Numerical Simulation of a Weakly Nonlinear Model for Internal Waves

BY

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THESIS

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To my sister, Sally.

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TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

LIST OF TABLES

LIST OF FIGURES

SUMMARY

In this thesis we numerically simulate the evolution of an internal wave in deep water, valid in the weakly nonlinear regime. We begin with a brief motivation and then, to familiarize the reader, give the background that led us to our research.

Following the introduction, the first two chapters focus on the development of our model. Chapter 2 outlines the classical formulation of the governing equations for the water-wave problem and in Chapter 3 we derive the model. In more detail, we begin Chapter 3 by reformulating the classical problem in terms of interface variables and interface operators to obtain a system of evolution equations. Next, we describe a Boundary Perturbation Method (the method of Operator Expansions–OE) to simulate the interface operators. We conclude the chapter and thus our derivation with the nondimensionalization of the evolution system.

The next two chapters focus on the numerics. In Chapter 4, we first describe our numerical method and then derive exact solutions for linear water waves and traveling waves that are used to test our code. We present numerical results in Chapter 5. We give the results of our tests (temporal and spatial convergence) and further analyze the traveling wave approximations. Finally, after having confirmed that our scheme is accurate and stable, we apply it to our model and check energy conservation.

In Chapter 6, we conclude with an overview and future research on two-fluid systems.

CHAPTER 1

INTRODUCTION

1.1 Motivation

An internal water wave propagates along an interface between two fluids with varying densities due to variations in temperature and salinity (17). Internal waves are ubiquitous in the world's oceans and, as seen by satellite (14), they have been observed to travel for hundreds of miles within the oceans. Internal waves are important both as a significant source of energy and momentum transportation, and in their interactions with ocean dynamics and topographies. In addition to their many engineering and other practical applications, they inspire many difficult questions in both theoretical and numerical analysis.

1.2 Background

The classical formulation of the internal water–wave problem uses Euler's equations with kinematic and dynamic boundary conditions at the free interface. Well–known approximation models include, most notably, the Korteweg–de Vries, Benjamin–Ono, Boussinesq, Nonlinear Schrödinger, and Intermediate Long Wave equations (24; 3). Alternately, one could reformulate the classical system as a boundary integral equation, a Hamiltonian system, or evolution equations involving operators (see, (12; 7) for recent examples).

In this thesis, we reformulate the classical system as evolution equations. Our evolution equations govern interface variables which involve interface differential operators, e.g., Dirichlet–

Neumann operators (DNOs). The interface variables, first introduced by Benjamin & Bridges (1; 2) and Craig & Groves (8), are the free interface and a weighted difference of velocity potentials at the interface. These are analogous to the surface variables identified by Zakharov (32) for a Hamiltonian formulation of the classical single–fluid water–wave problem. Zakharov's surface variables are the free surface, $\eta(x,t)$, and the velocity potential at the surface, $\xi(x,t) = \phi(x,\eta(x,t),t)$, and the Hamiltonian is described by the integral:

$$H^{\mathrm{Z}} = \frac{1}{2}\int\int |\nabla\phi|^2 \; dy\; dx + \frac{1}{2}\int g\eta^2 \; dx.$$

Craig and Sulem (9) (see also the work of Milder (20; 21) and Milder & Sharp (22; 23) in the setting of electromagnetic scattering) introduced the DNO to the water–wave problem to make Zakharov's formulation more explicit. The DNO,

$$G(\eta)[\xi] := \partial_{\vec{N}}\phi,$$

where $\vec{N}$ is an exterior normal, is an operator that maps Dirichlet boundary data to Neumann boundary data (26), more specifically, to the normal derivative of the potential at the surface. It is a linear operator in ξ and nonlinear in η. Knowing the DNO, the process of recovering the potential inside the fluid domain is straightforward and is accomplished through an integral equation involving Green's functions. With the DNO, the Hamiltonian becomes

$$H^{ZCS} = \frac{1}{2}\int \left[\xi G(\eta)[\xi] + g\eta^2\right] dx,$$

and the classical problem can now be explicitly expressed in terms of surface variables, (η, ξ). The surface (or in our case, interface) formulation is appealing for a number of reasons, not the least of which is that it reduces the problem dimension. In (9) Craig and Sulem used this new formulation to numerically simulate surface gravity waves. Our work is motivated by (9; 8) and can be considered an extension of their work to the two–fluid case.

1.3 Problem Description

Beyond the formulation of the governing equations, this thesis is a numerical study of time–dependent internal wave propagation and for this we focus on two–fluid systems which Koop & Butler (16) believe are sufficient to capture the physics of multi-stratified layers. For simplicity, we neglect forces other than gravity and assume infinite extent in both the upper and lower layers, but we note that our method can easily be extended to include additional forces (e.g., surface tension), finite layers, and rigid–lid assumptions. The model derivation is based on a perturbative approach (25) in the weakly nonlinear regime (see also (5; 24)).

Given our model equations, the evolution of the internal wave is simulated by a Fourier collocation spectral method (10) in the spatial variable and a fourth–order Runge–Kutta algorithm (4) for time–stepping. The main effort lies with the approximation of the operators, and once this is accomplished, the numerical method is fast, highly accurate, and easy to implement. The full literature of numerical schemes to simulate the motion of a free surface or interface in the Euler equations of ideal fluid motion is far too vast to recapitulate here. For the interested reader we recommend the survey articles of (18; 31; 30; 29; 13).

CHAPTER 2

CLASSICAL FORMULATION OF THE INTERNAL WATER-WAVE PROBLEM

2.1 Euler's Equations of Motion

Consider two-dimensional fluid flow in the deep ocean where an interface well below the ocean surface separates two layers of varying densities. Denote the Eulerian fluid velocities in the upper and lower regions by

$$\boldsymbol{u}' = [u'(x,y,t), v'(x,y,t), 0], \quad \boldsymbol{u} = [u(x,y,t), v(x,y,t), 0],$$

and the layer interface by $y = \eta(x,t)$. We define the fluid domains in the upper and lower regions by

$$S' := \{x \in \mathbb{R},\ y > \eta(x,t)\} \ \text{ and } \ S := \{x \in \mathbb{R},\ y < \eta(x,t)\}.$$

Note that two-dimensional implies one horizontal and one vertical dimension.

We assume the fluid in each layer is ideal. An ideal fluid is one that is incompressible, inviscid, and has constant density. Assuming that the fluids are immiscible, we denote the densities as ρ' and ρ, in the top and bottom layers, respectively. A less dense fluid floats on a more dense one, thus, $\rho > \rho'$. Inviscid flow is governed by Euler's equations of motion. These are coupled with an incompressibility condition (divergence of fluid velocity equals zero) to obtain the governing equations for our fluid flow:

$$\frac{D\boldsymbol{u'}}{Dt} = -\frac{1}{\rho'}\nabla p' + \boldsymbol{g}, \qquad \text{in } S' \tag{2.1a}$$

$$\nabla \cdot \boldsymbol{u'} = 0, \tag{2.1b}$$

and

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\nabla p + \boldsymbol{g}, \qquad \text{in } S \tag{2.2a}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.2b}$$

where $p' = p'(x, y, t)$ and $p = p(x, y, t)$ are pressure functions, $\boldsymbol{g}(x, y) = (0, -g)$ is the gravitational force, and $\frac{D}{Dt}$ is the rate of change following the fluid, or the material derivative.

2.2 Velocity Potential

We make one last assumption on the fluid flow: it is irrotational. Irrotationality (curl of fluid velocity equals zero) implies that there exists a velocity potential, ϕ' and ϕ, in each region such that

$$\boldsymbol{u}'(x, y, t) = \nabla\phi'(x, y, t), \qquad \text{in } S' \tag{2.3a}$$

$$\boldsymbol{u}(x, y, t) = \nabla\phi(x, y, t), \qquad \text{in } S. \tag{2.3b}$$

If we substitute (Equation 2.3a) and (Equation 2.3b) into (Equation 2.1b) and (Equation 2.2b), respectively, then the potential functions satisfy Laplace's equation in their domains of definition:

$$\Delta\phi' = 0, \qquad \text{in } S'$$

$$\Delta\phi = 0, \qquad \text{in } S.$$

Given the existence of the potentials, we wish for all remaining conditions to be formulated in terms of ϕ' and ϕ rather than $\boldsymbol{u}'$ and $\boldsymbol{u}$.

2.3 Boundary Conditions

To specify a unique solution we require lateral and vertical boundary conditions for our fluid flow. For the former we consider the classical λ–periodicity conditions

$$\eta(x+\lambda,t) = \eta(x,t), \quad \phi'(x+\lambda,y,t) = \phi'(x,y,t), \quad \phi(x+\lambda,y,t) = \phi(x,y,t), \quad \forall x \in \mathbb{R}.$$

As we have assumed fluids of infinite extent, we specify that the vertical velocity components of $\boldsymbol{u}'$ and $\boldsymbol{u}$, thus $\partial_y\phi'$ and $\partial_y\phi$, vanish far from the interface:

$$\partial_y\phi' \to 0, \qquad \text{as } y \to \infty$$

$$\partial_y\phi \to 0, \qquad \text{as } y \to -\infty.$$

There are two boundary conditions at the interface: a kinematic and dynamic condition. The kinematic condition states that a particle on the surface must stay on the surface or

$$F(x, y, t) = y - \eta(x, t). \tag{2.4}$$

For a particle on the surface this quantity is zero and hence the material derivative (rate of change following the fluid) is zero:

$$\frac{DF}{Dt} = \frac{d}{dt} F(x(t), y(t), t) = 0. \tag{2.5}$$

We first apply the chain rule to (Equation 2.5)

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial t} = 0,$$

and then substitute the expression for F (Equation 2.4):

$$-\partial_x \eta \frac{dx}{dt} + \frac{dy}{dt} - \partial_t \eta = 0.$$

The expressions $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are the horizontal and vertical velocity components, respectively, thus we arrive at

$$\partial_t \eta = \partial_y \phi' - \partial_x \eta \partial_x \phi', \qquad \text{at } y = \eta(x,t)$$

$$\partial_t \eta = \partial_y \phi - \partial_x \eta \partial_x \phi, \qquad \text{at } y = \eta(x,t).$$

The dynamic condition, also known as the Bernoulli pressure balance, states that in the absence of viscous forces, pressures at the interface must be equal. We return to Euler's momentum equations, (Equation 2.1a) and (Equation 2.2a), and rewrite these as

$$\partial_t \boldsymbol{u}' + (\boldsymbol{u}' \cdot \nabla)\boldsymbol{u}' = -\nabla \left(\frac{p'}{\rho'} + gy \right), \quad \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla \left(\frac{p}{\rho} + gy \right),$$

where g is the acceleration due to gravity. In terms of potentials, these become

$$\partial_t \nabla \phi' = -\nabla \left(\frac{p'}{\rho'} + \frac{1}{2} \left| \nabla \phi' \right|^2 + gy \right), \quad \partial_t \nabla \phi = -\nabla \left(\frac{p}{\rho} + \frac{1}{2} \left| \nabla \phi \right|^2 + gy \right).$$

Integration in the spatial variables gives

$$\partial_t \phi' + \frac{p'}{\rho'} + \frac{1}{2} \left| \nabla \phi' \right|^2 + gy = B(t), \quad \partial_t \phi + \frac{p}{\rho} + \frac{1}{2} \left| \nabla \phi \right|^2 + gy = D(t),$$

where B and D are arbitrary functions of time. As previously mentioned, inviscidness implies that $p' = p$ at the interface so if we conveniently choose B and D (which can be done given they are functions of time thus having no impact on the *gradient* of the velocity potential), such that

$$\frac{B(t)}{D(t)} = \frac{\rho}{\rho'},$$

then the pressure condition at the interface can be expressed as

$$\rho' \left(\partial_t \phi' + \frac{1}{2} \left| \nabla \phi' \right|^2 + g\eta \right) = \rho \left(\partial_t \phi + \frac{1}{2} \left| \nabla \phi \right|^2 + g\eta \right), \quad \text{at } y = \eta(x,t).$$

Collecting all of the constraints above, the full system of equations is given by

$$\Delta \phi' = 0, \qquad \text{in } S' \tag{2.6a}$$

$$\partial_y \phi' \to 0, \qquad y \to \infty \tag{2.6b}$$

$$\partial_t \eta = \partial_y \phi' - \partial_x \eta \ \partial_x \phi', \qquad \text{at } y = \eta \tag{2.6c}$$

$$\Delta \phi = 0, \qquad \text{in } S \tag{2.6d}$$

$$\partial_y \phi \to 0, \qquad y \to -\infty \tag{2.6e}$$

$$\partial_t \eta = \partial_y \phi - \partial_x \eta \ \partial_x \phi, \qquad \text{at } y = \eta \tag{2.6f}$$

$$\rho' \left(\partial_t \phi' + \frac{1}{2} \left| \nabla \phi' \right|^2 + g\eta \right) = \rho \left(\partial_t \phi + \frac{1}{2} \left| \nabla \phi \right|^2 + g\eta \right), \qquad \text{at } y = \eta \tag{2.6g}$$

which should be supplemented with initial conditions for η, ϕ, and ϕ':

$$n_0(x) = n(x,0), \quad \phi_0(x,y) = \phi(x,y,0), \quad \phi'_0(x,y) = \phi'(x,y,0).$$

The problem domain is depicted in Figure 1 over two periods and with a mean interface displacement equal to zero.

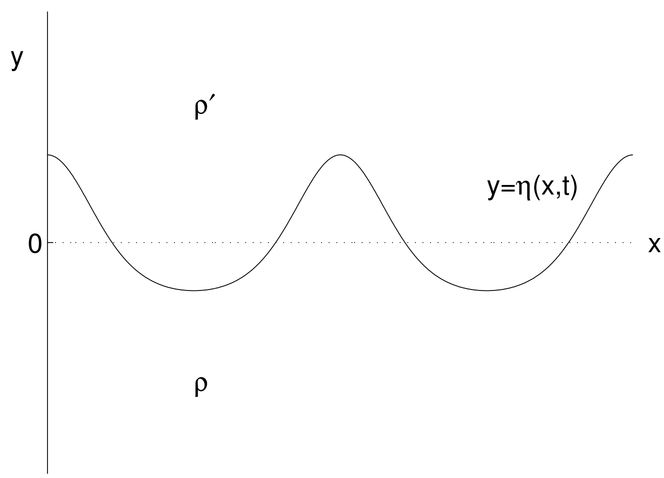

Figure 1: Domain of the internal water–wave problem.

CHAPTER 3

DERIVATION OF THE WEAKLY NONLINEAR MODEL

3.1 Evolution Equations

We begin by reformulating the system in (Equation 2.6) in terms of interfacial variables and operators (Dirichlet–Neumann operators). Following Benjamin & Bridges (1; 2) and Craig & Groves (8), we define

$$\xi(x,t) := \rho\Phi(x,t) - \rho'\Phi'(x,t),$$

where

$$\Phi(x,t) := \phi(x,\eta(x,t),t), \quad \Phi'(x,t) := \phi'(x,\eta(x,t),t),$$

so that ξ is a weighted difference of the potentials on the interface. Benjamin & Bridges showed that the variables (η,ξ) form a canonical set of variables in a Hamiltonian formulation of this two–layer flow (c.f., (32)) and we use these as evolution variables in our study. To find evolution equations for these, we first introduce Dirichlet-Neumann Operators (DNOs).

A DNO (20; 9; 26) maps Dirichlet data to Neumann data at the interface of a domain. To demonstrate this, consider the general boundary problem (independent of time) in the lower region:

$$
\begin{aligned}
&\Delta\phi = 0, && \text{in } S\\
&\partial_y\phi \to 0, && y \to -\infty\\
&\phi(x,\eta(x)) = \Phi(x), &&\\
&\phi(x+\lambda,y) = \phi(x,y). &&
\end{aligned}
$$

Upon finding a unique solution for ϕ, the normal derivative of ϕ evaluated at the interface can be found and this mapping from Φ to the normal derivative defines the DNO. Therefore, given our system (Equation 2.6), two useful definitions are

$$G'(\eta)[\Phi'] := \partial_{\vec{N}}\Phi', \tag{3.1a}$$

$$G(\eta)[\Phi] := \partial_{\vec{N}}\Phi, \tag{3.1b}$$

where $\vec{N}$ is the (unnormalized) normal vector to the interface. If we return to the implicit form of the interface (Equation 2.4) then

$$\vec{N} = \begin{pmatrix} -\partial_x \eta \\ \\ 1 \end{pmatrix},$$

where $\vec{N}$ is exterior to the lower fluid domain and interior to the upper, and we can alternatively write (Equation 3.1) as

$$G'(\eta)[\Phi'] := \nabla \Phi' \cdot \vec{N} = \nabla \phi'(x,y)|_{y=\eta(x)} \cdot (-\partial_x \eta, 1) = \left[-\partial_x \eta \; \partial_x \phi' + \partial_y \phi'\right]_{y=\eta(x)}, \tag{3.2a}$$

$$G(\eta)[\Phi] := \nabla \Phi \cdot \vec{N} = \nabla \phi(x,y)|_{y=\eta(x)} \cdot (-\partial_x \eta, 1) = \left[-\partial_x \eta \; \partial_x \phi + \partial_y \phi\right]_{y=\eta(x)}. \tag{3.2b}$$

With these definitions (Equation 2.6c) and (Equation 2.6f) become

$$\partial_t \eta = G'(\eta)[\Phi'], \tag{3.3a}$$

$$\partial_t \eta = G(\eta)[\Phi], \tag{3.3b}$$

which also imply that the DNOs are equal (i.e., there can be no jump in the normal component of velocity at the interface).

We reiterate that our goal is to reformulate the classical system in terms of (η, ξ), therefore we need to find an expression for the DNOs as operators on ξ rather than Φ and Φ'. Applying each operator to ξ we get

$$G'(\eta)[\xi] = G'(\eta)[\rho\Phi - \rho'\Phi'],$$

$$G(\eta)[\xi] = G(\eta)[\rho\Phi - \rho'\Phi'].$$

As DNOs are linear operators in ξ, we can rewrite the above as

$$G'(\eta)[\xi] = \rho G'(\eta)[\Phi] - \rho' G'(\eta)[\Phi'],$$

$$G(\eta)[\xi] = \rho G(\eta)[\Phi] - \rho' G(\eta)[\Phi'],$$

and the equality of the DNOs gives

$$G'(\eta)[\xi] = \rho G'(\eta)[\Phi] - \rho' G(\eta)[\Phi] = \left(\rho G'(\eta) - \rho' G(\eta)\right)[\Phi], \tag{3.4a}$$

$$G(\eta)[\xi] = \rho G'(\eta)[\Phi'] - \rho' G(\eta)[\Phi'] = \left(\rho G'(\eta) - \rho' G(\eta)\right)[\Phi']. \tag{3.4b}$$

If we define a new operator $B := \rho G'(\eta) - \rho' G(\eta)$, then (Equation 3.4) implies that

$$\Phi' = B^{-1}(\eta) G(\eta)[\xi], \tag{3.5a}$$

$$\Phi = B^{-1}(\eta) G'(\eta)[\xi]. \tag{3.5b}$$

Substituting (Equation 3.5) into (Equation 3.3) we arrive at two possible evolution equations for η:

$$\partial_t \eta = G'(\eta) B^{-1}(\eta) G(\eta)[\xi], \tag{3.6a}$$

$$\partial_t \eta = G(\eta) B^{-1}(\eta) G'(\eta)[\xi]. \tag{3.6b}$$

Seeking an equation for the time evolution of ξ we compute

$$\partial_t \xi = \partial_t \left[\rho \Phi - \rho' \Phi' \right],$$

for which an application of the chain rule gives

$$\partial_t \xi = \rho(\partial_y \phi \, \partial_t \eta + \partial_t \phi) - \rho'(\partial_y \phi' \, \partial_t \eta + \partial_t \phi'). \tag{3.7}$$

We rearrange the pressure condition (Equation 2.6g) as

$$\rho \partial_t \phi - \rho' \partial_t \phi' = -(\rho - \rho') g \eta - \frac{1}{2} \rho \left| \nabla \phi \right|^2 + \frac{1}{2} \rho' \left| \nabla \phi' \right|^2,$$

and substitute this into (Equation 3.7) to get

$$\partial_t \xi = -(\rho - \rho') g \eta - \frac{1}{2} \rho \left| \nabla \phi \right|^2 + \frac{1}{2} \rho' \left| \nabla \phi' \right|^2 + \rho \partial_y \phi \partial_t \eta - \rho' \partial_y \phi' \partial_t \eta. \tag{3.8}$$

If we replace $\partial_t \eta$ by either of the terms in (Equation 3.6) then we simply need expressions for $\partial_x \phi$, $\partial_x \phi'$, $\partial_y \phi$, and $\partial_y \phi'$ in terms of η and ξ, to close our system of equations. Another application of the chain rule delivers

$$\partial_x \Phi' = \partial_x \phi' + \partial_x \eta \partial_y \phi',$$

$$\partial_x \Phi = \partial_x \phi + \partial_x \eta \partial_y \phi,$$

and with (Equation 3.2) we have the systems

$$\begin{pmatrix} 1 & \partial_x \eta \\ -\partial_x \eta & 1 \end{pmatrix} \begin{pmatrix} \partial_x \phi \\ \partial_y \phi \end{pmatrix} = \begin{pmatrix} \partial_x \Phi \\ G(\eta)[\Phi] \end{pmatrix},$$

$$\begin{pmatrix} 1 & \partial_x \eta \\ -\partial_x \eta & 1 \end{pmatrix} \begin{pmatrix} \partial_x \phi' \\ \partial_y \phi' \end{pmatrix} = \begin{pmatrix} \partial_x \Phi' \\ G(\eta)[\Phi'] \end{pmatrix}.$$

Consequently, from a straightforward application of Gaussian elimination,

$$\begin{pmatrix} \partial_x \phi \\ \partial_y \phi \end{pmatrix} = \frac{1}{|\vec{N}|^2} \begin{pmatrix} \partial_x \Phi - (\partial_x \eta) G(\eta)[\Phi] \\ (\partial_x \eta) \partial_x \Phi + G(\eta)[\Phi] \end{pmatrix}, \tag{3.9a}$$

$$\begin{pmatrix} \partial_x \phi' \\ \partial_y \phi' \end{pmatrix} = \frac{1}{|\vec{N}|^2} \begin{pmatrix} \partial_x \Phi' - (\partial_x \eta) G'(\eta)[\Phi'] \\ (\partial_x \eta) \partial_x \Phi' + G'(\eta)[\Phi'] \end{pmatrix}, \tag{3.9b}$$

where

$$|\vec{N}|^2 = 1 + (\partial_x \eta)^2. \tag{3.10}$$

Recall (Equation 3.8). For the two terms involving $\partial_t \eta$ we select (Equation 3.6b) and (Equation 3.6a), respectively, and substitute these as well as the expressions for Φ, Φ', $\partial_x \phi$, $\partial_y \phi$, $\partial_x \phi'$, and $\partial_y \phi'$ from (Equation 3.5) and (Equation 3.9) into (Equation 3.8):

$$\begin{aligned}
\partial_t \xi = &- (\rho - \rho')g\eta - \frac{\rho}{2|\vec{N}|^4}\Big[\Big(\partial_x B^{-1}(\eta)G'(\eta)[\xi] - (\partial_x \eta)G(\eta)B^{-1}(\eta)G'(\eta)[\xi]\Big)^2 \\
&+ \Big((\partial_x \eta)\partial_x B^{-1}(\eta)G'(\eta)[\xi] + G(\eta)B^{-1}(\eta)G'(\eta)[\xi]\Big)^2\Big] \\
&+ \frac{\rho'}{2|\vec{N}|^4}\Big[\Big(\partial_x B^{-1}(\eta)G(\eta)[\xi] - (\partial_x \eta)G'(\eta)B^{-1}(\eta)G(\eta)[\xi]\Big)^2 \\
&+ \Big((\partial_x \eta)\partial_x B^{-1}(\eta)G(\eta)[\xi] + G'(\eta)B^{-1}(\eta)G(\eta)[\xi]\Big)^2\Big] \\
&+ \frac{\rho}{|\vec{N}|^2}\Big((\partial_x \eta)\partial_x B^{-1}(\eta)G'(\eta)[\xi] + G(\eta)B^{-1}(\eta)G'(\eta)[\xi]\Big)G'(\eta)B^{-1}(\eta)G(\eta)[\xi] \\
&- \frac{\rho'}{|\vec{N}|^2}\Big((\partial_x \eta)\partial_x B^{-1}(\eta)G(\eta)[\xi] + G'(\eta)B^{-1}(\eta)G(\eta)[\xi]\Big)G'(\eta)B^{-1}(\eta)G(\eta)[\xi].
\end{aligned}$$

After expanding the squared terms, we simplify:

$$\begin{aligned}
\partial_t \xi = &- (\rho - \rho')g\eta - \frac{\rho}{2|\vec{N}|^2}\Big[\Big(\partial_x B^{-1}(\eta)G'(\eta)[\xi]\Big)^2 + \Big(G(\eta)B^{-1}(\eta)G'(\eta)[\xi]\Big)^2\Big] \\
&+ \frac{\rho'}{2|\vec{N}|^2}\Big[\Big(\partial_x B^{-1}(\eta)G(\eta)[\xi]\Big)^2 + \Big(G'(\eta)B^{-1}(\eta)G(\eta)[\xi]\Big)^2\Big] \\
&+ \frac{\rho}{|\vec{N}|^2}\Big((\partial_x \eta)\partial_x B^{-1}(\eta)G'(\eta)[\xi] + G(\eta)B^{-1}(\eta)G'(\eta)[\xi]\Big)G(\eta)B^{-1}(\eta)G(\eta)[\xi] \\
&- \frac{\rho'}{|\vec{N}|^2}\Big((\partial_x \eta)\partial_x B^{-1}(\eta)G(\eta)[\xi] + G'(\eta)B^{-1}(\eta)G(\eta)[\xi]\Big)G'(\eta)B^{-1}(\eta)G(\eta)[\xi].
\end{aligned}$$

We manipulate the above expression further:

$$
\begin{aligned}
\partial_t \xi = &- (\rho - \rho')g\eta - \frac{\rho}{2|\vec{N}|^2}\left[\Big(\partial_x B^{-1}(\eta)G'(\eta)[\xi]\Big)^2 + \Big(G(\eta)B^{-1}(\eta)G'(\eta)[\xi]\Big)^2\right] \\
&+ \frac{\rho'}{2|\vec{N}|^2}\left[\Big(\partial_x B^{-1}(\eta)G(\eta)[\xi]\Big)^2 + \Big(G'(\eta)B^{-1}(\eta)G(\eta)[\xi]\Big)^2\right] \\
&+ \frac{\rho}{2|\vec{N}|^2}\left[2\Big(\partial_x\eta\Big)\Big(\partial_x B^{-1}(\eta)G'(\eta)[\xi]\Big)\Big(G(\eta)B^{-1}(\eta)G'(\eta)[\xi]\Big) + 2\Big(G(\eta)B^{-1}(\eta)G'(\eta)[\xi]\Big)^2\right] \\
&- \frac{\rho'}{2|\vec{N}|^2}\left[2\Big(\partial_x\eta\Big)\Big(\partial_x B^{-1}(\eta)G(\eta)[\xi]\Big)\Big(G'(\eta)B^{-1}(\eta)G(\eta)[\xi]\Big) + 2\Big(G'(\eta)B^{-1}(\eta)G(\eta)[\xi]\Big)^2\right].
\end{aligned}
$$

Finally, combining terms and substituting the expression for $|\vec{N}|$ from (Equation 3.10) yields an evolution equation for ξ:

$$
\begin{aligned}
\partial_t \xi = &- (\rho - \rho')g\eta + \frac{1}{2(1 + (\partial_x\eta)^2)}\Bigg[- \rho\Big(\partial_x B^{-1}(\eta)G'(\eta)[\xi]\Big)^2 \\
&+ \rho\Big(G(\eta)B^{-1}(\eta)G'(\eta)[\xi]\Big)^2 + 2\rho\Big(\partial_x\eta\Big)\Big(\partial_x B^{-1}(\eta)G'(\eta)[\xi]\Big)\Big(G(\eta)B^{-1}(\eta)G'(\eta)[\xi]\Big) \\
&+ \rho'\Big(\partial_x B^{-1}(\eta)G(\eta)[\xi]\Big)^2 - \rho'\Big(G'(\eta)B^{-1}(\eta)G(\eta)[\xi]\Big)^2 \\
&- 2\rho'\Big(\partial_x\eta\Big)\Big(\partial_x B^{-1}(\eta)G(\eta)[\xi]\Big)\Big(G'(\eta)B^{-1}(\eta)G(\eta)[\xi]\Big)\Bigg].
\end{aligned} \tag{3.11}
$$

3.2 Operator Expansions

The only specification remaining is the computation of the operators $G(\eta)$, $G'(\eta)$, and $B^{-1}(\eta)$. For this we choose a convenient and accurate Boundary Perturbation Method: the method of Operator Expansions (OE) (20; 21; 22; 23; 9). This approach is based on the smooth dependence of the DNO upon sufficiently small boundary deformations, e.g.,

$$\eta(x,t) = \varepsilon\tilde{\eta}(x,t), \quad \varepsilon \ll 1.$$

More specifically, Coifman & Meyer (6) (see also Nicholls & Reitich (26)) proved that the operators G and G' are analytic functions of Lipschitz deformations η and, thus, can be expanded as convergent Taylor series:

$$G(\eta)[\xi] = G(\varepsilon\tilde{\eta})[\xi] = \sum_{n=0}^{\infty} G_n(\tilde{\eta})[\xi]\varepsilon^n, \quad G'(\eta)[\xi] = G'(\varepsilon\tilde{\eta})[\xi] = \sum_{n=0}^{\infty} G'_n(\tilde{\eta})[\xi]\varepsilon^n. \tag{3.12}$$

These results can be extended to the operators B and $M := B^{-1}$ so that

$$M(\eta)[\xi] = M(\varepsilon\tilde{\eta})[\xi] = \sum_{n=0}^{\infty} M_n(\tilde{\eta})[\xi]\varepsilon^n.$$

The OE method delivers recursive formulas for the Taylor series terms (e.g., G_n) which can then be used in a *spectrally* accurate approximation, for instance,

$$G(\eta) \approx G^N(\eta) := \sum_{n=0}^{N} G_n(\tilde{\eta})[\xi]\varepsilon^n.$$

The derivation of formulas for the G_n have been given in a number of publications (20; 9; 26), but we include them here, for completeness. To begin, we note that

$$\phi'_p(x,y) = e^{ipx-|p|y}, \quad \phi_p(x,y) = e^{ipx+|p|y},$$

are $\lambda = 2\pi$–periodic solutions of (Equation 2.6a)–(Equation 2.6b) and (Equation 2.6d)–(Equation 2.6e), respectively, for any integer p. Focusing on the lower region, we insert ϕ_p into the definition of the DNO G (Equation 3.2b) to find

$$G(\varepsilon\tilde{\eta})\left[e^{ipx+|p|\varepsilon\tilde{\eta}}\right] = \left[\,|p| - \partial_x(\varepsilon\tilde{\eta})(ip)\right]e^{ipx+|p|\varepsilon\tilde{\eta}}.$$

Using the Taylor series for G (Equation 3.12) and expanding the exponentials in a Taylor series in ε, we obtain, after dropping the tildes,

$$\left(\sum_{n=0}^{\infty} G_n(\eta)\varepsilon^n\right)\left[e^{ipx}\sum_{n=0}^{\infty}\frac{(\eta\,|p|)^n}{n!}\varepsilon^n\right] = \left[\,|p| - \varepsilon(\partial_x\eta)(ip)\right]e^{ipx}\left[\sum_{n=0}^{\infty}\frac{(\eta\,|p|)^n}{n!}\varepsilon^n\right]. \qquad (3.13)$$

Equating the above left and right sides at order zero:

$$G_0\left[e^{ipx}\right] = |p|\,e^{ipx}. \qquad (3.14)$$

To derive a general expression for $n \geq 1$, we rearrange the double sum in (Equation 3.13):

$$\sum_{n=0}^{\infty} \left\{ \sum_{l=0}^{n} G_l(\eta) \left[e^{ipx} \frac{(\eta\,|p|)^{n-l}}{(n-l)!} \right] \right\} \varepsilon^n = |p|\, e^{ipx} \sum_{n=0}^{\infty} \frac{(\eta\,|p|)^n}{n!} \varepsilon^n - (\partial_x \eta)(ip) e^{ipx} \sum_{n=0}^{\infty} \frac{(\eta\,|p|)^n}{n!} \varepsilon^{n+1}.$$

Therefore, at order $n \geq 1$,

$$\sum_{l=0}^{n} G_l(\eta) \left[\frac{(\eta\,|p|)^{n-l}}{(n-l)!} e^{ipx} \right] = |p|\, \frac{(\eta\,|p|)^n}{n!} e^{ipx} - (\partial_x \eta)(ip) \frac{(\eta\,|p|)^{n-1}}{(n-1)!} e^{ipx},$$

or, solving for G_n,

$$G_n(\eta)\left[e^{ipx}\right] = \frac{\eta^n\,|p|^{n+1}}{n!} e^{ipx} - (\partial_x \eta)(ip) \frac{(\eta\,|p|)^{n-1}}{(n-1)!} e^{ipx} - \sum_{l=0}^{n-1} G_l(\eta) \left[\frac{(\eta\,|p|)^{n-l}}{(n-l)!} e^{ipx} \right]. \qquad (3.15)$$

At order one:

$$G_1(\eta)\left[e^{ipx}\right] = \Big(\eta\,|p|^2 - (\partial_x \eta)(ip)\Big) e^{ipx} - G_0\left[\eta\,|p|\, e^{ipx}\right]. \qquad (3.16a)$$

As we shall see, order zero and one operators suffice for a weakly nonlinear model; higher order terms can be easily recovered if desired from (Equation 3.15).

Similar manipulations for ϕ' and G' reveal that

$$G_0'\left[e^{ipx}\right] = -\,|p|\, e^{ipx}, \qquad (3.16b)$$

$$G_1'(\eta)\left[e^{ipx}\right] = \Big(\eta\,|p|^2 - (\partial_x \eta)(ip)\Big) e^{ipx} - G_0'(\eta)\Big[\eta(-\,|p|) e^{ipx}\Big]. \qquad (3.16c)$$

Remark: The above could also be found by observing that the solution in the upper region corresponds to replacing $|p|$ by -$|p|$ in the lower region solution.

Given the periodicity of ξ, we can write ξ as a Fourier series,

$$\xi(x) = \sum_{p=-\infty}^{\infty} \hat{\xi}_p e^{ipx},$$

where the p–th Fourier coefficient is given by

$$\hat{\xi}_p = \frac{1}{2\pi} \int_0^{2\pi} \xi(x) e^{-ipx}\, dx.$$

Using (Equation 3.14), we can find an expression for $G_0[\xi]$:

$$G_0[\xi] = G_0\Big[\sum_p \hat{\xi}_p e^{ipx}\Big] = \sum_p \hat{\xi}_p G_0\left[e^{ipx}\right] = \sum_p \hat{\xi}_p\, |p|\, e^{ipx}.$$

Further, if we define the Fourier multiplier $D := \dfrac{1}{i}\partial_x$ then

$$D[\xi] = \frac{1}{i}\partial_x \sum_p \hat{\xi}_p e^{ipx} = \frac{1}{i}\sum_p \hat{\xi}_p \partial_x e^{ipx} = \sum_p \hat{\xi}_p (p) e^{ipx}.$$

Thus, the differential operator D corresponds to p in Fourier space. To achieve the desired expression, we define another differential operator $|D|$ as the mapping to $|p|$ such that

$$|D|\,[\xi] = \sum_p \hat{\xi}_p\, |p|\, e^{ipx}.$$

Now we can rewrite $G_0[\xi]$ as

$$G_0[\xi] = \sum_p \hat{\xi}_p \left|p\right| e^{ipx} = \left|D\right| \xi. \tag{3.17a}$$

Similarly, to obtain expressions for $G_0'[\xi]$, $G_1(\eta)[\xi]$, and $G_1'(\eta)[\xi]$, we make note that

$$-\left|D\right|[\xi] = \sum_p \hat{\xi}_p(-\left|p\right|)e^{ipx}, \quad \partial_x[\xi] = \sum_p \hat{\xi}_p(ip)e^{ipx}, \quad \text{and} \quad \left|D\right|^2[\xi] = \sum_p \hat{\xi}_p \left|p\right|^2 e^{ipx}.$$

Recalling (Equation 3.16), we find expressions for the remaining operators:

$$G_0'[\xi] = G_0'\Big[\sum_p \hat{\xi}_p e^{ipx}\Big] = \sum_p \hat{\xi}_p G_0'\left[e^{ipx}\right] = \sum_p \hat{\xi}_p(-|p|)e^{ipx} = -|D|\,\xi, \tag{3.17b}$$

$$\begin{aligned}
G_1(\eta)[\xi] &= G_1(\eta)\Big[\sum_p \hat{\xi}_p e^{ipx}\Big] = \sum_p \hat{\xi}_p G_1(\eta)\left[e^{ipx}\right] \\
&= \sum_p \hat{\xi}_p \Big\{ \Big(\eta\,|p|^2 - (\partial_x\eta)(ip)\Big)\, e^{ipx} - G_0\left[\eta\,|p|\, e^{ipx}\right] \Big\} \\
&= \eta \sum_p \hat{\xi}_p\,|p|^2\, e^{ipx} - (\partial_x\eta)\sum_p \hat{\xi}_p(ip)e^{ipx} - \sum_p \hat{\xi}_p G_0\left[\eta\,|p|\,e^{ipx}\right] \\
&= \eta \sum_p \hat{\xi}_p\,|p|^2\, e^{ipx} - (\partial_x\eta)\sum_p \hat{\xi}_p(ip)e^{ipx} - G_0\Big[\eta \sum_p \hat{\xi}_p\,|p|\,e^{ipx}\Big] \\
&= \eta\,|D|^2\,\xi - (\partial_x\eta)(\partial_x\xi) - |D|\left[\eta\,|D|\,\xi\right],
\end{aligned} \tag{3.17c}$$

$$\begin{aligned}
G_1'(\eta)[\xi] &= G_1'(\eta)\Big[\sum_p \hat{\xi}_p e^{ipx}\Big] = \sum_p \hat{\xi}_p G_1'(\eta)\left[e^{ipx}\right] \\
&= \sum_p \hat{\xi}_p \Big\{ \Big(\eta\,|p|^2 - (\partial_x\eta)(ip)\Big)\, e^{ipx} - G_0'\left[\eta(-|p|)e^{ipx}\right] \Big\} \\
&= \eta \sum_p \hat{\xi}_p\,|p|^2\, e^{ipx} - (\partial_x\eta)\sum_p \hat{\xi}_p(ip)e^{ipx} - \sum_p \hat{\xi}_p G_0'\left[\eta(-|p|)e^{ipx}\right] \\
&= \eta \sum_p \hat{\xi}_p\,|p|^2\, e^{ipx} - (\partial_x\eta)\sum_p \hat{\xi}_p(ip)e^{ipx} - G_0'\Big[\eta \sum_p \hat{\xi}_p(-|p|)e^{ipx}\Big] \\
&= \eta\,|D|^2\,\xi - (\partial_x\eta)(\partial_x\xi) + |D|\left[\eta(-|D|)\xi\right] \\
&= \eta\,|D|^2\,\xi - (\partial_x\eta)(\partial_x\xi) - |D|\left[\eta\,|D|\,\xi\right].
\end{aligned} \tag{3.17d}$$

Remark: In regards to the last terms of G_1 and G_1', we make note that for an arbitrary periodic function, ζ:

$$G_0[\zeta] = |D|\,\zeta, \quad G_0'[\zeta] = -|D|\,\zeta.$$

Generally at order $n \geq 1$,

$$G_n(\eta)[\xi] = \frac{\eta^n}{n!} |D|^{n+1} \xi - (\partial_x \eta)\partial_x \frac{(\eta |D|)^{n-1}}{(n-1)!} \xi - \sum_{l=0}^{n-1} G_l(\eta) \frac{(\eta |D|)^{n-l}}{(n-l)!} \xi,$$
$$G'_n(\eta)[\xi] = \frac{\eta^n}{n!}(-|D|)^{n+1}\xi - (\partial_x \eta)\partial_x \frac{\big(\eta(-|D|)\big)^{n-1}}{(n-1)!}\xi - \sum_{l=0}^{n-1} G'_l(\eta) \frac{\big(\eta(-|D|)\big)^{n-l}}{(n-l)!}\xi.$$

To find forms for the M_n recall that $M = B^{-1}$ and $B = \rho G' - \rho' G$. We begin with the fact that

$$MM^{-1} = I.$$

This implies that

$$MB = M[\rho G' - \rho' G] = I,$$

which, upon expansion, can be written as

$$\left(\sum_{n=0}^{\infty} M_n(\eta)\varepsilon^n\right) \left[\rho \sum_{n=0}^{\infty} G'_n(\eta)[\xi]\varepsilon^n - \rho' \sum_{n=0}^{\infty} G_n(\eta)[\xi]\varepsilon^n\right] = I.$$

After collecting powers of ε we have

$$\sum_{n=0}^{\infty} \left\{ \rho \sum_{l=0}^{n} M_l(\eta) G'_{n-l}(\eta)[\xi] - \rho' \sum_{l=0}^{n} M_l(\eta) G_{n-l}(\eta)[\xi] \right\} \varepsilon^n = I, \tag{3.18}$$

and equating at order zero:

$$\rho M_0 G'_0[\xi] - \rho' M_0 G_0[\xi] = I.$$

Given $G_0'[\xi] = -G_0'[\xi]$, i.e. (Equation 3.17), we can combine and solve for M_0:

$$M_0[\xi] = -\frac{1}{\rho + \rho'} G_0^{-1}[\xi].$$

We define the Fourier multiplier $D^{-1} := i \int dx$. This integral operator is the right inverse of D, but is only the left inverse up to a constant. To render this a well–defined operator we restrict the range to the set of zero–mean functions. Applying this operator to ξ gives

$$D^{-1}[\xi] = i \int \sum_p \hat{\xi}_p e^{ipx}\, dx = i \sum_p \hat{\xi}_p \int e^{ipx}\, dx = i \sum_p \hat{\xi}_p \left(\frac{1}{ip}\right) e^{ipx}\, dx = \sum_p \hat{\xi}_p \left(\frac{1}{p}\right) e^{ipx}\, dx.$$

If we additionally define

$$|D|^{-1}[\xi] = \sum_p \hat{\xi}_p \left(\frac{1}{|p|}\right) e^{ipx}\, dx,$$

then

$$M_0[\xi] = -\frac{1}{\rho + \rho'} |D|^{-1} \xi. \tag{3.19}$$

At order $n \geq 1$, we equate the left and right sides of (Equation 3.18) and solve for M_n to get

$$M_n(\eta) = -M_0 \left\{ \rho \sum_{l=0}^{n-1} M_l G_{n-l}'(\eta)[\xi] - \rho' \sum_{l=0}^{n-1} M_l G_{n-l}(\eta)[\xi] \right\},$$

or,

$$M_n(\eta) = -M_0 \left\{ \sum_{l=0}^{n-1} \Big(\rho G_{n-l}'(\eta) - \rho' G_{n-l}(\eta)\Big) M_l(\eta)[\xi] \right\}.$$

Therefore, at order one,

$$M_1(\eta)[\xi] = -M_0\Big[\Big(\rho G_1'(\eta) - \rho' G_1(\eta)\Big) M_0[\xi]\Big],$$

and given $G_1'(\eta) = G_1(\eta)$, i.e (Equation 3.17),

$$M_1(\eta)[\xi] = -(\rho - \rho')M_0 G_1(\eta) M_0[\xi].$$

Finally, substituting the expressions for G_1 and M_0 from (Equation 3.17) and (Equation 3.19) gives

$$\begin{aligned}
M_1(\eta)[\xi] &= -\frac{\rho - \rho'}{(\rho + \rho')^2} |D|^{-1} \left[\Big(\eta |D|^2 - (\partial_x \eta)(\partial_x) - |D| [\eta |D|]\Big)\Big(|D|^{-1} [\xi]\Big)\right] \\
&= -\frac{\rho - \rho'}{(\rho + \rho')^2} |D|^{-1} \left[\eta |D| [\xi] - (\partial_x \eta)(\partial_x |D|^{-1} \xi) - |D| \left[\eta |D| |D|^{-1} [\xi]\right]\right] \\
&= -\frac{\rho - \rho'}{(\rho + \rho')^2} |D|^{-1} \left[\eta |D| \xi - (\partial_x \eta)(\partial_x |D|^{-1} \xi) - |D| [\eta \cdot \xi]\right].
\end{aligned}$$

In summary, all order zero and one operators are:

$$G_0[\xi] = |D|\,\xi, \tag{3.20a}$$

$$G_0'[\xi] = -\,|D|\,\xi, \tag{3.20b}$$

$$G_1(\eta)[\xi] = \eta\,|D|^2\,\xi - (\partial_x\eta)(\partial_x\xi) - |D|\,[\eta\,|D|\,\xi], \tag{3.20c}$$

$$G_1'(\eta)[\xi] = \eta\,|D|^2\,\xi - (\partial_x\eta)(\partial_x\xi) - |D|\,[\eta\,|D|\,\xi], \tag{3.20d}$$

$$M_0[\xi] = -\frac{1}{\rho+\rho'}\,|D|^{-1}\,\xi, \tag{3.20e}$$

$$M_1[\xi] = -\frac{\rho-\rho'}{(\rho+\rho')^2}\,|D|^{-1}\left[\eta\,|D|\,\xi - (\partial_x\eta)(\partial_x\,|D|^{-1}\,\xi) - |D|\,[\eta\cdot\xi]\right]. \tag{3.20f}$$

As our eventual goal is to find a weakly nonlinear model accurate to order two, the following operator combinations (derived from (Equation 3.20)) will serve our purposes:

$$G_0'M_0G_0[\xi] =\frac{1}{\rho+\rho'}\,|D|\,\xi, \tag{3.21a}$$

$$G_0'M_0G_1(\eta)[\xi] =\frac{1}{\rho+\rho'}\left[\eta\,|D|^2\,\xi - (\partial_x\eta)(\partial_x\xi) - |D|\,[\eta\,|D|\,\xi]\right], \tag{3.21b}$$

$$G_0'M_1G_0(\eta)[\xi] =\frac{\rho-\rho'}{(\rho+\rho')^2}\left[\eta\,|D|^2\,\xi - (\partial_x\eta)(\partial_x\xi) - |D|\,[\eta\,|D|\,\xi]\right], \tag{3.21c}$$

$$G_1'M_0G_0(\eta)[\xi] =-\,\frac{1}{\rho+\rho'}\left[\eta\,|D|^2\,\xi - (\partial_x\eta)(\partial_x\xi) - |D|\,[\eta\,|D|\,\xi]\right], \tag{3.21d}$$

$$G_0M_0G_0'[\xi] =\frac{1}{\rho+\rho'}\,|D|\,\xi, \tag{3.21e}$$

$$M_0G_0'[\xi] =\frac{1}{\rho+\rho'}\xi, \tag{3.21f}$$

$$M_0G_0[\xi] =-\,\frac{1}{\rho+\rho'}\xi. \tag{3.21g}$$

3.3 Nondimensionalization

At this point it is convenient to nondimensionalize our nonlinear system of equations governing (η, ξ), i.e. (Equation 3.6) & (Equation 3.11). We select (Equation 3.6a) for the evolution of η and recall that we denoted the operator B^{-1} by M, so we have

$$\partial_t \eta = G'(\eta)M(\eta)G(\eta)[\xi] \tag{3.22a}$$

$$\begin{aligned}\partial_t \xi = &-(\rho-\rho')g\eta + \frac{1}{2(1+(\partial_x\eta)^2)}\Big[-\rho\Big(\partial_x M(\eta)G'(\eta)[\xi]\Big)^2 \\ &+\rho\Big(G(\eta)M(\eta)G'(\eta)[\xi]\Big)^2 + 2\rho\Big(\partial_x\eta\Big)\Big(\partial_x M(\eta)G'(\eta)[\xi]\Big)\Big(G(\eta)M(\eta)G'(\eta)[\xi]\Big) \\ &+\rho'\Big(\partial_x M(\eta)G(\eta)[\xi]\Big)^2 - \rho'\Big(G'(\eta)M(\eta)G(\eta)[\xi]\Big)^2 \\ &-2\rho'\Big(\partial_x\eta\Big)\Big(\partial_x M(\eta)G(\eta)[\xi]\Big)\Big(G'(\eta)M(\eta)G(\eta)[\xi]\Big)\Big].\end{aligned} \tag{3.22b}$$

Let $L = \lambda/2\pi$ and a be a typical length and amplitude, respectively, and consider the classical scalings (17):

$$x = L\bar{x}, \quad y = L\bar{y}, \quad \eta = a\bar{\eta}.$$

A conventional scaling for time is the ratio of a length parameter to a linear wave speed. To determine the linear wave speed we solve the linearized (about $\eta = 0$) equations of (Equation 3.22). These equations are

$$\partial_t \eta = G_0' M_0 G_0[\xi]$$

$$\partial_t \xi = -(\rho-\rho')g\eta,$$

or, substituting (Equation 3.21a),

$$\partial_t \eta = \frac{1}{\rho + \rho'} \left| D \right| \xi \tag{3.23a}$$

$$\partial_t \xi = -(\rho - \rho') g \eta. \tag{3.23b}$$

Using the Fourier series representations of η and ξ leads to

$$\frac{d}{dt} \begin{pmatrix} \hat{\eta}_p(t) \\ \hat{\xi}_p(t) \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{|p|}{\rho + \rho'} \\ -(\rho - \rho')g & 0 \end{pmatrix} \begin{pmatrix} \hat{\eta}_p(t) \\ \hat{\xi}_p(t) \end{pmatrix},$$

where $\hat{\eta}_p$ and $\hat{\xi}_p$ are the p–th Fourier coefficients of η and ξ, respectively. The eigenvalues and respective eigenvectors of the coefficient matrix are

$$\mu_1 = i\sqrt{\frac{\rho - \rho'}{\rho + \rho'} g \, |p|}, \quad \boldsymbol{\chi}_1 = \begin{pmatrix} -i\sqrt{|p|} \\ \sqrt{g(\rho^2 - (\rho')^2)} \end{pmatrix},$$

$$\mu_2 = -i\sqrt{\frac{\rho - \rho'}{\rho + \rho'} g \, |p|}, \quad \boldsymbol{\chi}_2 = \begin{pmatrix} i\sqrt{|p|} \\ \sqrt{g(\rho^2 - (\rho')^2)} \end{pmatrix},$$

which give two solutions to the homogeneous linear system:

$$\boldsymbol{\chi}_1 e^{\mu_1 t}, \quad \boldsymbol{\chi}_2 e^{\mu_2 t}.$$

To obtain real-valued solutions we use the pair (μ_1, χ_1):

$$\boldsymbol{\chi}_1 e^{\mu_1 t} = \begin{pmatrix} \sqrt{|p|}\sin(\omega_p t) \\ \\ \sqrt{g(\rho^2 - (\rho')^2)}\cos(\omega_p t) \end{pmatrix} + i \begin{pmatrix} -\sqrt{|p|}\cos(\omega_p t) \\ \\ \sqrt{g(\rho^2 - (\rho')^2)}\sin(\omega_p t) \end{pmatrix},$$

where $\omega_p = \sqrt{\dfrac{\rho - \rho'}{\rho + \rho'} g\,|p|}$.

We choose the real and imaginary parts which implies that

$$\begin{pmatrix} \hat{\eta}_p(t) \\ \\ \hat{\xi}_p(t) \end{pmatrix} = c_1 \begin{pmatrix} \sqrt{|p|}\sin(\omega_p t) \\ \\ \sqrt{g(\rho^2 - (\rho')^2)}\cos(\omega_p t) \end{pmatrix} + c_2 \begin{pmatrix} -\sqrt{|p|}\cos(\omega_p t) \\ \\ \sqrt{g(\rho^2 - (\rho')^2)}\sin(\omega_p t) \end{pmatrix}$$

is a general solution. Given initial conditions $(\hat{\eta}_p(0), \hat{\xi}_p(0))$, we find

$$\begin{pmatrix} c_1 \\ \\ c_2 \end{pmatrix} = \begin{pmatrix} \dfrac{\hat{\xi}_p(0)}{\sqrt{g(\rho^2 - (\rho')^2)}} \\ -\dfrac{\hat{\eta}_p(0)}{\sqrt{|p|}} \end{pmatrix},$$

so that the exact solution for wavenumbers $p > 0$ is

$$\begin{pmatrix} \hat{\eta}_p(t) \\ \\ \hat{\xi}_p(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega_p t)\hat{\eta}_p(0) + \dfrac{|p|}{(\rho + \rho')\omega_p}\sin(\omega_p t)\hat{\xi}_p(0) \\ -\dfrac{(\rho + \rho')\omega_p}{|p|}\sin(\omega_p t)\hat{\eta}_p(0) + \cos(\omega_p t)\hat{\xi}_p(0) \end{pmatrix}.$$

Therefore, the dispersion relation is given by

$$\omega_p^2 = \frac{\rho - \rho'}{\rho + \rho'} g \left| p \right|,$$

and a (typical) wave speed (in the positive x direction) is

$$c_L = \frac{\omega_p}{|p|} = \omega_p L = \sqrt{\frac{(\rho - \rho')gL}{\rho + \rho'}}.$$

Thus, we scale time by

$$t = T\bar{t}, \quad T := \sqrt{\frac{(\rho + \rho')L}{(\rho - \rho')g}} = \sqrt{\frac{(1 + R)L}{(1 - R)g}},$$

where $R := \rho'/\rho$ is the (dimensionless) density ratio ($0 < R < 1$).

The differential operators are scaled as

$$\partial_x = \frac{1}{L}\partial_{\bar{x}}, \quad \partial_t = \frac{1}{T}\partial_{\bar{t}}, \quad |D| = \frac{1}{L}|\bar{D}|, \quad |D|^2 = \frac{1}{L^2}|\bar{D}|^2,$$

and all that remains to scale is the density–weighted potential difference. For this, we denote the unknown scaling by K:

$$\xi = K\bar{\xi}.$$

We determine an appropriate scaling by substituting the above scalings into (Equation 3.23),

$$\begin{aligned}\frac{1}{T}\partial_{\bar{t}} a\bar{\eta} &= \frac{1}{(\rho+\rho')L}\left|D\right| K\bar{\xi} \\ \frac{1}{T}\partial_{\bar{t}} K\bar{\xi} &= -(\rho-\rho')ga\bar{\eta},\end{aligned}$$

and simplify,

$$\begin{aligned}\partial_{\bar{t}}\bar{\eta} &= \frac{TK}{(\rho+\rho')aL}\left|D\right|\bar{\xi} \\ \partial_{\bar{t}}\bar{\xi} &= -\frac{T}{K}(\rho-\rho')ga\bar{\eta}.\end{aligned}$$

As the purpose of nondimensionalization is to simplify equations, we desire the (positive) coefficients of $(\bar{\eta},\bar{\xi})$ to equal one:

$$\begin{aligned}\frac{TK}{(\rho+\rho')aL} &= 1 \\ \frac{T}{K}(\rho-\rho')ga &= 1.\end{aligned}$$

We solve to find our above choice for T is confirmed and

$$K = a\sqrt{(\rho-\rho')(\rho+\rho')gL}\,.$$

Changing notation, we scale the potential as

$$\xi = aX\bar{\xi}, \quad X := \sqrt{(\rho - \rho')(\rho + \rho')gL}\,.$$

and define the dimensionless steepness parameter

$$\alpha := a/L.$$

We return to (Equation 3.22) and insert in it the above scalings:

$$\frac{1}{T}\partial_{\bar{t}}(a\bar{\eta}) = G'MG(a\bar{\eta})[aX\bar{\xi}] \tag{3.24a}$$

$$\begin{aligned}\frac{1}{T}\partial_{\bar{t}}(aX\bar{\xi}) = -(\rho - \rho')ga\bar{\eta} + \frac{1}{2\left(1 + (\frac{1}{L}\partial_{\bar{x}}a\bar{\eta})^2\right)}\Bigg[&- \rho\Big(\frac{1}{L}\partial_{\bar{x}}MG'(a\bar{\eta})[aX\bar{\xi}]\Big)^2 \\ &+ \rho\Big(GMG'(a\bar{\eta})[aX\bar{\xi}]\Big)^2 + 2\rho\Big(\frac{1}{L}\partial_{\bar{x}}a\bar{\eta}\Big)\Big(\frac{1}{L}\partial_{\bar{x}}MG'(a\bar{\eta})[aX\bar{\xi}]\Big)\Big(GMG'(a\bar{\eta})[aX\bar{\xi}]\Big) \\ &+ \rho'\Big(\frac{1}{L}\partial_{\bar{x}}MG(a\bar{\eta})[aX\bar{\xi}]\Big)^2 - \rho'\Big(G'MG(a\bar{\eta})[aX\bar{\xi}]\Big)^2 \\ &- 2\rho'\Big(\frac{1}{L}\partial_{\bar{x}}(a\bar{\eta})\Big)\left(\frac{1}{L}\partial_{\bar{x}}MG(a\bar{\eta})[aX\bar{\xi}]\right)\Big(G'MG(a\bar{\eta})[aX\bar{\xi}]\Big)\Bigg].\end{aligned} \tag{3.24b}$$

Remark: We condensed the notation for operator combinations and the operators are still dimensional.

We multiply (Equation 3.24a) by $\frac{T}{a}$ and (Equation 3.24b) by $\frac{T}{aX}$ and rearrange:

$$\partial_{\bar{t}}\bar{\eta} = (TX)G'MG(a\bar{\eta})[\bar{\xi}] \tag{3.25a}$$

$$\begin{aligned}\partial_{\bar{t}}\bar{\xi} = -\frac{T}{X}(\rho-\rho')g\bar{\eta} + \frac{aTX}{2\left(1+(\frac{1}{L}\partial_{\bar{x}}a\bar{\eta})^2\right)}\Bigg[&-\rho\Big(\frac{1}{L}\partial_{\bar{x}}MG'(a\bar{\eta})[\bar{\xi}]\Big)^2 \\ &+\rho\Big(GMG'(a\bar{\eta})[\bar{\xi}]\Big)^2 + 2\rho\Big(\frac{1}{L}\partial_{\bar{x}}a\bar{\eta}\Big)\Big(\frac{1}{L}\partial_{\bar{x}}MG'(a\bar{\eta})[\bar{\xi}]\Big)\Big(GMG'(a\bar{\eta})[\bar{\xi}]\Big) \\ &+\rho'\Big(\frac{1}{L}\partial_{\bar{x}}MG(a\bar{\eta})[\bar{\xi}]\Big)^2 - \rho'\Big(G'MG(a\bar{\eta})[\bar{\xi}]\Big)^2 \\ &-2\rho'\Big(\frac{1}{L}\partial_{\bar{x}}a\bar{\eta}\Big)\Big(\frac{1}{L}\partial_{\bar{x}}MG(a\bar{\eta})[\bar{\xi}]\Big)\Big(G'MG(a\bar{\eta})[\bar{\xi}]\Big)\Bigg].\end{aligned} \tag{3.25b}$$

We can expand:

$$G'MG(a\bar{\eta})[\bar{\xi}] = \frac{1}{L}\sum_{n=0}^{\infty}(\overline{G'MG})_n(\bar{\eta})[\bar{\xi}]\alpha^n, \quad GMG'(a\bar{\eta})[\bar{\xi}] = \frac{1}{L}\sum_{n=0}^{\infty}(\overline{GMG'})_n(\bar{\eta})[\bar{\xi}]\alpha^n,$$
$$MG(a\bar{\eta})[\bar{\xi}] = \sum_{n=0}^{\infty}(\overline{MG})_n(\bar{\eta})[\bar{\xi}]\alpha^n, \quad MG'(a\bar{\eta})[\bar{\xi}] = \sum_{n=0}^{\infty}(\overline{MG'})_n(\bar{\eta})[\bar{\xi}]\alpha^n,$$

where the bar notation for the operators (e.g. $\overline{MG}$) denotes a dimensionless operator. We insert the expansions into (Equation 3.25), drop the “bar” notation, and simplify given $\frac{T}{X} = \frac{1}{(\rho-\rho')g}$ and $TX = (\rho+\rho')L$:

$$\begin{aligned}
\partial_t \eta &= (\rho+\rho')(G'MG)_0[\xi] + \alpha(\rho+\rho')(G'MG)_1(\eta)[\xi] + \mathcal{O}(\alpha^2) \\
\partial_t \xi &= -\eta + \alpha\left(\frac{\rho+\rho'}{2[1+(\alpha\partial_x\eta)^2]}\right)\Bigg[-\rho\left(\partial_x(MG')_0[\xi]\right)^2 + \rho\left((GMG')_0[\xi]\right)^2 \\
&\qquad + \rho'\left(\partial_x(MG)_0[\xi]\right)^2 - \rho'\left((G'MG)_0[\xi]\right)^2\Bigg] + \mathcal{O}(\alpha^2).
\end{aligned}$$

Using the original operator notation, these equations can be expressed as

$$\begin{aligned}
\partial_t \eta &= (\rho+\rho')G_0'M_0G_0[\xi] + \alpha(\rho+\rho')\Big[G_0'M_0G_1(\eta)[\xi] + G_0'M_1G_0(\eta)[\xi] + G_1'M_0G_0(\eta)[\xi]\Big] + \mathcal{O}(\alpha^2) \\
\partial_t \xi &= -\eta + \alpha\left(\frac{\rho+\rho'}{2[1+(\alpha\partial_x\eta)^2]}\right)\Bigg[-\rho\left(\partial_x M_0G_0'[\xi]\right)^2 + \rho\left(G_0M_0G_0')[\xi]\right)^2 \\
&\qquad + \rho'\left(\partial_x M_0G_0[\xi]\right)^2 - \rho'\left(G_0'M_0G_0[\xi]\right)^2\Bigg] + \mathcal{O}(\alpha^2),
\end{aligned}$$

and a final substitution of the expressions from (Equation 3.21) gives

$$\begin{aligned}
\partial_t \eta &= |D|\,\xi + \alpha\left(\frac{\rho-\rho'}{\rho+\rho'}\right)\Big[\eta\,|D|^2\,\xi - (\partial_x\eta)(\partial_x\xi) - |D|\left[\eta\,|D|\,\xi\right]\Big] + \mathcal{O}(\alpha^2) \\
\partial_t \xi &= -\eta + \alpha\left(\frac{\rho-\rho'}{2(\rho+\rho')[1+(\alpha\partial_x\eta)^2]}\right)\Big[(|D|\,\xi)^2 - (\partial_x\xi)^2\Big] + \mathcal{O}(\alpha^2).
\end{aligned}$$

We complete the derivation of our model by assuming that $\alpha \ll 1$ and neglecting terms of order $\mathcal{O}(\alpha^2)$. Thus, we arrive at our nondimensional weakly nonlinear model:

$$\partial_t \eta = |D|\,\xi + \alpha\left(\frac{\rho-\rho'}{\rho+\rho'}\right)\left[\eta\,|D|^2\,\xi - (\partial_x\eta)(\partial_x\xi) - |D|\left[\eta\,|D|\,\xi\right]\right] \tag{3.26a}$$

$$\partial_t \xi = -\eta + \alpha\left(\frac{\rho-\rho'}{2(\rho+\rho')}\right)\left[(|D|\,\xi)^2 - (\partial_x\xi)^2\right]. \tag{3.26b}$$

CHAPTER 4

NUMERICAL METHOD

4.1 Numerical Approximations

For a numerical approximation of solutions to (Equation 3.26) it is natural, in light of nonlinearities and the lateral ($\lambda = 2\pi$) periodicity of solutions, to utilize a Fourier collocation approach (10). We denote the equally spaced collocation points by

$$x_j = j(\Delta x) = j\left(\frac{2\pi}{N_x}\right), \quad j = 0, 1, \ldots, N_x - 1$$

and approximate the solution $(\eta(x,t), \xi(x,t))$ by a (truncated) Fourier series of complex exponentials:

$$\eta^{N_x}(x,t) := \sum_{p=-N_x/2}^{N_x/2-1} \hat{\eta}_p^{N_x}(t) e^{ipx}, \quad \xi^{N_x}(x,t) := \sum_{p=-N_x/2}^{N_x/2-1} \hat{\xi}_p^{N_x}(t) e^{ipx},$$

where N_x is even. For a collocation method, we want the solution to satisfy the system (Equation 3.26) at the collocation points. In other words, we require that the residuals (errors),

$$\begin{aligned} R_{N_x}^{\eta}(x,t) &= \partial_t \eta^{N_x}(x,t) - |D|\, \xi^{N_x}(x,t) - \alpha\left(\frac{\rho - \rho'}{\rho + \rho'}\right)\Big[\eta^{N_x}(x,t)\, |D|^2\, \xi^{N_x}(x,t) \\ &\qquad - \left(\partial_x \eta^{N_x}(x,t)\right)\left(\partial_x \xi^{N_x}(x,t)\right) - |D|\left[\eta^{N_x}(x,t)\, |D|\, \xi^{N_x}(x,t)\right]\Big] \\ R_{N_x}^{\xi}(x,t) &= \partial_t \xi^{N_x}(x,t) + \eta^{N_x}(x,t) - \alpha\left(\frac{\rho - \rho'}{2(\rho + \rho')}\right)\left[\left(|D|\, \xi^{N_x}(x,t)\right)^2 - \left(\partial_x \xi^{N_x}(x,t)\right)^2\right], \end{aligned}$$

vanish at the grid points. Effectively, this means we enforce (Equation 3.26) at the collocation points, x_j. This results in a system of $(2N_x)$ ordinary differential equations for the $\eta^{N_x}(x_j,t)$ and $\xi^{N_x}(x_j,t)$:

$$
\begin{aligned}
\frac{d}{dt}\eta^{N_x}(x_j,t) &= |D|\,\xi^{N_x}(x,t)|_{x=x_j} + \alpha\left(\frac{\rho-\rho'}{\rho+\rho'}\right)\Big[\eta^{N_x}(x_j,t)\,|D|^2\,\xi^{N_x}(x,t)|_{x=x_j} \\
&\quad - \left(\partial_x\eta^{N_x}(x,t)|_{x=x_j}\right)\left(\partial_x\xi^{N_x}(x,t)|_{x=x_j}\right) - |D|\left[\eta^{N_x}(x,t)\,|D|\,\xi^{N_x}(x,t)\right]_{x=x_j}\Big]
\end{aligned}
\tag{4.1a}
$$

$$
\frac{d}{dt}\xi^{N_x}(x_j,t) = -\eta^{N_x}(x_j,t) + \alpha\left(\frac{\rho-\rho'}{2(\rho+\rho')}\right)\left[\left(|D|\,\xi^{N_x}(x,t)|_{x=x_j}\right)^2 - \left(\partial_x\xi^{N_x}(x,t)|_{x=x_j}\right)^2\right].
\tag{4.1b}
$$

For our computations, derivatives and Fourier multipliers are applied in Fourier space, e.g.,

$$
|D|\,\eta \approx |D|\,\eta^{N_x} = \sum_{p=-N_x/2}^{N_x/2-1} |p|\,\hat{\eta}_p^{N_x}(t)e^{ipx}.
$$

Products are performed pointwise in physical space and the Discrete Fourier Transform, accelerated by the Fast Fourier Transform algorithm, is used to transform between physical and Fourier space (10). Note that we only make use of the Fourier approximation for computing derivatives in physical space and nonlinear terms are trivial to evaluate because products are also done in physical space. This is the appeal of the collocation method, as opposed to the Galerkin or Tau method, or other Fourier spectral methods.

We approximate $\eta^{N_x}(x_j, t)$ and $\xi^{N_x}(x_j, t)$ with a fourth–order Runge–Kutta scheme (RK4) (4), which delivers approximate solutions at the time levels

$$t_m = m(\Delta t) = m\left(\frac{\tau}{N_t}\right), \quad m = 0, 1, \ldots, N_t$$

where τ is the total time. If we write system (Equation 4.1) as

$$\frac{d}{dt}\begin{pmatrix} \eta^{N_x}(x_j, t) \\ \\ \xi^{N_x}(x_j, t) \end{pmatrix} = f\begin{pmatrix} \eta^{N_x}(x_j, t) \\ \\ \xi^{N_x}(x_j, t) \end{pmatrix},$$

and let

$$w_m = \begin{pmatrix} \eta^{N_x}(x_j, t_m) \\ \\ \xi^{N_x}(x_j, t_m) \end{pmatrix},$$

then the RK4 algorithm is given by

$$w_{m+1} = w_m + \frac{\Delta t}{6}(s_1 + 2s_2 + 2s_3 + s_4), \qquad m = 0, 1, ..., N_t,$$

where

$$s_1 = f(w_m), \quad s_2 = f(w_m + .5\Delta t s_1), \quad s_3 = f(w_m + .5\Delta t s_2), \quad s_4 = f(w_m + \Delta t s_3).$$

The term

$$\frac{1}{6}(s_1 + 2s_2 + 2s_3 + s_4)$$

can be thought of as a weighted average of slopes (s_1, s_2, s_3, s_4) on the interval $[t_m, t_m + \Delta t]$, thus, the term

$$\frac{\Delta t}{6}(s_1 + 2s_2 + 2s_3 + s_4)$$

approximates the change of w for that time interval. We favor the RK4 method for its simplicity: you only need an initial condition, w_0, to begin and it is straightforward to program. Moreover, it is extremely accurate as a fourth-order method.

4.2 Exact Solutions for Convergence Study

To test our algorithm we examine temporal and spatial convergence against exact solutions of the linearized equations ((Equation 3.26) with $\mathcal{O}(\alpha)$ terms eliminated) and an exact traveling wave solution of the full model equations (Equation 3.26).

4.2.1 Exact Solutions for Linear Water Waves

The linearization of (Equation 3.26) (where terms of order $\mathcal{O}(\alpha)$ are ignored) is easily seen to be

$$\partial_t \eta = |D|\, \xi \tag{4.2a}$$

$$\partial_t \xi = -\eta, \tag{4.2b}$$

which we denote the "Linear Model." Taking the Fourier transform gives the following system of ordinary differential equations:

$$\frac{d}{dt}\begin{pmatrix}\hat{\eta}_p(t)\\ \hat{\xi}_p(t)\end{pmatrix} = \begin{pmatrix}0 & |p|\\ -1 & 0\end{pmatrix}\begin{pmatrix}\hat{\eta}_p(t)\\ \hat{\xi}_p(t)\end{pmatrix}.$$

Let

$$A_p = \begin{pmatrix}0 & |p|\\ -1 & 0\end{pmatrix}.$$

The eigenvalues of A_p, $p \neq 0$, are

$$\mu_1 = i\sqrt{|p|},$$

$$\mu_2 = -i\sqrt{|p|},$$

with corresponding eigenvectors

$$\boldsymbol{\chi}_1 = \begin{pmatrix}-i\sqrt{|p|}\\ 1\end{pmatrix},$$

$$\boldsymbol{\chi}_2 = \begin{pmatrix}i\sqrt{|p|}\\ 1\end{pmatrix}.$$

As outlined in § 3.3, a general solution is

$$\begin{pmatrix} \hat{\eta}_p(t) \\ \\ \hat{\xi}_p(t) \end{pmatrix} = c_1 \begin{pmatrix} \sqrt{|p|}\sin(\sqrt{|p|}t) \\ \\ \cos(\sqrt{|p|}t) \end{pmatrix} + c_2 \begin{pmatrix} -\sqrt{|p|}\cos(\sqrt{|p|}t) \\ \\ \sin(\sqrt{|p|}t) \end{pmatrix},$$

and with initial conditions $(\hat{\eta}_p(0), \hat{\xi}_p(0))$, we find

$$\begin{pmatrix} c_1 \\ \\ c_2 \end{pmatrix} = \begin{pmatrix} \hat{\xi}_p(0) \\ \\ -\frac{1}{\sqrt{|p|}}\hat{\eta}_p(0) \end{pmatrix}.$$

Therefore, the exact solution for wavenumbers $p > 0$ is

$$\begin{pmatrix} \hat{\eta}_p(t) \\ \\ \hat{\xi}_p(t) \end{pmatrix} = \begin{pmatrix} \hat{\eta}_p(0)\cos(\omega_p t) + \hat{\xi}_p(0)\omega_p \sin(\omega_p t) \\ \\ \hat{\xi}_p(0)\cos(\omega_p t) - \hat{\eta}_p(0)\frac{1}{\omega_p}\sin(\omega_p t) \end{pmatrix}, \tag{4.3a}$$

where the dispersion relation is given by

$$\omega_p^2 = |p|\,.$$

At wavenumber $p = 0$ we can solve the system

$$\begin{aligned} \frac{d}{dt}\hat{\eta}_0(t) &= 0 \\ \frac{d}{dt}\hat{\xi}_0(t) &= -\hat{\eta}_0(t), \end{aligned}$$

or determine the limiting case of (Equation 4.3a) as $\omega_p \to 0$, to get the formulas

$$\begin{pmatrix} \hat{\eta}_0(t) \\ \\ \hat{\xi}_0(t) \end{pmatrix} = \begin{pmatrix} \hat{\eta}_0(0) \\ \\ \hat{\xi}_0(0) - t\hat{\eta}_0(0) \end{pmatrix}. \tag{4.3b}$$

4.2.2 Traveling Wave Solutions

To find a class of exact solutions to the *full* set of model equations, (Equation 3.26), we begin by changing to a reference frame moving uniformly with velocity $c \in \mathbb{R}$. If we let

$$\tilde{x} = x + ct, \quad \tilde{t} = t,$$

then we desire solutions

$$\tilde{\eta}(\tilde{x}, \tilde{t}) = \eta(x + ct, t), \quad \tilde{\xi}(\tilde{x}, \tilde{t}) = \xi(x + ct, t).$$

Rewriting differential operators in terms of the new variables,

$$\partial_x = \partial_{\tilde{x}}, \quad \partial_t = c\partial_{\tilde{x}} + \partial_{\tilde{t}},$$

the model equations (Equation 3.26) become

$$\begin{aligned} \partial_{\tilde{t}}\tilde{\eta} + c\partial_{\tilde{x}}\tilde{\eta} &= |\tilde{D}|\tilde{\xi} + \alpha\left(\frac{\rho - \rho'}{\rho + \rho'}\right)\left[\tilde{\eta}|\tilde{D}|^2\tilde{\xi} - \left(\partial_{\tilde{x}}\tilde{\eta}\right)\left(\partial_{\tilde{x}}\tilde{\xi}\right) - |\tilde{D}|\left[\tilde{\eta}|\tilde{D}|\tilde{\xi}\right]\right] \\ \partial_{\tilde{t}}\tilde{\xi} + c\partial_{\tilde{x}}\tilde{\xi} &= -\tilde{\eta} + \alpha\left(\frac{\rho - \rho'}{2(\rho + \rho')}\right)\left[\left(|\tilde{D}|\tilde{\xi}\right)^2 - \left(\partial_{\tilde{x}}\tilde{\xi}\right)^2\right]. \end{aligned}$$

Seeking steady solutions ($\partial_{\tilde{t}}\tilde{\eta} = \partial_{\tilde{t}}\tilde{\xi} = 0$) in this frame will yield traveling wave solutions which satisfy, upon dropping the tildes,

$$c\partial_x \eta - |D|\,\xi = \alpha \left(\frac{\rho - \rho'}{\rho + \rho'}\right) \left[\eta\,|D|^2\,\xi - (\partial_x \eta)(\partial_x \xi) - |D|\left[\eta\,|D|\,\xi\right]\right] \tag{4.4a}$$

$$c\partial_x \xi + \eta = \alpha \left(\frac{\rho - \rho'}{2(\rho + \rho')}\right) \left[(|D|\,\xi)^2 - (\partial_x \xi)^2\right]. \tag{4.4b}$$

Note that the linear terms were moved to the left-hand side of the equations. To find solutions to (Equation 4.4), we expand the solution (η, ξ, c) in Taylor series in the (small) parameter α:

$$\eta(x;\alpha) = \sum_{n=0}^{\infty} \eta_n(x)\alpha^n, \quad \xi(x;\alpha) = \sum_{n=0}^{\infty} \xi_n(x)\alpha^n, \quad c(\alpha) = \sum_{n=0}^{\infty} c_n \alpha^n. \tag{4.5}$$

These expansions will be viewed as an exact convergent series that will be summed numerically. Inserting the expansions (Equation 4.5) into (Equation 4.4) and carrying out convolutions in perturbation order gives

$$\begin{aligned} c_0 \sum_{n=0}^{\infty} \partial_x \eta_n \alpha^n - \sum_{n=0}^{\infty} |D|\,\xi_n \alpha^n = \left(\frac{\rho - \rho'}{\rho + \rho'}\right) \sum_{n=0}^{\infty} \Bigg[\sum_{l=0}^{n} \eta_{n-l}\,|D|^2\,\xi_l - \sum_{l=0}^{n} (\partial_x \eta_{n-l})(\partial_x \xi_l) \\ - \sum_{l=0}^{n} |D|\left[\eta_{n-l}\,|D|\,\xi_l\right]\Bigg] \alpha^{n+1} - \sum_{n=1}^{\infty}\left[\sum_{l=0}^{n-1} c_{n-l}\partial_x \eta_l\right]\alpha^n \end{aligned} \tag{4.6a}$$

$$\begin{aligned} c_0 \sum_{n=0}^{\infty} \partial_x \xi_n \alpha^n + \sum_{l=0}^{n} \eta_n \alpha^n = \left(\frac{\rho - \rho'}{2(\rho + \rho')}\right) \sum_{n=0}^{\infty} \Bigg[\sum_{l=0}^{n} (|D|\,\xi_{n-l})(|D|\,\xi_l) \\ - \sum_{l=0}^{n} (\partial_x \xi_{n-l})(\partial_x \xi_l)\Bigg] \alpha^{n+1} - \sum_{n=1}^{\infty}\left[\sum_{l=0}^{n-1} c_{n-l}\partial_x \xi_l\right]\alpha^n. \end{aligned} \tag{4.6b}$$

At order $n = 0$ we find

$$c_0 \partial_x \eta_0 - |D|\, \xi_0 = 0$$

$$c_0 \partial_x \xi_0 + \eta_0 = 0,$$

or, alternatively,

$$A_{c_0,p} \hat{u}_{0,p} = \begin{pmatrix} c_0(ip) & -|p| \\ 1 & c_0(ip) \end{pmatrix} \begin{pmatrix} \hat{\eta}_{0,p} \\ \hat{\xi}_{0,p} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.7}$$

where

$$\hat{u}_{n,p} = \begin{pmatrix} \hat{\eta}_{n,p} \\ \hat{\xi}_{n,p} \end{pmatrix}.$$

To obtain a non–trivial solution, $A_{c_0,p}$ must be singular, that is, the determinant function

$$\Lambda_{c_0,p} = -(c_0 p)^2 + |p|$$

must vanish. To arrange this we select a non–zero wavenumber p_0, and then choose

$$c_0 = \frac{\sqrt{|p_0|}}{p_0}.$$

With the pair (c_0, p_0), we can solve (Equation 4.7) to find a non–trivial solution:

$$\hat{u}_{0,p_0} = \gamma \begin{pmatrix} |p_0| \\ i c_0 p_0 \end{pmatrix},$$

for any $\gamma \in \mathbb{C}$. As we wish to have real–valued solutions, at wavenumber $p = -p_0$ we select the complex conjugate:

$$\hat{u}_{0,-p_0} = \bar{\hat{u}}_{0,p_0}.$$

At wavenumber $p = 0$, (Equation 4.7) reduces to

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{\eta}_{0,0} \\ \hat{\xi}_{0,0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For consistency, we enforce

$$\hat{\eta}_{0,0} = 0,$$

which simply sets the mean water level since

$$\hat{\eta}_{0,0} = \frac{1}{2\pi} \int_0^{2\pi} \eta(x)\, dx = 0.$$

There are no restrictions on $\hat{\xi}_{0,0}$ so, for convenience, we choose

$$\hat{\xi}_{0,0} = 0.$$

We note that this is no great restriction as it is the *gradient* of the potential which is important in the evolution equations so any additive constant term is effectively meaningless. When $p \neq 0, \pm p_0$, $A_{c_0,p}$ is non–singular and the unique solution is trivial:

$$\hat{u}_{0,p} = \mathbf{0}.$$

At order $n \geq 1$, (Equation 4.6) implies that

$$\begin{aligned}
c_0 \partial_x \eta_n - |D|\, \xi_n &= \left(\frac{\rho - \rho'}{\rho + \rho'}\right) \left[\sum_{l=0}^{n-1} \eta_{n-1-l}\, |D|^2\, \xi_l - \sum_{l=0}^{n-1} (\partial_x \eta_{n-1-l})(\partial_x \xi_l) \right. \\
&\quad \left. - \sum_{l=0}^{n-1} |D| \left[\eta_{n-1-l}\, |D|\, \xi_l \right] \right] - c_n \partial_x \eta_0 - \sum_{l=1}^{n-1} c_{n-l} \partial_x \eta_l \\
c_0 \partial_x \xi_n + \eta_n &= \left(\frac{\rho - \rho'}{2(\rho + \rho')}\right) \left[\sum_{l=0}^{n-1} (|D|\, \xi_{n-1-l})(|D|\, \xi_l) - \sum_{l=0}^{n-1} (\partial_x \xi_{n-1-l})(\partial_x \xi_l) \right] \\
&\quad - c_n \partial_x \xi_0 - \sum_{l=1}^{n-1} c_{n-l} \partial_x \xi_l.
\end{aligned}$$

Again, appealing to the Fourier series for (η_n, ξ_n), we can express this as

$$A_{c_0,p} \hat{u}_{n,p} = \hat{Q}_{n,p} - c_n (ip) \hat{u}_{0,p}, \tag{4.8}$$

where

$$Q_n^{\eta} = \left(\frac{\rho-\rho'}{\rho+\rho'}\right)\left[\sum_{l=0}^{n-1}\eta_{n-1-l}\,|D|^2\,\xi_l - \sum_{l=0}^{n-1}(\partial_x\eta_{n-1-l})(\partial_x\xi_l) - \sum_{l=0}^{n-1}|D|\left[\eta_{n-1-l}\,|D|\,\xi_l\right]\right]$$
$$- \sum_{l=1}^{n-1} c_{n-l}(\partial_x\eta_l) \tag{4.9a}$$
$$Q_n^{\xi} = \left(\frac{\rho-\rho'}{2(\rho+\rho')}\right)\left[\sum_{l=0}^{n-1}(|D|\,\xi_{n-1-l})(|D|\,\xi_l) - \sum_{l=0}^{n-1}(\partial_x\xi_{n-1-l})(\partial_x\xi_l)\right]$$
$$- \sum_{l=1}^{n-1} c_{n-l}(\partial_x\xi_l), \tag{4.9b}$$

and $\hat{Q}_{n,p}$ denotes the expressions (Equation 4.9) in Fourier space.

Given our choice of c_0, the matrix $A_{c_0,p}$ is, of course, singular at wavenumbers $p = \pm p_0$. Additionally, $A_{c_0,0}$ is also singular, so in order to solve (Equation 4.8) we consider three cases: $p = \pm p_0$, $p = 0$, and $p \neq 0, \pm p_0$.

Case 1: $p = \pm p_0$

As we have mentioned, A_{c_0,p_0} is singular, however, the undetermined parameter c_n can be used to find a solution. Row reducing (Equation 4.8) gives the augmented matrix

$$\left(\begin{array}{cc|c} c_0(ip_0) & -|p_0| & \hat{Q}^{\eta}_{n,p_0} - c_n ip_0\hat{\eta}_{0,p_0} \\ 0 & 0 & \hat{Q}^{\eta}_{n,p_0} - c_n ip_0\hat{\eta}_{0,p_0} - c_0 ip_0 \hat{Q}^{\xi}_{n,p_0} - c_n c_0 p_0^2 \hat{\xi}_{0,p_0} \end{array}\right),$$

and to ensure consistency of the system we pick:

$$c_n = \frac{\hat{Q}^{\eta}_{n,p_0} - c_0 ip_0\hat{Q}^{\xi}_{n,p_0}}{ip_0\hat{\eta}_{0,p_0} + c_0 p_0^2\hat{\xi}_{0,p_0}}.$$

For uniqueness, we implement Stokes' strategy (see (27; 28)) by requiring that $\hat{\eta}_n$ be L^2–orthogonal to $\hat{\eta}_0$. Since $\hat{\eta}_{0,p} = 0$ for all $p \neq \pm p_0$ this condition amounts to

$$\hat{\eta}_{n,p_0} = \hat{\eta}_{n,-p_0} = 0.$$

With these choices

$$\hat{\xi}_{n,p_0} = \frac{\hat{Q}^{\xi}_{n,p_0} - c_n(ip_0)\hat{\xi}_{0,p_0}}{ic_0 p_0},$$

and $\hat{\xi}_{n,-p_0} = \bar{\hat{\xi}}_{n,p_0}$.

Case 2: $p = 0$

At order $n = 0$ we set $\hat{u}_{0,0} = 0$, therefore, (Equation 4.8) reduces to

$$A_{c_0,0}\hat{u}_{n,0} = \hat{Q}_{n,0},$$

or,

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{\eta}_{n,0} \\ \hat{\xi}_{n,0} \end{pmatrix} = \begin{pmatrix} \hat{Q}^{\eta}_{n,0} \\ \hat{Q}^{\xi}_{n,0} \end{pmatrix}. \tag{4.10}$$

To show that $\hat{Q}^{\eta}_{n,0} = 0$ (see (15)) so our system is consistent recall (Equation 4.9a) and consider the expression for fixed indices n and l. For an arbitrary pair (n, l) (and suppressing this notation):

$$Q^{\eta} = \left(\frac{\rho - \rho'}{\rho + \rho'}\right) \left[\eta \,|D|^2\, \xi - (\partial_x \eta)(\partial_x \xi) - |D| \left[\eta \,|D|\, \xi\right]\right] - c\,\partial_x \eta. \tag{4.11}$$

Making note that

$$\partial_x(\eta\,\partial_x\xi) = (\partial_x\eta)(\partial_x\xi) + \eta\,\partial_x^2\xi,$$

and

$$|D|^2 = \left(\frac{1}{i}\partial_x\right)^2 = -\partial_x^2,$$

we rewrite (Equation 4.11) as

$$Q^\eta = -\partial_x(\Psi\eta\,\partial_x\xi) - |D|\left[\Psi\eta\,|D|\,\xi\right] - \partial_x(c\,\eta),$$

where $\Psi = \dfrac{\rho-\rho'}{\rho+\rho'}$. The terms

$$\Psi\eta\,\partial_x\xi, \quad \Psi\eta\,|D|\,\xi, \quad c\,\eta,$$

can be represented by a Fourier series, without loss of generality,

$$f(x) = \hat{f}_0 + \sum_p \hat{f}_p e^{ipx},$$

for which applying differential operators (with respect to x) map $\hat{f}_0$ to zero. Therefore,

$$\hat{Q}^\eta_{n,0} = 0, \quad \forall n.$$

Returning to (Equation 4.10), the second equation reduces to

$$\hat{\eta}_{n,0} = \hat{Q}^{\xi}_{n,0},$$

and to specify a unique solution we follow the strategy of the $n = 0$ case and choose

$$\hat{\xi}_{n,0} = 0.$$

Case 3: $p \neq 0,\ \pm p_0$

In this case (Equation 4.8) reduces to

$$A_{c_0,p}\hat{u}_{n,p} = \hat{Q}_{n,p},$$

and, as $A_{c_0,p}$ is non–singular, this can be solved uniquely:

$$\hat{u}_{n,p} = \frac{1}{\Lambda_{c_0,p}} \begin{pmatrix} ic_0p\hat{Q}^{\eta}_{n,p} + |p|\,\hat{Q}^{\xi}_{n,p} \\ -\hat{Q}^{\eta}_{n,p} + ic_0p\hat{Q}^{\xi}_{n,p} \end{pmatrix}.$$

To summarize, the traveling wave solutions, $\hat{u}_{n,p} = \begin{pmatrix} \hat{\eta}_{n,p} \\ \hat{\xi}_{n,p} \end{pmatrix}$, are

$$\hat{u}_{0,0} = \mathbf{0}, \tag{4.12a}$$

$$\hat{u}_{0,p} = \mathbf{0}, \tag{4.12b}$$

$$\hat{u}_{0,p_0} = \bar{\hat{u}}_{0,-p_0} = \gamma \begin{pmatrix} |p_0| \\ ic_0 p_0 \end{pmatrix}, \tag{4.12c}$$

and for $n \geq 1$,

$$\hat{u}_{n,0} = \begin{pmatrix} \hat{Q}^{\xi}_{n,0} \\ 0 \end{pmatrix}, \tag{4.12d}$$

$$\hat{u}_{n,p} = \frac{1}{\Lambda_{c_0,p}} \begin{pmatrix} ic_0 p \hat{Q}^{\eta}_{n,p} + |p| \, \hat{Q}^{\xi}_{n,p} \\ -\hat{Q}^{\eta}_{n,p} + ic_0 p \hat{Q}^{\xi}_{n,p} \end{pmatrix}, \quad p \neq 0, \ p \neq \pm p_0 \tag{4.12e}$$

$$\hat{u}_{n,p_0} = \bar{\hat{u}}_{n,-p_0} = \begin{pmatrix} 0 \\ \dfrac{\hat{Q}^{\xi}_{n,p_0} - c_n(ip_0)\hat{\xi}_{0,p_0}}{ic_0 p_0} \end{pmatrix}. \tag{4.12f}$$

CHAPTER 5

NUMERICAL RESULTS

We validate our code by studying convergence of our numerical solutions to the exact solutions we derived in the previous chapter, (Equation 4.3) or (Equation 4.12). Denoting either of these two families of exact solutions by $\{\eta_{\text{exact}}, \xi_{\text{exact}}\}$. We measure the errors

$$\eta_{\text{error}} := \left|\eta_{\text{exact}}(x,\tau) - \eta^{N_x}(x,\tau)\right|_{L_x^\infty}, \tag{5.1a}$$

$$\xi_{\text{error}} := \left|\xi_{\text{exact}}(x,\tau) - \xi^{N_x}(x,\tau)\right|_{L_x^\infty}, \tag{5.1b}$$

where $\tau = N_t(\Delta t)$ is the final time of our simulation and L_x^∞, the L–infinity norm with respect to x, is the maximum error over the collocation points. For all simulations we choose the values $\tau = 10$, $\lambda = 2\pi$, while the initial conditions for the Linear Model (Equation 4.2) are

$$\eta_0(x) = e^{\cos(x)}, \quad \xi_0(x) = e^{\sin(x)},$$

and the traveling wave found in § 4.2.2 serves as the initial condition for the weakly nonlinear model (Equation 3.26). The traveling wave solution is approximated by

$$\eta^N(x;\alpha) := \sum_{n=0}^{N} \eta_n(x)\alpha^n, \quad \xi^N(x;\alpha) := \sum_{n=0}^{N} \xi_n(x)\alpha^n, \quad c^N(\alpha) := \sum_{n=0}^{N} c_n\alpha^n, \tag{5.2}$$

for which we set $N = 20$ and $\alpha = 0.01$. For an error measurement, the numerically evolved traveling wave is compared to the initial wave shifted:

$$\eta_{\text{error}} := \left|\eta_{\text{exact}}(x - c^N\tau, \tau) - \eta^N(x,\tau)\right|_{L_x^\infty}, \tag{5.3a}$$

$$\xi_{\text{error}} := \left|\xi_{\text{exact}}(x - c^N\tau, \tau) - \xi^N(x,\tau)\right|_{L_x^\infty}. \tag{5.3b}$$

5.1 Temporal Convergence

To examine temporal convergence we determine the errors for a fixed, highly resolved, spatial discretization $N_x = 64$ ($\Delta x \approx 0.09817$), as Δt is refined. For physical parameters we choose

$$\lambda = 2\pi, \quad \tau = 10, \quad \rho' = 1, \quad \rho = 2, \quad p_0 = 1, \quad \alpha = 0.01,$$

and use $N = 20$ for the traveling wave solutions. The RK4 time–stepping algorithm has a global error of $\mathcal{O}((\Delta t)^4)$ (4), therefore, we should observe a slope of four on a log–log plot of error versus Δt:

$$\text{error} \approx C(\Delta t)^4$$

$$\log(\text{error}) \approx 4(\log \Delta t) + \log C,$$

for some constant C. Using a least squares algorithm to fit our "experimental" data we find values that are almost exactly four. Figure 2 and Figure 3 further demonstrate that the desired slope was achieved for the Linear Model and traveling waves, respectively. In Table I and Table II we present the raw data of these experiments.

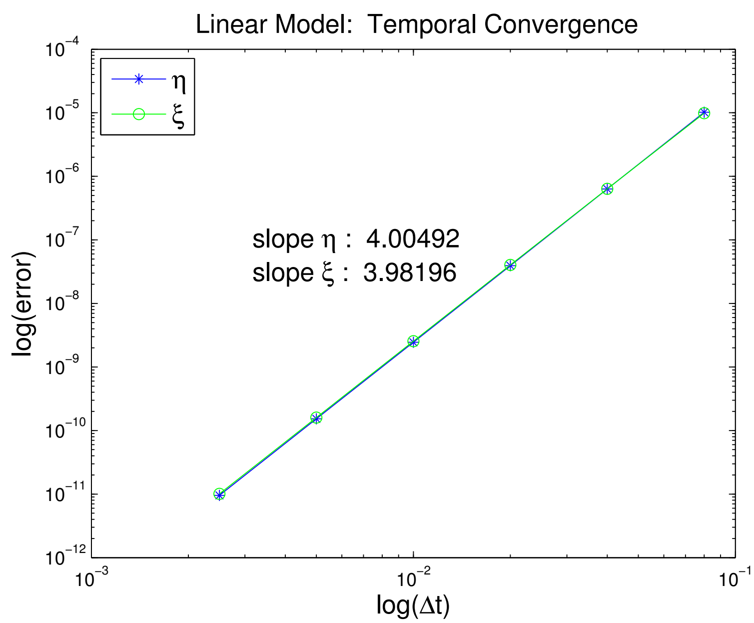

Figure 2: Temporal convergence for the Linear Model.

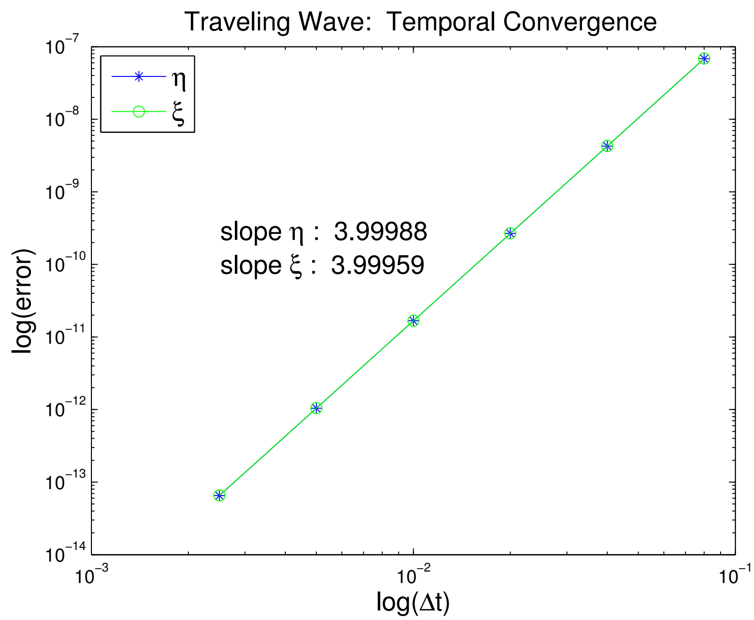

Figure 3: Temporal convergence for traveling waves.

Δt	η_{error}	ξ_{error}
.08	1.0218×10^{-5}	9.8617×10^{-6}
.04	6.3132×10^{-7}	6.3697×10^{-7}
.02	3.9311×10^{-8}	4.0419×10^{-8}
.01	2.4518×10^{-9}	2.5447×10^{-9}
.005	1.5307×10^{-10}	1.5970×10^{-10}
.0025	9.5579×10^{-12}	1.0088×10^{-11}

TABLE I: ERROR RESULTS FOR THE LINEAR MODEL: TEMPORAL CONVERGENCE ANALYSIS.

Δt	η_{error}	ξ_{error}
.08	1.4728×10^{-9}	2.5312×10^{-9}
.04	9.2089×10^{-11}	1.5809×10^{-11}
.02	5.7574×10^{-12}	9.8819×10^{-12}
.01	3.5993×10^{-13}	6.1774×10^{-13}
.005	2.2483×10^{-14}	3.8649×10^{-14}
.0025	1.4413×10^{-15}	2.4319×10^{-15}

TABLE II: ERROR RESULTS FOR TRAVELING WAVES: TEMPORAL CONVERGENCE ANALYSIS.

5.2 Spatial Convergence

For our investigation of spatial convergence we fix the temporal discretization to the highly resolved value $N_t = 4000$ ($\Delta t = 0.0025$) and simulate solutions for $N_x = 2^j$, $j = 2, ..., 6$. Once again we select

$$\lambda = 2\pi, \quad \tau = 10, \quad \rho' = 1, \quad \rho = 2, \quad p_0 = 1, \quad \alpha = 0.01,$$

and $N = 20$ for the traveling waves. In our simulations we find the spectral convergence results that such methods generate for smooth solutions (10). The error results are presented graphically in Figure 4 and Figure 5, and as raw data in Table III and Table IV.

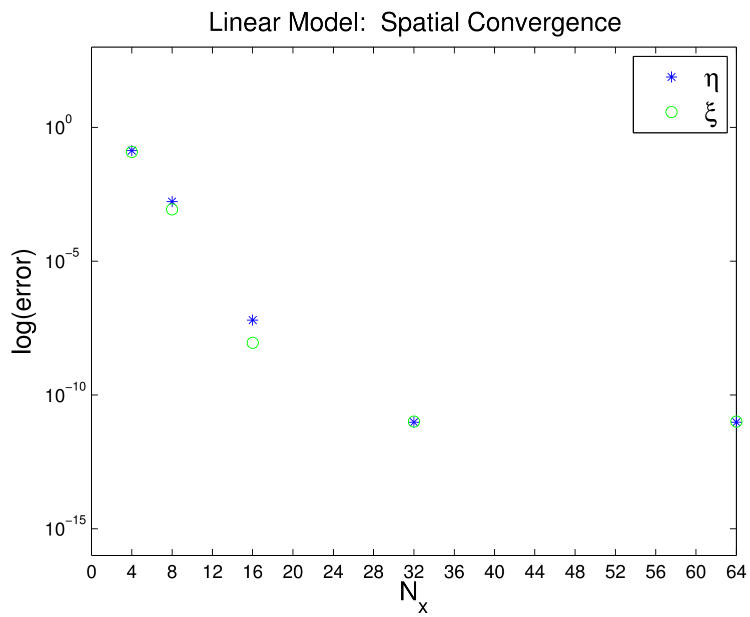

Figure 4: Spatial convergence for the Linear Model.

N_x	η_{error}	ξ_{error}
4	1.3438×10^{-1}	1.2002×10^{-1}
8	1.6651×10^{-3}	8.5868×10^{-4}
16	6.2002×10^{-8}	8.8060×10^{-9}
32	9.5586×10^{-12}	1.0088×10^{-11}
64	9.5584×10^{-12}	1.0090×10^{-11}

TABLE III: ERROR RESULTS FOR THE LINEAR MODEL: SPATIAL CONVERGENCE ANALYSIS.

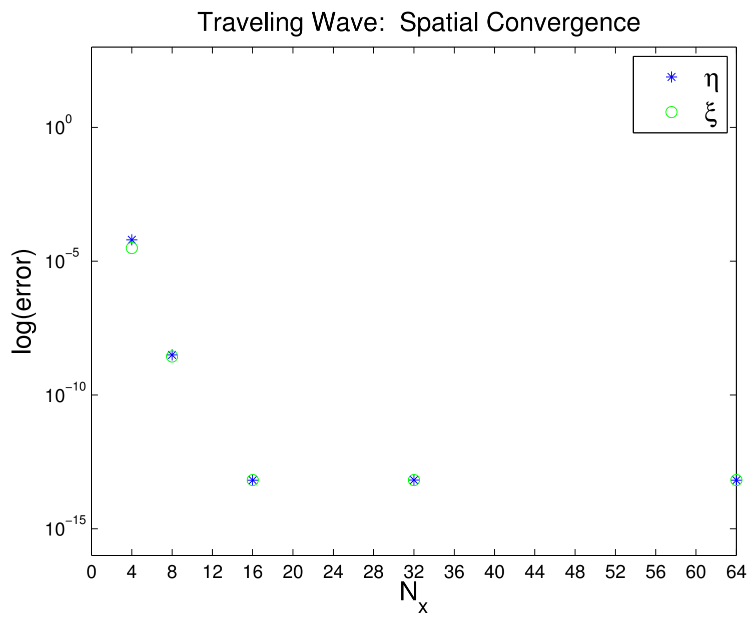

Figure 5: Spatial convergence for the traveling waves.

N_x	$\eta_{\mathbf{error}}$	$\xi_{\mathbf{error}}$
4	6.0995×10^{-5}	2.7558×10^{-5}
8	3.0202×10^{-11}	2.7069×10^{-11}
16	6.5090×10^{-13}	6.5192×10^{-13}
32	6.5177×10^{-13}	6.5145×10^{-13}
64	6.5140×10^{-13}	6.5124×10^{-13}

TABLE IV: ERROR RESULTS FOR TRAVELING WAVES: SPATIAL CONVERGENCE ANALYSIS.

5.3 Traveling Wave Approximations

As a final check of our code, we examine the behavior of the approximations of the traveling wave solutions (Equation 5.2). Specifically, we vary the nonlinear parameter α and consider the approximations $\big(\eta^N(x;\alpha), \xi^N(x;\alpha)\big)$. Additionally, we determine the speed, $c^N(x;\alpha)$, and maximum amplitude height of (η^N, ξ^N) for the various α. For our simulations we select

$$\lambda = 2\pi, \quad N_x = 128, \quad N = 20, \quad \rho' = 1, \quad \rho = 2, \quad p_0 = 1,$$

and let α range from 0 to .05 in increments of .001. Figure 6 and Figure 7 show that as the nonlinear parameter increases the waveform of the interface becomes more developed, as expected. Figure 8 and Figure 9 show that for increasing values of α the speed versus amplitude increases, again as expected.

5.4 Weakly Nonlinear Model

To conclude our numerical experiments we examine the wave–packet profiles considered by Craig & Sulem (9) in their seminal work on OE methods for the classical single–fluid water–wave problem. For this we specify the nondimensional initial data

$$\eta_0(x) = e^{-(4/3)(x-\pi)^2}\cos(10x), \quad \xi_0(x) = 0, \tag{5.4}$$

with $\alpha = 0.01$. In these simulations we select final times $\tau = 10$ and $\tau = 50$, and numerical parameters $N_t = 4000$ and $N_x = 128$ ($\Delta t = 0.0025$ and $\Delta x \approx 0.04909$) which deliver excellent accuracy and stability. To validate these computations we compute the total energy (Hamilto-

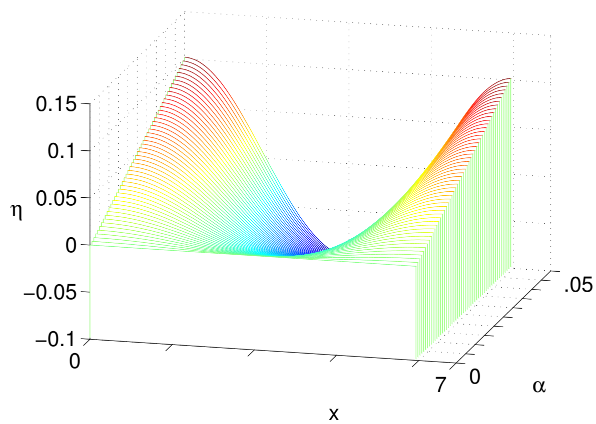

Figure 6: Illustration of η for $\alpha = [0 : .001 : .05]$

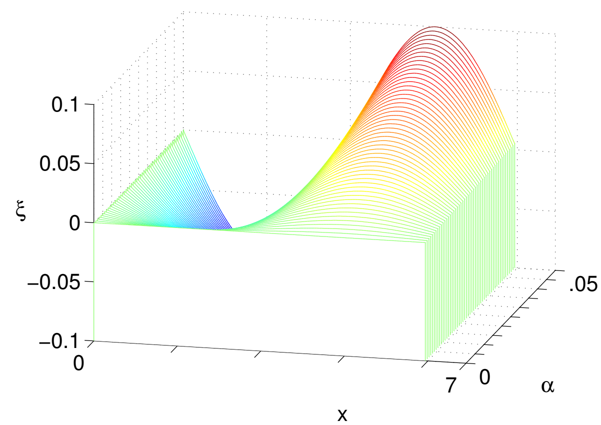

Figure 7: Illustration of ξ for $\alpha = [0 : .001 : .05]$

nian) of our model system. To obtain this we recall (8) that the Hamiltonian of the *full* set of evolution equations (Equation 3.22) is

$$H_{\text{full}}(\eta, \xi) = \frac{1}{2} \int_0^\lambda \left[\xi \left(G'(\eta) M(\eta) G(\eta) \xi \right) + g(\rho - \rho') \eta^2 \right] dx,$$

where the first term of the integrand is the kinetic energy of the system and the second gives the potential energy. The total energy for solutions of our model system (Equation 3.26) can

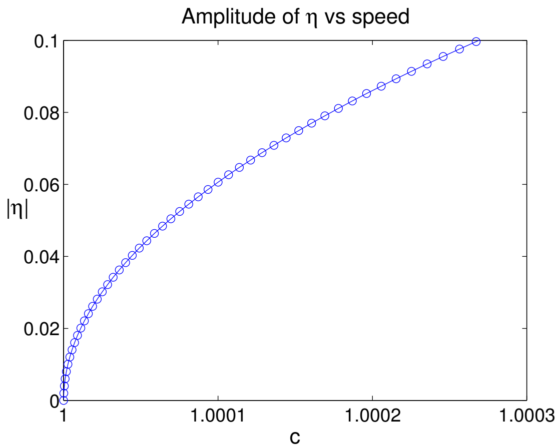

Figure 8: Maximum amplitude of η versus wave speed for $\alpha = [0 : .001 : .05]$

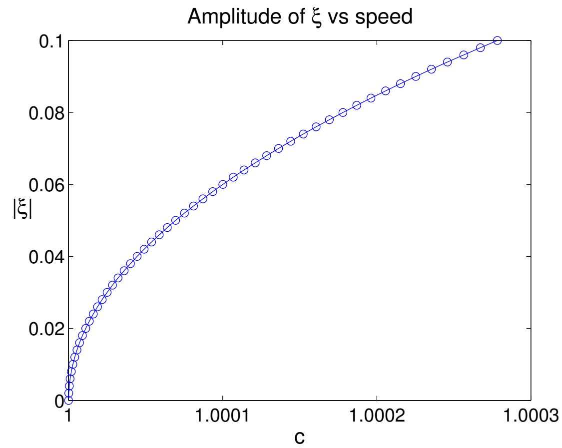

Figure 9: Maximum amplitude of ξ versus wave speed for $\alpha = [0 : .001 : .05]$

be recovered by nondimensionalization (see § 3.3) and retaining only order one terms in the expansion of the operators, resulting in

$$H_1(\bar\eta,\bar\xi) = \frac{1}{2}\int_0^{\frac{\lambda}{L}} \Big[aX\bar\xi\Big(G_0' M_0 G_0[aX\bar\xi] + G_0' M_0 G_1(a\bar\eta)[aX\bar\xi] + G_0' M_1 G_0(a\bar\eta)[aX\bar\xi] + G_1' M_0 G_0(a\bar\eta)[aX\bar\xi]\Big) + g(\rho-\rho')(a\bar\eta)^2 \Big] L d\bar x.$$

We recall that $L = \lambda/2\pi$, simplify, and rearrange to get (dropping scaling notation)

$$\frac{1}{a^2(\rho-\rho')gL} H_1(\eta,\xi) = \frac{1}{2}\int_0^{2\pi} \left[\xi\Big(|D|\,\xi + \alpha\left(\frac{\rho-\rho'}{\rho+\rho'}\right)\left[\eta\,|D|^2\,\xi - (\partial_x\eta)(\partial_x\xi) - |D|\left[\eta\,|D|\,\xi\right]\right]\Big) + \eta^2 \right] dx.$$

We define

$$H(\eta, \xi) := \frac{1}{a^2(\rho - \rho')gL} H_1(\eta, \xi),$$

and remark that a constant multiple of the Hamiltonian $H_1(\eta, \xi)$ will also be conserved. That is, we can check the conservation of H defined by

$$H(\eta, \xi) := \frac{1}{2} \int_0^{2\pi} \left[\xi \left(|D|\, \xi + \alpha \left(\frac{\rho - \rho'}{\rho + \rho'} \right) \left[\eta\, |D|^2\, \xi - (\partial_x \eta)(\partial_x \xi) - |D| \left[\eta\, |D|\, \xi \right] \right] \right) + \eta^2 \right] dx. \tag{5.5}$$

We make note that $H(\eta, \xi)$ is effortless to calculate if we represent the integrand of (Equation 5.5) by its Fourier series: the value of the integral is obtained by scaling the zeroeth Fourier coefficient. We define the relative energy defect by

$$E_{\text{error}} := \frac{|H(\eta_0(x), \xi_0(x)) - H(\eta(x, T), \xi(x, T))|}{H^0(\eta_0(x), \xi_0(x))},$$

and display in Table VI values of this error measured after $T = 10$ and $T = 50$ units of time. We notice that as the nonlinear parameter α is increased to 0.1 our code becomes unreliable because solutions blow up and some type of filtering mechanism is required (as it was in the work of Craig & Sulem (9)).

For our final simulation we choose values of the density parameters suggested by the literature (see Mercier, Vasseur, and Dauxios (19)). For this we select

$$\lambda = 2\pi, \quad \rho' = 1.0079, \quad \rho = 1.0201, \quad p_0 = 1, \quad \alpha = 0.01,$$

$\boldsymbol{\alpha}$	$\boldsymbol{E}_{\textbf{error}}$
0.001	1.4108×10^{-11}
0.01	1.4201×10^{-11}
0.1	1.3550×10^{-1}

TABLE V: ENERGY DEFECT VERSUS NONLINEARITY PARAMETER α, $\tau = 10$.

$\boldsymbol{\alpha}$	$\boldsymbol{E}_{\textbf{error}}$
0.001	7.0536×10^{-11}
0.01	7.0881×10^{-11}
0.1	Unstable

TABLE VI: ENERGY DEFECT VERSUS NONLINEARITY PARAMETER α, $\tau = 50$.

and set $N_x = 128$ and $N_t = 4000$. In Figure 10 and Figure 11 we depict the evolution of η and ξ, respectively, as t increases from $t_0 = 0$ to $\tau = 10$. In Figure 12 and Figure 13, we depict the evolution as t increases from $t_0 = 0$ to $\tau = 50$. Both qualitatively show the same behavior observed by Craig & Sulem (9) for the classical water–wave problem.

We conclude with a few remarks on programming. The programming was done in MATLAB on a MacBook Pro, which has a single processor (Intel Core i5) that runs at a speed of 2.4 GHz. We demonstrate the speed in Table VII, where we give total simulation times, in seconds, along with energy errors for various N_x. The other parameters are:

$$\lambda = 2\pi, \quad \Delta t = .0025, \quad \tau = 100, \quad \rho' = 1, \quad \rho = 2, \quad p_0 = 1, \quad \alpha = .01.$$

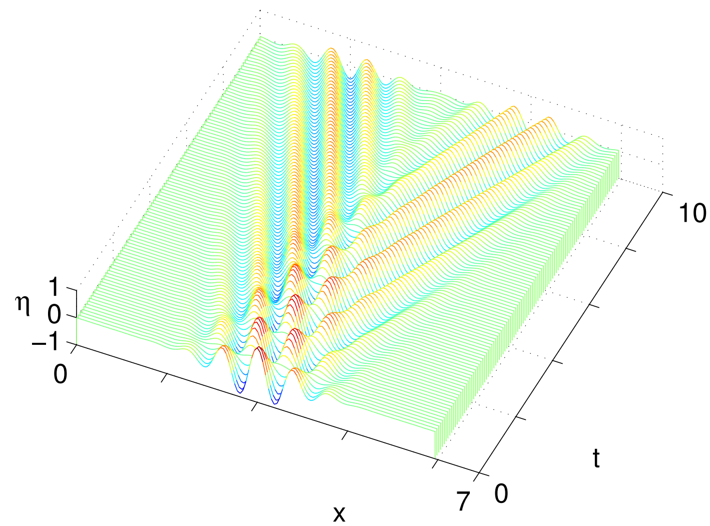

Figure 10: Evolution of the interface η for the weakly nonlinear model with the initial wave packet (Equation 5.4), $\tau = 10$.

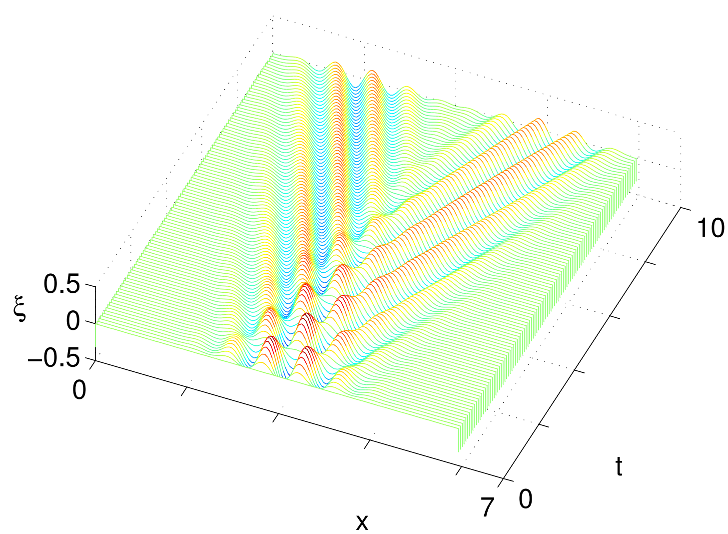

Figure 11: Evolution of the interface quantity ξ for the weakly nonlinear model with the initial wave packet (Equation 5.4), $\tau = 10$.

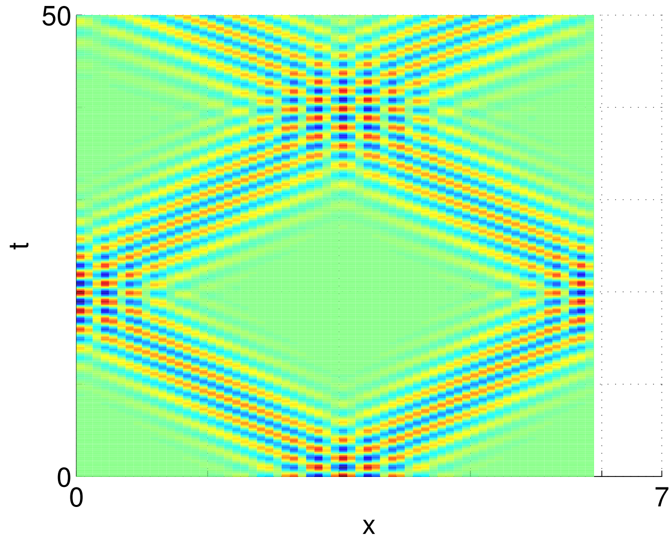

Figure 12: Evolution of the interface η for the weakly nonlinear model with the initial wave packet (Equation 5.4), $\tau = 50$.

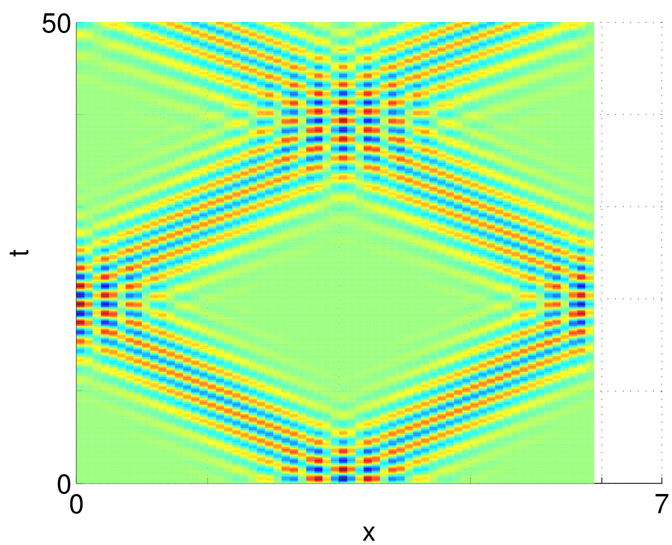

Figure 13: Evolution of the interface quantity ξ for the weakly nonlinear model with the initial wave packet (Equation 5.4), $\tau = 50$.

$\boldsymbol{N_x}$	**Run time**	$\boldsymbol{E}_{\mathbf{error}}$
32	22.2 s	$\mathcal{O}(10^{-2})$
64	23.3 s	$\mathcal{O}(10^{-6})$
128	26.9 s	$\mathcal{O}(10^{-10})$
256	33.8 s	$\mathcal{O}(10^{-10})$
512	48.4 s	$\mathcal{O}(10^{-10})$
1028	184.2 s	$\mathcal{O}(10^{-10})$

TABLE VII: CLOCK TIMES FOR RK4 ALGORITHM, $\tau = 100$

CHAPTER 6

CONCLUSION

6.1 Overview

To numerically simulate the evolution of an internal wave generated by an initial disturbance we began by recasting the classical formulation in terms of interface variables, $\eta(x,t)$ and $\xi(x,t)$. In order to explicitly express the problem in terms of ξ we introduced Dirichlet-Neumann Operators (DNOs), linear operators in ξ which map Dirichlet boundary data (ξ) to Neumann boundary data (normal derivatives of potentials at the interface). Our interface separates two layers of fluid so we initially defined two DNOs, G and G'. Now, with the DNOs, the kinematic and dynamic boundary conditions at the interface essentially gave rise to a coupled system of nonlinear partial differential equations (evolution equations) with two unknowns, η and ξ:

$$\begin{aligned}
\partial_t \eta =& G'(\eta)M(\eta)G(\eta)[\xi] \\
\partial_t \xi =& -(\rho-\rho')g\eta + \frac{1}{2(1+(\partial_x\eta)^2)}\Big[-\rho\Big(\partial_x M(\eta)G'(\eta)[\xi]\Big)^2 \\
&+\rho\Big(G(\eta)M(\eta)G'(\eta)[\xi]\Big)^2 + 2\rho\Big(\partial_x\eta\Big)\Big(\partial_x M(\eta)G'(\eta)[\xi]\Big)\Big(G(\eta)M(\eta)G'(\eta)[\xi]\Big) \\
&+\rho'\Big(\partial_x M(\eta)G(\eta)[\xi]\Big)^2 - \rho'\Big(G'(\eta)M(\eta)G(\eta)[\xi]\Big)^2 \\
&-2\rho'\Big(\partial_x\eta\Big)\Big(\partial_x M(\eta)G(\eta)[\xi]\Big)\Big(G'(\eta)M(\eta)G(\eta)[\xi]\Big)\Big].
\end{aligned}$$

where M is an inverse operator involving the operators, G and G'.

Our next goal was to find a simpler set of evolution equations to model the full equations. We decided upon a perturbative approach for a model valid in the weakly nonlinear regime. To find explicit expressions for the operators we used a Boundary Perturbation Method: Operator Expansions. For a weakly nonlinear regime one nonlinear correction beyond the linearized equations is retained and so, using Fourier complex exponentials as our basis functions, we derived the first two orders of the operators: G, G', and M.

The final phase of our model derivation was nondimensionalization, which introduced a (small) nonlinear steepness parameter, α. Expanding operator combinations in α and substituting the explicit formulas for the operators resulted in our nondimensional weakly nonlinear model:

$$\partial_t \eta = |D|\,\xi + \alpha \left(\frac{\rho - \rho'}{\rho + \rho'}\right) \left[\eta\,|D|^2\,\xi - (\partial_x \eta)(\partial_x \xi) - |D|\,[\eta\,|D|\,\xi]\right] \tag{6.1a}$$

$$\partial_t \xi = -\eta + \alpha \left(\frac{\rho - \rho'}{2(\rho + \rho')}\right) \left[(|D|\,\xi)^2 - (\partial_x \xi)^2\right], \tag{6.1b}$$

for which we defined the Fourier multiplier $|D|$. This model is original and one of the main contributions of this thesis. We note that if $\rho' = 0$, the equations give a nondimensional weakly nonlinear model for a surface wave in deep water. One can refer to Craig and Sulem (9), who numerically simulated such gravity waves, though with a bottom boundary.

To solve (Equation 6.1) we used a Fourier collocation spectral method in the spatial variable and a fourth-order Runge-Kutta time-stepping algorithm. For both, it is well-known what to expect in regards to convergence and so we used this knowledge along with derived exact solutions to test our code. In particular, we investigated spatial and temporal convergence for exact solutions of the linearized equations ((Equation 6.1) with $\mathcal{O}(\alpha)$ terms eliminated) and an exact traveling wave solution of (Equation 6.1). We appealed to perturbation theory again to approximate a traveling wave solution and expanded in the existing small parameter α:

$$\eta(x;\alpha) = \sum_{n=0}^{\infty} \eta_n(x)\alpha^n, \quad \xi(x;\alpha) = \sum_{n=0}^{\infty} \xi_n(x)\alpha^n, \quad c(\alpha) = \sum_{n=0}^{\infty} c_n\alpha^n,$$

where $c \in \mathbb{R}$ is the constant velocity of the wave. For an additional test of our code we studied the behavior of these approximations (amplitudes and speed while varying α) to conclude that our numerical approximations were reasonable. These traveling wave solutions are also an original contribution and we refer the interested reader to Appendix A for a different approach: the derivation of a dimensional model and the corresponding linear and traveling wave solutions.

Our convergence studies gave excellent results. We achieved the spectral accuracy (fast convergence) expected with smooth functions: by $N_x = 32$ errors were $\mathcal{O}(10^{-11})$. Also, we achieved slopes very close to four, as expected with a fourth-order time-stepping algorithm. Comfortable with the validity of our code, we then applied our numerical scheme to our model (Equation 6.1). We verified this simulation with energy comparisons of initial and final waveforms. For long run times, we found that energy was conserved sufficiently, $\mathcal{O}(10^{-11})$, for $\alpha = \{.001, .01, .1\}$. For considerably long run times, we observed instabilities for $\alpha = .1$ due to amplified errors in high wavenumbers. This can be resolved by using a low-pass filter. We make a final remark that the group velocity, c_g, of the initial wave packet (Equation 5.4) was numerically observed,

$$c_g = \partial_p \, \omega = \partial_p \sqrt{\frac{\rho - \rho'}{\rho + \rho'} g \, |p|} = \frac{1}{2} \sqrt{\frac{g(\rho - \rho')}{10(\rho + \rho')}} \, ,$$

and conclude that our numerical scheme is stable and highly accurate.

6.2 Future Projects

Now that we are convinced our numerical scheme is sufficient and our results will be published (see (11)), we can focus on other facets of the internal-wave problem. We will begin with developing a general code which computes order n operators and, in turn, order n operator combinations for higher-order models. Next, we would like to extend to three-dimensions, that is, a two-dimensional ocean floor, first in the weakly nonlinear regime and then for higher-order models.

Another idea is to move out of deep water and closer to the surface so that the upper layer is of finite depth. At first we would impose a rigid-lid assumption, but ultimately we would want to include surface effects. Alternatively, we could change the physical parameters to resemble a lake and have two layers of finite depth. In this set-up, one could have bottom topography and/or surface effects.

Much further down the road, we would like to experiment with different density stratifications (profiles) or, in a different direction, focus on theoretical results, such as well-posedness of model equations and stability of numerical algorithms. The possibilities are endless and we look forward to providing further contributions.

APPENDICES

Appendix A

DIMENSIONAL EQUATIONS

A.1 Weakly Nonlinear Model

Recall our evolution equations:

$$\partial_t \eta = G'(\eta)M(\eta)G(\eta)[\xi] \tag{A.1a}$$

$$\begin{aligned}\partial_t \xi = &-(\rho-\rho')g\eta + \frac{1}{2(1+(\partial_x\eta)^2)}\Big\{ -\rho\left(\partial_x M(\eta)G'(\eta)[\xi]\right)^2 \\ &+\rho\left(G(\eta)M(\eta)G'(\eta)[\xi]\right)^2 + 2\rho(\partial_x\eta)\left(\partial_x M(\eta)G'(\eta)[\xi]\right)\left(G(\eta)M(\eta)G'(\eta)[\xi]\right) \\ &+\rho'\left(\partial_x M(\eta)G(\eta)[\xi]\right)^2 - \rho'\left(G'(\eta)M(\eta)G(\eta)[\xi]\right)^2 \\ &-2\rho'(\partial_x\eta)\left(\partial_x M(\eta)G(\eta)[\xi]\right)\left(G'(\eta)M(\eta)G(\eta)[\xi]\right)\Big\}.\end{aligned} \tag{A.1b}$$

To derive dimensional weakly nonlinear equations accurate to order two we assume (η,ξ) are of the same order:

$$\eta(x,t) = \varepsilon\tilde{\eta}(x,t),$$

$$\xi(x,t) = \varepsilon\tilde{\xi}(x,t),$$

where $\varepsilon \ll 1$. We can expand:

$$G(\varepsilon\tilde{\eta})[\varepsilon\tilde{\xi}] = \sum_{n=0}^{\infty} G_n(\tilde{\eta})[\varepsilon\tilde{\xi}]\varepsilon^n, \quad G'(\varepsilon\tilde{\eta})[\varepsilon\tilde{\xi}] = \sum_{n=0}^{\infty} G'_n(\tilde{\eta})[\varepsilon\tilde{\xi}]\varepsilon^n, \quad M(\varepsilon\tilde{\eta})[\varepsilon\tilde{\xi}] = \sum_{n=0}^{\infty} M_n(\tilde{\eta})[\varepsilon\tilde{\xi}]\epsilon^n,$$

and substitute into (Equation A.1) to get

$$\begin{aligned}
\partial_t \varepsilon\tilde{\eta} &= \sum_{n=0}^{\infty} \Big\{ \sum_{l=0}^{n}\sum_{k=0}^{n-l} G'_{n-l-k} M_k G_l(\tilde{\eta})[\varepsilon\tilde{\xi}] \Big\} \varepsilon^n \\
\partial_t \varepsilon\tilde{\xi} &= -(\rho-\rho')g\varepsilon\tilde{\eta} + \frac{1}{2(1+(\partial_x\varepsilon\tilde{\eta})^2)} \Big\{ -\rho \Big(\sum_{n=0}^{\infty} \Big\{ \sum_{l=0}^{n} \partial_x M_l G'_{n-l}(\tilde{\eta})[\varepsilon\tilde{\xi}] \Big\} \varepsilon^n \Big)^2 \\
&\quad + \rho \Big(\sum_{n=0}^{\infty} \Big\{ \sum_{l=0}^{n}\sum_{k=0}^{n-l} G_{n-l-k} M_k G'_l(\tilde{\eta})[\varepsilon\tilde{\xi}] \Big\} \varepsilon^n \Big)^2 \\
&\quad + 2\rho(\partial_x\varepsilon\tilde{\eta}) \sum_{n=0}^{\infty} \Big\{ \sum_{l=0}^{n} \partial_x M_l G'_{n-l}(\tilde{\eta})[\varepsilon\tilde{\xi}] \Big\} \epsilon^n \sum_{n=0}^{\infty} \Big\{ \sum_{l=0}^{n}\sum_{k=0}^{n-l} G_{n-l-k} M_k G'_l(\tilde{\eta})[\varepsilon\tilde{\xi}] \Big\} \varepsilon^n \\
&\quad + \rho' \Big(\sum_{n=0}^{\infty} \Big\{ \sum_{l=0}^{n} \partial_x M_l G_{n-l}(\tilde{\eta})[\varepsilon\tilde{\xi}] \Big\} \varepsilon^n \Big)^2 \\
&\quad - \rho' \Big(\sum_{n=0}^{\infty} \Big\{ \sum_{l=0}^{n}\sum_{k=0}^{n-l} G'_{n-l-k} M_k G_l(\tilde{\eta})[\varepsilon\tilde{\xi}] \Big\} \epsilon^n \Big)^2 \\
&\quad - 2\rho'(\partial_x\varepsilon\tilde{\eta}) \sum_{n=0}^{\infty} \Big\{ \sum_{l=0}^{n} \partial_x M_l G_{n-l}(\tilde{\eta})[\varepsilon\tilde{\xi}] \Big\} \varepsilon^n \sum_{n=0}^{\infty} \Big\{ \sum_{l=0}^{n}\sum_{k=0}^{n-l} G'_{n-l-k} M_k G_l(\tilde{\eta})[\varepsilon\tilde{\xi}] \Big\} \varepsilon^n \Big\}.
\end{aligned}$$

Collecting orders of ε we have

$$\begin{aligned}
\partial_t \varepsilon\tilde{\eta} &= G'_0 M_0 G_0[\varepsilon\tilde{\xi}] + \varepsilon \Big\{ G'_0 M_0 G_1(\tilde{\eta}) + G'_0 M_1 G_0(\tilde{\eta}) + G'_1 M_0 G_0(\tilde{\eta}) \Big\} [\varepsilon\tilde{\xi}] + \mathcal{O}(\varepsilon^3) \\
\partial_t \varepsilon\tilde{\xi} &= -(\rho-\rho')g\varepsilon\tilde{\eta} + \frac{1}{2} \Big\{ -\rho(\partial_x M_0 G'_0[\varepsilon\tilde{\xi}])^2 + \rho(G_0 M_0 G'_0[\varepsilon\tilde{\xi}])^2 \\
&\quad + \rho'(\partial_x M_0 G_0[\varepsilon\tilde{\xi}])^2 - \rho'(G'_0 M_0 G_0[\varepsilon\tilde{\xi}])^2 \Big\} + \mathcal{O}(\varepsilon^3),
\end{aligned}$$

which simplifies to

$$\partial_t\tilde{\eta} = G_0' M_0 G_0[\tilde{\xi}] + \varepsilon\Big\{G_0' M_0 G_1(\tilde{\eta}) + G_0' M_1 G_0(\tilde{\eta}) + G_1' M_0 G_0(\tilde{\eta})\Big\}[\tilde{\xi}] + \mathcal{O}(\varepsilon^2)$$

$$\partial_t\tilde{\xi} = -(\rho-\rho')g\tilde{\eta} + \frac{1}{2}\varepsilon\Big\{ -\rho(\partial_x M_0 G_0'[\tilde{\xi}])^2 + \rho(G_0 M_0 G_0'[\tilde{\xi}])^2$$

$$+ \rho'(\partial_x M_0 G_0[\tilde{\xi}])^2 - \rho'(G_0' M_0 G_0[\tilde{\xi}])^2\Big\} + \mathcal{O}(\varepsilon^2).$$

A resulting dimensional model (sans tilde notation) is

$$\partial_t\eta = \frac{1}{\rho+\rho'}\left|D\right|\xi + \frac{\rho-\rho'}{(\rho+\rho')^2}\Big[\eta\left|D\right|^2\xi - (\partial_x\eta)(\partial_x\xi) - \left|D\right|\big[\eta\left|D\right|\xi\big]\Big]$$

$$\partial_t\xi = -(\rho-\rho')g\eta + \frac{\rho-\rho'}{2(\rho+\rho')^2}\Big[(\left|D\right|\xi)^2 - (\partial_x\xi)^2\Big].$$

A.2 Linear Water-Wave Solutions

The linearized equations of our dimensional model are

$$\partial_t\eta = \frac{1}{\rho+\rho'}\left|D\right|\xi$$

$$\partial_t\xi = -(\rho-\rho')g\eta,$$

which we solved in § 3.3. The exact solution for wavenumbers $p > 0$ is

$$\begin{pmatrix}\hat{\eta}_p(t)\\ \hat{\xi}_p(t)\end{pmatrix} = \begin{pmatrix}\cos(\omega_p t)\hat{\eta}_p(0) + \dfrac{|p|}{(\rho+\rho')\omega_p}\sin(\omega_p t)\hat{\xi}_p(0)\\ -\dfrac{(\rho+\rho')\omega_p}{|p|}\sin(\omega_p t)\hat{\eta}_p(0) + \cos(\omega_p t)\hat{\xi}_p(0)\end{pmatrix},$$

Appendix A (Continued)

where the dispersion relation is given by

$$\omega_p^2 = \frac{\rho - \rho'}{\rho + \rho'} g \, |p| \, .$$

At wavenumber $p = 0$ we have the formulas

$$\begin{pmatrix} \hat{\eta}_0(t) \\ \\ \hat{\xi}_0(t) \end{pmatrix} = \begin{pmatrix} \hat{\eta}_0(0) \\ \\ -g(\rho - \rho')t \, \hat{\eta}_0(0) + \hat{\xi}_0(0) \end{pmatrix} .$$

A.3 Traveling Wave Solutions

The traveling wave solutions satisfy

$$\begin{aligned} c\partial_x \eta - \frac{1}{\rho + \rho'} |D| \, \xi &= \frac{\rho - \rho'}{(\rho + \rho')^2} \left[\eta \, |D|^2 \, \xi - (\partial_x \eta)(\partial_x \xi) - |D| \, [\eta \, |D| \, \xi] \right] \\ c\partial_x \xi + (\rho - \rho') g \eta &= \frac{\rho - \rho'}{2(\rho + \rho')^2} \left[(|D| \, \xi)^2 - (\partial_x \xi)^2 \right] , \end{aligned} \tag{A.2a}$$

and we expand the solution $(\eta, \, \xi, \, c)$ in Taylor series in the (small) wave height parameter δ:

$$\eta(x; \delta) = \sum_{n=1}^{\infty} \eta_n(x) \delta^n, \quad \xi(x; \delta) = \sum_{n=1}^{\infty} \xi_n(x) \delta^n, \quad c(\delta) = \sum_{n=0}^{\infty} c_n \delta^n.$$

We insert these expansions into (Equation A.2) and equate at matching orders of δ. First we define

$$A_{c_0,p} \hat{u}_{n,p} := \begin{pmatrix} c_0(ip) & -\dfrac{|p|}{\rho + \rho'} \\ \\ (\rho - \rho')g & c_0(ip) \end{pmatrix} \begin{pmatrix} \hat{\eta}_{n,p} \\ \\ \hat{\xi}_{n,p} \end{pmatrix} ,$$

and then consider the following cases.

Remark: Note that the indexing is different than in the nondimensional analysis.

Case 1: $n = 1$, $p = \pm p_0$

$$A_{c_0,p_0}\hat{u}_{1,p_0} = \begin{pmatrix} c_0(ip_0) & -\dfrac{|p|}{\rho+\rho'} \\ (\rho-\rho')g & c_0(ip) \end{pmatrix} \begin{pmatrix} \hat{\eta}_{1,p_0} \\ \hat{\xi}_{1,p_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Given the determinant function

$$\Lambda_{c_0,p_0} := -(c_0p_0)^2 + \frac{|p_0|}{\rho+\rho'}(\rho-\rho')g,$$

we choose

$$c_0 = \frac{\sqrt{\dfrac{(\rho-\rho')g\,|p_0|}{\rho+\rho'}}}{p_0}, \quad p_0 \neq 0.$$

Thus

$$\hat{u}_{1,p_0} = \gamma \begin{pmatrix} \dfrac{|p_0|}{\rho+\rho'} \\ ic_0p_0 \end{pmatrix}, \quad \gamma \in \boldsymbol{C}$$

and

$$\hat{u}_{1,-p_0} = \bar{\hat{u}}_{1,p_0}.$$

Case 2: $n = 1$, $p = 0$

As in the nondimensional analysis, we enforce

$$\hat{\eta}_{1,0} = 0,$$

and choose

$$\hat{\xi}_{1,0} = 0.$$

Case 3: $n = 1$, $p \neq 0$, $\pm p_0$

When $p \neq p_0$, $p \neq 0$, $A_{c_0,p}$ is non-singular and the unique solution is trivial:

$$\hat{u}_{1,p} = \mathbf{0}.$$

Case 4: $n > 1$, $p = \pm p_0$

$$A_{c_0,p_0}\hat{u}_{n,p_0} = \hat{Q}_{n,p_0} - c_{n-1}(ip_0)\hat{u}_{1,p_0},$$

Appendix A (Continued)

where

$$
\begin{aligned}
Q_n^{\eta} &= \left(\frac{\rho-\rho'}{(\rho+\rho')^2}\right)\left\{\sum_{l=1}^{n-1}\eta_{n-l}\left|D\right|^2\xi_l - \sum_{l=1}^{n-1}(\partial_x\eta_{n-l})(\partial_x\xi_l) - \sum_{l=1}^{n-1}\left|D\right|\left[\eta_{n-l}\left|D\right|\xi_l\right]\right\}\\
&\quad - \sum_{l=2}^{n-1}c_{n-l}(\partial_x\eta_l)\\
Q_n^{\xi} &= \left(\frac{\rho-\rho'}{2(\rho+\rho')^2}\right)\left\{\sum_{l=1}^{n-1}(\left|D\right|\xi_{n-l})(\left|D\right|\xi_l) - \sum_{l=1}^{n-1}(\partial_x\xi_{n-l})(\partial_x\xi_l)\right\}\\
&\quad - \sum_{l=2}^{n-1}c_{n-l}(\partial_x\xi_l),
\end{aligned}
$$

and $\hat{Q}$ represents the expressions in Fourier space. If we choose

$$
c_{n-1} = \frac{(\rho-\rho')g\hat{Q}^{\eta}_{n,p_0} - ic_0p_0\hat{Q}^{\xi}_{n,p_0}}{(\rho-\rho')g(ip_0)\hat{\eta}_{1,p_0} + p_0^2c_0\hat{\xi}_{1,p_0}},
$$

and

$$
\hat{\eta}_{n,p_0} = \bar{\hat{\eta}}_{n,-p_0} = 0,
$$

then

$$
\hat{\xi}_{n,p_0} = \bar{\hat{\xi}}_{n,-p_0} = \frac{\hat{Q}^{\xi}_{n,p_0} - c_{n-1}(ip_0)\hat{\xi}_{1,p_0}}{ic_0p_0}.
$$

Case 5: $n > 1$, $p = 0$

$$
A_{c_0,0}\hat{u}_{n,0} = \hat{Q}_{n,0}
$$

Appendix A (Continued)

implies

$$\hat{\eta}_{n,0} = \frac{\hat{Q}^{\xi}_{n,0}}{(\rho - \rho')g},$$

and we choose

$$\hat{\xi}_{n,0} = 0.$$

Case 6: $n > 1$, $p \neq 0$, $p \neq \pm p_0$

$$A_{c_0,p}\hat{u}_{n,p} = \hat{Q}_{n,p}$$

can be solved uniquely:

$$\hat{u}_{n,p} = \frac{1}{\Lambda_{c_0,p}} \begin{pmatrix} ic_0 p\hat{Q}^{\eta}_{n,p} + \dfrac{|p|}{\rho + \rho'}\hat{Q}^{\xi}_{n,p} \\ -(\rho - \rho')g\hat{Q}^{\eta}_{n,p} + ic_0 p\hat{Q}^{\xi}_{n,p} \end{pmatrix}.$$

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