $z_i = 1$ for all $i \in A$. Since $d(A) = 1 > |F|p^{(1-\alpha)s} = \tau_s$, $A \in H_s$ or $d(B) > \tau_b$ for some $B \in \binom{A}{b}$. If $A \in H_s$, then $A \in Z_s$ and $A \subset V(Z_s)$, so assume the second case. Choose a minimal set $B \in \binom{A}{b}$ with $d(B) > \tau_b$. Then $B \in H_b$, so $B \in Z_b$ and hence $A \cap V(Z_b) \ne \emptyset$. □

5 Proof of Lemma 7

We break the proof of Lemma 7 into two steps. In step 1, we prove Lemma 15, which is a slight variant of Lemma 7 when $f \ge \Delta^\epsilon$. In Section 5.2, we will use the same argument as in [1] to show that the proof of Lemma 7 can be reduced to Lemma 15.

5.1 Step 1: $f \ge \Delta^\epsilon$

Lemma 15. *Fix $k \ge 2$ and $\epsilon \in (0, \frac{1}{k-1})$. Let G be a rank k hypergraph, and let $\mathcal{H}$ be a finite family of fixed hypergraphs. Suppose that*

- *G is $(\Delta, \omega_2, \ldots, \omega_k)$-sparse, where $\omega_j = \omega_j(\Delta) = \Delta^{o(1)}$ for all $2 \le j \le k$.*

- *$\Delta_H(G) \le \Delta^{\frac{v(H)-1}{k-1} - v(H)\epsilon}$ for all $H \in \mathcal{H}$.*

Then $V(G)$ can be partitioned into $O(\Delta^{\frac{1}{k-1}-\epsilon})$ parts such that the hypergraph induced by each part is $\mathcal{H}$-free and has maximum j-degree at most $2\Delta^{(j-1)\epsilon}\omega_j$, for each $j \in [k]$.

Proof. Let $N = \max_{H \in \mathcal{H}} v(H)$. Color the vertices of G uniformly at random with $r = \Delta^{\frac{1}{k-1}-\epsilon}$ colors. Fix u, and for each $v \in V(G)$, let z_v be a random indicator variable which is 1 if v receives the same color as u and 0 otherwise. Note that $\Pr[z_v = 1] = 1/r$.

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For each $H \in \mathcal{H}$, choose $v_H \in V(H)$ such that $\Delta_{H,v_H}(G) = \Delta_H(G)$. Define a $(v(H)-1)$-uniform hypergraph

$$T_H(u) = \{V(A) - u : A \subset G \text{ with } A \cong_\phi H \text{ and } \phi(u) = v_H\}.$$

Since to each $A \subset G$ with $A \cong_\phi H$ and $\phi(u) = v_H$ we may associate the set $V(A) - u \in T_H(u)$,

$$v(T_H(u)) \leq (N-1)|T_H(u)|(N-1) \leq \Delta_H(G).$$

Also, let

$$T'_H(u) = \{A \in T_H(u) : \forall v \in A, z_v = 1\}.$$

We define $k + |\mathcal{H}|$ bad events for each u.

- $A_{u,j}$: For $1 \leq j \leq k$, $A_{u,j}$ denotes the event

$$Z_{u,j} := \sum_{A \in L_{H,j}(u)} \prod_{i \in A} z_i \geq 2\Delta^{(j-1)\epsilon} \omega_j.$$

- $B_{u,H}$: For each $H \in \mathcal{H}$, $B_{u,H}$ denotes the event $\tau(T'_H(u)) > (v(H)-1)^2 (c/\alpha_H)^{v(H)}$, where

$$\alpha_H = 1 - \frac{v(H)\epsilon - \frac{v(H)-1}{k-1}}{(\epsilon - \frac{1}{k-1})(v(H)-1)} = \frac{\epsilon}{(\frac{1}{k-1} - \epsilon)(v(H)-1)} > 0$$

and

$$c = \frac{(N^2 + 3N)2^N}{\frac{1}{k-1} - \epsilon} > 3N2^N.$$

To bound the probability of $A_{u,j}$, we apply Theorem 12 to the $(j-1)$-uniform hypergraph $L_{H,j}(u)$. Let $A \subset V(L_{H,j}(u))$ with $|A| \leq j - 1$. Then

$$M_A = d_j(A \cup \{u\})/r^{j-|A|-1} \leq \Delta_{j,|A|+1}(G)/r^{j-|A|-1}$$
$$\leq \Delta^{\frac{j-|A|-1}{k-1}} \omega_j / r^{j-|A|-1}$$
$$= \Delta^{(j-|A|-1)\epsilon} \omega_j.$$

Therefore $\mathbf{E}[Z_u] = M_0 \leq \Delta^{(j-1)\epsilon} \omega_j$ and $M_1 \leq \Delta^{(j-2)\epsilon} \omega_j$. Setting $\lambda = (k+3N+2)\log\Delta$, we obtain constants a and b such that

$$\Pr[A_{u,j}] = \Pr[Z_{u,j} \geq 2\Delta^{(j-1)\epsilon}\omega_j] < \Pr[Z_{u,j} \geq \Delta^{(j-1)\epsilon}\omega_j + a\Delta^{\epsilon(j-1)-\epsilon/2}\lambda^{j-1}]$$
$$\leq b(j\Delta)^{j-2}\Delta^{-(j+3N+2)}$$
$$< \Delta^{-3N}.$$

To bound $B_{u,H}$, we apply Lemma 13 with $F = T_H(u)$, $F' = T'_H(u)$, and $p = 1/r$. Since

$$|T_H(u)|p^{(1-\alpha_H)(v(H)-1)} \leq \Delta^{\frac{v(H)-1}{k-1}-v(H)\epsilon}p^{(1-\alpha_H)(v(H)-1)} = 1,$$

$\alpha_H \in (0,1)$, and $c > 3N2^N > e2^N N\alpha_H > e2^{v(H)-1}(v(H)-1)\alpha_H$, Lemma 13 implies

$$\begin{aligned}
\Pr[B_{u,H}] &\leq (v(H)-1)^2 v(T_H(u))^{v(H)-2}p^c \\
&= (v(H)-1)^2 v(T_H(u))^{v(H)-2}\Delta^{(-N^2-3N)2^N} \\
&< (v(H)-1)^2 v(T_H(u))^{v(H)-2}\Delta^{-N^2-3N} \\
&< N^2(N\Delta^{\frac{N-1}{k-1}})^{N-1}\Delta^{-N^2-3N} \\
&< \Delta^{-3N}.
\end{aligned}$$

Each event $A_{u,j}$ is determined by the colors assigned to the vertices in $N_j(u) \cup \{u\}$. Each event $B_{u,H}$ is determined by the colors assigned to the vertices in $T_H(u) \cup \{u\}$. Thus $A_{u,j}$ depends on:

- Events of the form $A_{v,i}$ where $N_j(u) \cap N_i(v) \neq \emptyset$. There are at most

$$(j\Delta^{\frac{j-1}{k-1}}\omega_j)(\sum_{l=1}^k l\Delta^{\frac{l-1}{k-1}}\omega_l) < (j\Delta)(k^2\Delta) < \Delta^N$$

  such events.

- Events of the form $B_{v,H}$, where $N_j(u) \cap V(T_H(v)) \neq \emptyset$. Since $|N_j(u)| \leq j\Delta$ and $|V(T(v,H))| \leq N\Delta^{\frac{N-1}{k-1}}$, there are at most $(j\Delta)|\mathcal{H}|\Delta^{\frac{N-1}{k-1}} \leq \Delta^N$ such events.

Also, $B_{u,H}$ depends on:

- Events of the form $A_{v,j}$ where $V(T_H(u)) \cap N_j(v) \neq \emptyset$. There at most $(k^2\Delta)N\Delta^{\frac{N-1}{k-1}} \leq \Delta^N$ such events.

- Events of the form $B_{v,H'}$, where $V(T_H(u)) \cap V(T_{H'}(v)) \neq \emptyset$. There are at most $|\mathcal{H}|N^2\Delta^{2\frac{N-1}{k-1}} \leq \Delta^{2N}$ such events.

Since the probability of each event is at most Δ^{-3N}, the Local Lemma implies that there exists a coloring of $V(G)$ with r colors so that none of the events $A_{u,j}$ or $B_{u,H}$ occur.

Fix a color, and consider the subhypergraph G' induced by the vertices which received that color. For each $u \in V(G')$ and each $H \in \mathcal{H}$, let $K(u,H)$ be a minimum sized transversal of $T'(u,H)$. Create a simple graph W on $V(G')$ with edge set $\{\{u,v\} : u \in V(G'), v \in K(u,H)$ for some $H \in \mathcal{H}\}$. Consider any subgraph of W on n' vertices.

Since no event $B_{u,H}$ occurs, the number of edges in this subgraph is at most $n'|\mathcal{H}|(N-1)^2(c/\alpha)^N$; it therefore contains a vertex with degree at most $2|\mathcal{H}|(N-1)^2(c/\alpha)^N$. W is therefore $2|\mathcal{H}|(N-1)^2(c/\alpha)^N$-degenerate and can thus be properly colored with $2|\mathcal{H}|(N-1)^2(c/\alpha)^N + 1$ new colors. Since each of the $K(u,H)$ is a transversal, the subhypergraph induced by each of these new colors is $\mathcal{H}$-free. Repeating this for each of the original r colors results in a partition of F into at most

$$r(2|\mathcal{H}|(N-1)^2(c/\alpha)^N + 1) = O(\Delta^{\frac{1}{k-1}-\epsilon})$$

parts, where each part is $\mathcal{H}$-free and has maximum j-degree at most $2\Delta^{(j-1)\epsilon}\omega_j$.   □

5.2   Step 2: $f < \Delta^\epsilon$

When $f < \Delta^\epsilon$, we use the same random halving argument as in [1]. We recursively divide the hypergraph into two parts until we obtain a set of hypergraphs, each with maximum degree small enough to apply Lemma 15. The halving step is accomplished with Lemma 17, while Proposition 18 is used to analyze the recursion.

Proposition 16. *Let G be a rank k, $(\Delta, 2\omega, \dots, 2\omega)$-sparse hypergraph. Then for any $A \subset V(G)$ and $h \geq |A|$, G contains at most $2^{(h+1)T}\omega^T\Delta^{\frac{h-|A|}{k-1}}$ connected hypergraphs on h vertices which contain A, where $T = 2^h$.*

Proof. Starting with any edge that intersects A, we try to greedily grow a connected subgraph which contains A and $h-|A|$ other vertices. At step i, we add an edge of size j_i which contains $l_i \geq 1$ vertices already in the subgraph and $a_i \geq 0$ vertices in A which are not already in the subgraph. There are at most $2^h\Delta_{j_i,l_i+a_i}$ choices for this edge. Since one edge is added at every step, this process terminates after at most $T = 2^h$ steps, resulting in a total of $2^{hT}\prod_{i=1}^{T}\Delta_{j_i,l_i+a_i}$ subgraphs containing A. Since at each step we add $j_i - (l_i + a_i)$ vertices outside of A to the subgraph, $\sum_{i=1}^{T} j_i - (l_i + a_i) = h - |A|$. Thus

$$2^{hT}\prod_{i=1}^{T}\Delta_{j_i,l_i+a_i} \leq 2^{(h+1)T}\omega^T\prod_{i=1}^{T}\Delta^{\frac{j_i-(l_i+a_i)}{k-1}} = 2^{(h+1)T}\omega^T\Delta^{\frac{h-|A|}{k-1}}.$$

□

Lemma 17. *Let G be a rank k hypergraph, and let $\mathcal{H}$ be a finite family of fixed, connected hypergraphs. Suppose that G is $(\Delta, 2\omega_2, \dots, 2\omega_k)$-sparse, where $\omega_j = \omega_j(\Delta) = \Delta^{o(1)}$ for all $2 \leq j \leq k$. Then for Δ sufficiently large, there exists a partition of $V(G)$ into two subhypergraphs G_1 and G_2 such that*

- *For each $i = 1, 2$ and $1 \leq l < j \leq k$, we have $\Delta_{j,l}(G_i) \leq \Delta_{j,l}(G)/2^{j-l} + \Delta^{\frac{j-l}{k-1}-\frac{1}{2k}}$*

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- *For each $i = 1, 2$ and $H \in \mathcal{H}$, we have $\Delta_H(G_i) \leq \Delta_H(G)/2^{v(H)-1} + \Delta^{\frac{v(H)-1}{k-1} - \frac{1}{2k}}$.*

Proof. Let $N = \max_{H \in \mathcal{H}} v(H)$. Color the vertices of G uniformly at random with the colors 1 and 2. For each $A \subset V(G)$, let $d'_j(A)$ denote the j-degree of A in the subhypergraph induced by the minimum color of a vertex in A.

For each $H \in \mathcal{H}$, choose $v_H \in V(H)$ such that $\Delta_{H,v_H}(G) = \Delta_H(G)$. For each $u \in V(G)$, define a $(v(H) - 1)$-uniform hypergraph

$$T_H(u) = \{V(A) - u : A \subset G \text{ with } A \cong_\phi H \text{ and } \phi(u) = v_H\}.$$

Also, for each $H \in \mathcal{H}$, let

$$T'_H(u) = \{A \in T_H(u) : \forall v \in A, v \text{ receives the same color as } u\}.$$

Define the following bad events:

- $C_{A,j}$: For each $A \subset V(G)$ with $d_j(A) > 0$, $C_{A,j}$ denotes the event

$$d'_j(A) > \Delta_{j,|A|}(G)/2^{j-|A|} + \Delta^{\frac{j-|A|}{k-1} - \frac{1}{2k}}.$$

- $B_{u,H}$: For each $H \in \mathcal{H}$, $B_{u,H}$ denotes the event

$$|T'_H(u)| > \Delta_H(G)/2^{v(H)-1} + \Delta^{\frac{v(H)-1}{k-1} - \frac{1}{2k}}.$$

We use Theorem 11 to bound the probability of each event. The random variable $d'_j(A)$ is determined by the colors of the vertices in $N_j(A)$. If $v \in N_j(A)$, changing v's color affects $d'_j(A)$ by at most $d_j(A \cup \{v\}) \leq \Delta_{j,|A|+1}$. Also,

$$\sum_{v \in N_j(A)} d_j(A \cup \{v\})^2 \leq \Delta_{j,|A|+1}(G) \sum_{v \in N_j(A)} d_j(A \cup \{v\}) < \Delta_{j,|A|+1}(G) j d_j(A)$$

$$\leq j \Delta_{j,|A|+1}(G) \Delta_{j,|A|}(G)$$

$$\leq 4j \Delta^{\frac{2j-2|A|-1}{k-1}} \omega_j^2.$$

Consequently, Theorem 11 and $\omega_j(\Delta) = \Delta^{o(1)}$ imply

$$\Pr[C_{A,j}] < \Pr[d'_j(A) > \Delta_{j,|A|}(G)/2^{j-|A|} + (4j\Delta^{\frac{2j-2|A|-1}{k-1}} \omega_j^2 2N \log \Delta)^{1/2}]$$

$$\leq 2e^{-4N \log \Delta}$$

$$< \Delta^{-3N}.$$

Let $T_H(u, v)$ denote the set of copies of H in G containing both u and v. The random variable $|T'_H(u)|$ is determined by the colors of the vertices in $V(T_H(u))$. Changing the

color of $v \in V(T_H(u))$ affects $|T'_H(u)|$ by at most $|T_H(u,v)|$, which, by Proposition 16 (with $|A| = 2$), is at most $\omega\Delta^{\frac{v(H)-2}{k-1}}$, where $\omega = 2^{(v(H)+1)2^{v(H)}} \max_{j=1}^k \omega_j^{2^{v(H)}} = \Delta^{o(1)}$. Note also that Proposition 16 (with $|A| = 1$) implies $|T_H(u)| < \omega\Delta^{\frac{v(H)-1}{k-1}}$. Since

$$\sum_{v \in V(T(u,H))} |T_H(u,v)|^2 \le \omega\Delta^{\frac{v(H)-2}{k-1}} \sum_{v \in V(T(u,H))} |T_H(u,v)|$$

$$\le \omega\Delta^{\frac{v(H)-2}{k-1}} v(H)|T_H(u)|$$

$$\le v(H)\Delta^{\frac{v(H)-2}{k-1}+\frac{v(H)-1}{k-1}}\omega^2$$

$$= v(H)\Delta^{\frac{2v(H)-3}{k-1}}\omega^2,$$

$$\Pr[B_{u,H}] < \Pr[T'_H(u) > \Delta_H/2^{v(H)-1} + (v(H)\Delta^{\frac{2v(H)-3}{k-1}}\omega^2 2N \log\Delta)^{1/2}]$$

$$\le 2e^{-4N\log\Delta}$$

$$< \Delta^{-3N}.$$

The event $C_{A,j}$ is determined by the colors of the vertices in $N_j(A) \cup A$. The event $B_{u,H}$ is determined by the colors of the vertices in $T_H(u) \cup \{u\}$. Thus $C_{A,j}$ depends on:

- Events of the form $C_{B,i}$, where $N_j(A) \cap N_i(B) \ne \emptyset$. A vertex $u \in N_j(A)$ is in at most $i\Delta 2^i$ sets $N_i(B)$, so there are at most $(j\Delta)k^2\Delta^2 2^k < \Delta^{2N}$ such events.

- Events of the form $B_{u,H}$, where $N_j(A) \cap V(T_H(u)) \ne \emptyset$. Since $|V(T_H(u))| \le N\Delta^{\frac{N-1}{k-1}}$, there are at most $(j\Delta)|\mathcal{H}|N\Delta^{\frac{N-1}{k-1}} < \Delta^{2N}$ such events.

Also, $B_{u,H}$ depends on:

- Events of the form $B_{v,H'}$, were $V(T_H(u)) \cap V(T_{H'}(v)) \ne \emptyset$. There are at most $|\mathcal{H}|N^2\Delta^{2\frac{N-1}{k-1}} < \Delta^{2N}$ such events.

- Events of the form $C_{A,j}$, were $N_j(A) \cap V(T_H(u)) \ne \emptyset$. There are most $(N\Delta^{\frac{N-1}{k-1}})(k^2\Delta 2^k) < \Delta^{2N}$ such events.

Since the probability of each event is at most Δ^{-3N}, the Local Lemma implies that there exists a 2-coloring of $V(G)$ such that no event $C_{A,j}$ or $B_{u,H}$ holds. $\qquad\square$

Proposition 18. *Let $a, b, m \ge 1$ be fixed and $s_0 = d_0^{a/b}g$ for $g > 0$. Suppose that the sequences d_t and s_t have initial values d_0 and s_0 and satisfy*

$$d_{t+1} = \frac{d_t}{2^b} + d_t^{1-1/m} \quad and \quad s_{t+1} = \frac{s_t}{2^a} + d_t^{a/b-1/m}.$$

Then there exists $D > 0$ such that

$$d_t \le 2d_0 2^{-bt} \quad and \quad s_t \le d_t^{a/b}g + d_t^{a/b-1/(2m)}$$

for all $d_t \ge D$.

Proof. Note that for any constant c, as $d_0 \to \infty$,

$$(c + d_0^{1/m} 2^{-bt/m})^m = d_0 2^{-bt} + \sum_{i=0}^{m-1} \binom{m}{i} c^{m-i} d_0^{i/m} 2^{-\frac{bti}{m}} = d_0 2^{-bt} + o(d_0) \le 2 d_0 2^{-bt}.$$

To prove the first inequality, we will thus prove by induction the tighter bound (for D sufficiently large)

$$d_t \le \left(\frac{2^b}{m(2^{b/m} - 1)} + d_0^{1/m} 2^{-bt/m} \right)^m.$$

This is clear for d_0, so assume the bound for d_t. Then

$$d_{t+1} = \frac{d_t}{2^b} + d_t^{1-1/m} \le \frac{1}{2^b} \left(\frac{2^b}{m} + d_t^{1/m} \right)^m \le \frac{1}{2^b} \left(\frac{2^b}{m} + \frac{2^b}{m(2^{b/m} - 1)} + d_0^{1/m} 2^{-bt/m} \right)^m$$

$$= \left(\frac{2^b}{m(2^{b/m} - 1)} + d_0^{1/m} 2^{-b(t+1)/m} \right)^m.$$

For the second inequality, we first prove by induction $s_{t+1} = \frac{s_0}{x^{t+1}} + \sum_{k=0}^{t} \frac{d_k^y}{x^{t-k}}$, where $x = 2^a$ and $y = a/b - 1/m$. This is clear for $t = 0$, so assume the bound for s_t. Then

$$s_{t+1} = \frac{s_t}{x} + d_t^y = \frac{s_0}{x^{t+1}} + \frac{1}{x} \sum_{k=0}^{t-1} \frac{d_k^y}{x^{t-1-k}} + d_t^y = \frac{s_0}{x^{t+1}} + \sum_{k=0}^{t} \frac{d_k^y}{x^{t-k}}, \tag{4}$$

completing the induction. Using $d_k \le 2 d_0 2^{-bk}$ and $a - by = b/m \ge 0$,

$$\sum_{k=0}^{t} \frac{d_k^y}{x^{t-k}} \le \frac{(2d_0)^y}{x^t} \sum_{k=0}^{t} \left(\frac{x}{2^{by}} \right)^k = \frac{(2d_0)^y}{x^t} \sum_{k=0}^{t} 2^{(a-by)k} < \frac{(2d_0)^y}{x^t} \frac{2^{(t+1)(a-by)}}{2^{a-by} - 1}$$

$$= \frac{(2d_0)^y 2^{(t+1)(a-by) - at}}{2^{b/m} - 1} \tag{5}$$

By definition of d_t, $d_t \ge d_0 2^{-bt}$. Using this with (4) and with (5),

$$s_t < s_0 2^{-at} + \frac{(2d_0)^y 2^{a-by-tby}}{2^{b/m} - 1} \le s_0 \left(\frac{d_t}{d_0} \right)^{a/b} + \frac{(2d_t 2^{bt})^y 2^{a-by-tby}}{2^{b/m} - 1}$$

$$\le d_0^{a/b} g \left(\frac{d_t}{d_0} \right)^{a/b} + \frac{d_t^y 2^{y+a-by}}{2^{b/m} - 1}$$

$$= d_t^{a/b} g + \frac{d_t^{a/b-1/m} 2^{y+a-by}}{2^{b/m} - 1}$$

$$< d_t^{a/b} g + d_t^{a/b-1/(2m)},$$

where the last inequality assumes

$$d_t \ge D > \left(\frac{2^{y+a-by}}{2^{b/m} - 1} \right)^{2m}.$$

$\square$

Proof of Lemma 7. Define sequences

$$d_{t+1} = \frac{d_t}{2^{k-1}} + d_t^{1-\frac{1}{2k}}$$

$$r_{j,l,t+1} = \frac{r_{j,l,t}}{2^{j-l}} + d_t^{\frac{j-l}{k-1}-\frac{1}{2k}}, 1 \le l < j \le k$$

$$s_{H,t+1} = \frac{s_{H,t}}{2^{v(H)}-1} + d_t^{\frac{v(H)-1}{k-1}-\frac{1}{2k}}, H \in \mathcal{H},$$

where $d_0 = \Delta$, $r_{j,l,0} = \Delta^{\frac{j-l}{k-1}}\omega_j(\Delta)$, and $s_{H,0} = \Delta^{\frac{v(H)-1}{k-1}}/f^{v(H)}$. Since $r_{j,l,0} \le 2\Delta^{\frac{j-l}{k-1}}\omega_j(\Delta)$, we may apply Lemma 17 to G to obtain hypergraphs $G_{1,1}$ and $G_{1,2}$, each with maximum degree at most $d_1 = r_{k,1,1}$, maximum (j,l)-degree at most $r_{j,l,1}$, and maximum H-degree at most $s_{H,1}$. We apply this halving step a total of T times, where $T = \lceil \frac{1}{k-1}\log_2 \frac{2d_0}{f^{4Nk}}\rceil$ and $N = \max_{H\in\mathcal{H}} v(H)$. Specifically, at each step, we apply Lemma 17 (with parameters $d_t = r_{k,1,t}$, $r_{j,l,t}$, $s_{H,t}$) to each of the hypergraphs $G_{t,1},\dots,G_{t,2^t}$. Since $\omega_j(\Delta) = f^{o(1)}$, and for $t \le T$,

$$d_t \ge d_0 2^{-(k-1)T} \ge d_0 2^{-(k-1)(1+\frac{1}{k-1}\log_2 \frac{2d_0}{f^{4Nk}})} = 2^{-k}f^{4Nk}, \tag{6}$$

$\omega_j(\Delta) = d_t^{o(1)}$. Also, recall that f is sufficiently large, so by (6), we may assume that if $t \le T$, then d_t is sufficiently large to apply Proposition 18. Thus Proposition 18 (with each $r_{j,l,t}$ in the role of s_t, $g = \omega_j(\Delta), b = k-1, a = j-l$, and $m = 2k$) implies

$$r_{j,l,t} \le d_t^{\frac{j-l}{k-1}}\omega_j(\Delta) + d_t^{\frac{j-l}{k-1}-\frac{1}{4k}} \le 2d_t^{\frac{j-l}{k-1}}\omega_j(\Delta).$$

Thus each $G_{t,i}$ is $(d_t, 2\omega_2,\dots,2\omega_k)$-sparse, so we may apply Lemma 17 to obtain 2^{t+1} new hypergraphs $G_{t+1,1},\dots,G_{t+1,2^{t+1}}$ such that each $G_{t+1,i}$ has maximum degree $d_{t+1} = r_{k,1,t+1}$, maximum (j,l)-degree $r_{j,l,t+1}$, and maximum H-degree $s_{H,t+1}$. In the final step, we obtain 2^T hypergraphs $G_{T,1},\dots,G_{T,2^T}$, each with maximum degree d_T, maximum (j,l)-degree $r_{j,l,T}$ and maximum H-degree $s_{H,T}$. By Proposition 18,

$$d_T \le 2d_0 2^{-(k-1)T} \le f^{4Nk} < 2^k f^{4Nk},$$

and

$$r_{j,l,T} \le 2d_T^{\frac{j-l}{k-1}}\omega_j(\Delta) \le (2^k f^{4Nk})^{\frac{j-l}{k-1}}\omega_j(\Delta).$$

Proposition 18 (with each $s_{H,T}$ in the role of s_T, $g = 1/f^{v(H)}, b = k-1, a = v(H)-1$, and $m = 2k$) also yields

$$
\begin{aligned}
s_{H,T} &\le d_T^{\frac{v(H)-1}{k-1}}/f^{v(H)} + d_T^{\frac{v(H)-1}{k-1}-\frac{1}{4k}} \\
&\le (f^{4Nk})^{\frac{v(H)-1}{k-1}}/f^{v(H)} + (f^{4Nk})^{\frac{v(H)-1}{k-1}-\frac{1}{4k}} \\
&\le 2(f^{4Nk})^{\frac{v(H)-1}{k-1}}/f^{v(H)} \\
&= 2(f^{4Nk})^{\frac{v(H)-1}{k-1}-\frac{v(H)}{4Nk}} \\
&< (2^k f^{4Nk})^{\frac{v(H)-1}{k-1}-\frac{v(H)}{4Nk}}.
\end{aligned}
$$

We may therefore apply Lemma 15 (with $\Delta = 2^k f^{4Nk}, \omega_j(\Delta)$, and $\epsilon = \frac{1}{4Nk}$) to partition each of the hypergraphs $G_{T,1}, \ldots, G_{T,2^T}$ into $O(f^{4Nk(\frac{1}{k-1} - \frac{1}{4Nk})})$ parts such that the hypergraph induced by each part is $\mathcal{H}$-free and has maximum j-degree at most $2(2^k f^{4Nk})^{\frac{j-1}{4Nk}}\omega_j < 2^{2k} f^{j-1}\omega_j$ for each $1 \le j \le k$. Summing over each of the 2^T hypergraphs, this results in a total of

$$2^T O(f^{\frac{4Nk}{k-1} - 1}) \le 2^{1 + \frac{1}{k-1} \log_2 \frac{2d_0}{f^{4Nk}}} O(f^{\frac{4Nk}{k-1} - 1}) = O(\Delta^{\frac{1}{k-1}}/f)$$

$\mathcal{H}$-free hypergraphs, each with maximum j-degree at most $2^{2k} f^{j-1}\omega_j(\Delta)$.

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