Math 220: Exam 2
Fall 2007

Problem 1. (20 pts) Find the Laplace transform of the given function

$$a)\quad f(t) = \begin{matrix} e^t, & \text{if } 0 < t < 1 \\ t^2, & \text{if } t > 1 \end{matrix}$$

$$b)\quad f(t) = t \sin t$$

Problem 2. (20 pts) Determine the inverse Laplace transform of the given function.

$$a)\quad F(s) = \frac{1}{s(s^2+1)}$$

$$b)\quad F(s) = \left(\frac{s+7}{s^2+6s+10}\right)e^{-3s}$$

Problem 3. (15 pts) Solve the initial value problem

$$y' + 2y = u(t-4); \quad y(0) = 5$$

Problem 4. (30 pts) Solve the initial value problem

$$\begin{aligned} x' &= 2x + y - e^{2t}; & x(0) &= 1 \\ y' &= x + 2y; & y(0) &= -1 \end{aligned}$$

Problem 5. (15pts) Determine the form of a particular solution to the equation. DO NOT FIND the unknown coefficients.

$$y'' + 6y' + 5y = (t^2+1)e^t + (t-3)e^{-t} + \cos t$$

Problem 1.

a) $f(t) = e^t + (t^2 - e^t)u(t-1)$.

Set $g(t) = t^2 - e^t$ then $g(t+1) = (t+1)^2 - e^{t+1} = t^2 + 2t + 1 - ee^t$.

$\mathcal{L}\{f\} = \mathcal{L}\{e^t\} + \mathcal{L}\{g(t+1)\}e^{-s} = \dfrac{1}{s-1} + e^{-s}(\dfrac{2}{s^3} + \dfrac{2}{s^2} + \dfrac{1}{s} - \dfrac{e}{s-1})$

b) $\mathcal{L}\{f\} = (-1)\dfrac{d}{ds}(\mathcal{L}\{\sin t\}) = (-1)\dfrac{d}{ds}(\dfrac{1}{s^2+1}) = \dfrac{2s}{(s^2+1)^2}$

Problem 2.

a) $F(s) = \dfrac{1}{s} - \dfrac{s}{s^2+1}$, thus $f(t) = 1 - \cos t$

b) $F(s) = (\dfrac{s+3}{(s+3)^2+1} + \dfrac{4}{(s+3)^2+1})e^{-3s}$

Set $G(s) = \dfrac{s+3}{(s+3)^2+1} + \dfrac{4}{(s+3)^2+1}$ then $g(t) = \mathcal{L}^{-1}\{G\} = e^{-3t}\cos t + 4e^{-3t}\sin t$.

$f(t) = g(t-3)u(t-3) = (e^{-3(t-3)}\cos(t-3) + 4e^{-3(t-3)}\sin(t-3))u(t-3)$

Problem 3.

Set $Y = \mathcal{L}\{y\}$ then $\mathcal{L}\{y'\} = sY - 5$ and we get

$$sY - 5 + 2Y = \frac{e^{-4s}}{s}$$

$$Y = \frac{5}{s+2} + \frac{e^{-4s}}{s(s+2)}$$

Set $F(s) = \dfrac{1}{s(s+2)} = \dfrac{1}{2}(\dfrac{1}{s} - \dfrac{1}{s+2})$ then $f(t) = \mathcal{L}^{-1}\{F\} = \dfrac{1}{2}(1 - e^{-2t})$.

$$y = \mathcal{L}^{-1}\{Y\} = 5e^{-2t} + f(t-4)u(t-4) = 5e^{-2t} + \frac{1}{2}(1 - e^{-2(t-4)})u(t-4)$$

Problem 5.

Characteristic equation $r^2 + 6r + 5 = 0$ has roots -1 and -5.

$$y_p(t) = (At^2 + Bt + C)e^t + t(Dt + E)e^{-t} + F\cos t + G\sin t$$