Distortion and stabilisation of Lipschitz functions

<u>Definition</u> Suppose X is an infinite-dimensional Banach space and $f : S_X \to \mathbb{R}$ a mapping from the unit sphere of X.
We say that f <u>stabilises</u> if for all $\varepsilon > 0$ there is an infinite-dimensional subspace $Y \subseteq X$ such that

$$\operatorname{osc}(f, S_Y) = \sup \left\{ |f(y_1) - f(y_2)| \;\middle|\; y_i \in S_Y \right\} < \varepsilon .$$

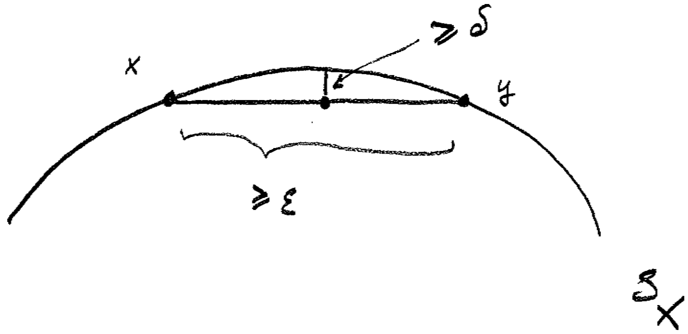
To avoid repetitions we shall assume all spaces and subspaces are infinite-dimensional unless explicitly stated otherwise.

<u>Definition</u> Let X be a Banach space and $\lambda > 1$. We say that X is <u>λ-distortable</u> if there is an equivalent norm $|||\cdot|||$ on X, i.e., for some C, $C^{-1}\|x\| \le |||x||| \le C\|x\|$ for all $x \in X$, such that for all subspaces $Y \subseteq X$,

$$\sup \left\{ \frac{|||y_1|||}{|||y_2|||} \;\middle|\; y_1, y_2 \in S_Y \right\} \ge \lambda .$$

<u>Definition</u> A Banach space X is said to be <u>uniformly convex</u> if for all $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that

$$\forall x, y \in S_X \quad \left(\|x-y\| \geq \varepsilon \;\longrightarrow\; \left\|\frac{x+y}{2}\right\| \leq 1-\delta \right)$$



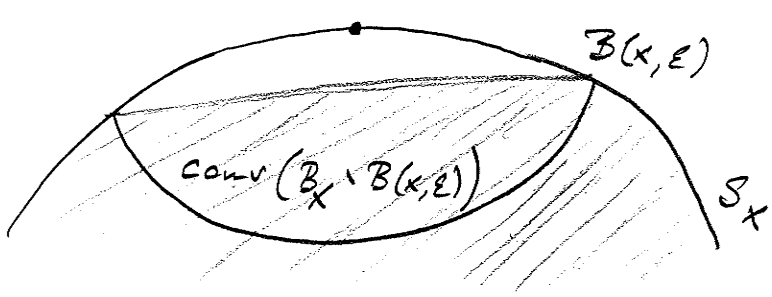
In this case we call $\delta_X : \,]0,1[\, \longrightarrow \,]0,1[$

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \left\|\frac{x+y}{2}\right\| \;\middle|\; x, y \in S_X, \|x-y\| \geq \varepsilon \right\}$$

the <u>modulus</u> of <u>uniform</u> <u>convexity</u> of X.

<u>Lemma</u> Assume X is uniformly convex. Then for every $0 < \varepsilon < 1$ there is $\eta = \eta(\varepsilon) > 0$ such that for all $x \in S_X$,

$$\operatorname{dist}\left(x, \operatorname{conv}\left(B_X \setminus B(x,\varepsilon)\right)\right) \geq \eta(\varepsilon).$$



<u>Proof</u> Assume towards a contradiction that the lemma fails for some ε. Then we can can find sequences (x_n), (z_n) such that

$$x_n \in S_X \quad , \quad z_n \in \operatorname{conv}\left(B_X \setminus B(x_n, \varepsilon)\right),$$

and $\|x_n - z_n\| \to 0$.

Thus for some $y_i^{(n)} \in B_X \setminus B(x_n, \varepsilon)$ and scalars $a_i^{(n)} \geq 0$ for $1 \leq i \leq k_n$ we have

$$\sum_{i=1}^{k_n} a_i^{(n)} = 1 \quad \text{and} \quad z_n = \sum_{i=1}^{k_n} a_i^{(n)} y_i^{(n)}.$$

Now, choose $x_n^* \in S_{X^*}$ with $x_n^*(x_n) = 1$.

Thus

$$\sum_{i=1}^{k_n} a_i^{(n)} x_n^*(y_i^{(n)}) = x_n^*(z_n) \longrightarrow 1$$

and hence we can find $1 \leq i_n \leq k_n$ such that $x_n^*\left(y_{i_n}^{(n)}\right) \to 1$. But then $\|y_{i_n}^{(n)}\| \to 1$ and

$$\liminf_n \|x_n + y_{i_n}^{(n)}\| \geq \liminf_n x_n^*\left(x_n + y_{i_n}^{(n)}\right) = 2$$

$> 2(1 - \delta(\varepsilon))$, contradicting uniform convexity since $\|x_n - y_{i_n}^{(n)}\| \geq \varepsilon$. $\square$

<u>Proposition</u> Suppose X is a Banach space with a λ-distorted norm for some $\lambda > 1$. Then there is a Lipschitz function $f : S_X \to \mathbb{R}$ that does not stabilise.

<u>Proof</u> Suppose $|||\cdot|||$ is a λ-distorted norm on X and define

$$f : S_{(X, \|\cdot\|)} \longrightarrow \mathbb{R}$$

by $f(x) = |||x|||$. Clearly, if $K^{-1}\|\cdot\| \leq |||\cdot||| \leq K\|\cdot\|$, then $|f(x) - f(y)| = \big| |||x||| - |||y||| \big| \leq |||x-y|||$

$\leq K\|x-y\|$, so f is K-Lipschitz.

Let now $\lambda > \delta > 1$ and $\varepsilon = \frac{\delta - 1}{K} > 0$. Then if $Y \subseteq X$ is a subspace we can find $x, y \in S_Y$ such that

$\frac{|||x|||}{|||y|||} \geq \delta$, whereby $|||x||| - |||y||| \geq (\delta - 1)K^{-1} = \varepsilon$.

So f does not stabilise. $\square$

<u>Theorem</u> (Milman)

Suppose X is a uniformly convex space and $f : S_X \to \mathbb{R}$ a Lipschitz function that does not stabilise. Then X has a λ-distortable subspace for some $\lambda > 1$.

<u>Proof</u> Suppose that for some $\varepsilon > 0$ and all subspaces $Y \subseteq X$ we have $\operatorname{osc}(f, S_Y) > \varepsilon$.

Write $f = g_1 + g_2$, where $g_1, g_2 : S_X \to \mathbb{R}$ are the Lipschitz functions

$$g_1(x) = \frac{1}{2}\left(f(x) + f(-x)\right), \quad g_2(x) = \frac{1}{2}\left(f(x) - f(-x)\right).$$

Suppose now that for some Y we have $\operatorname{osc}(|g_2|, S_Y) \leq \frac{\varepsilon}{4}$. Then, as g_2 is antisymmetric and hence takes value 0 on S_Y, we must have $|g_2|(S_Y) \subseteq [0, \frac{\varepsilon}{4}]$ and thus $g_2(S_Y) \subseteq [-\varepsilon/4, \varepsilon/4]$. But then we must have $\operatorname{osc}(g_1, S_Z) > \frac{\varepsilon}{4}$ for all $Z \subseteq Y$, or else we would have $\operatorname{osc}(g_1 + g_2, S_Z) \leq \varepsilon$.

In any case, we thus see that there must be a subspace $Y \subseteq X$ such that some symmetric Lipschitz function $h : S_Y \to \mathbb{R}$ having oscillation $> \frac{\varepsilon}{4}$ on all subspaces $Z \subseteq Y$.

By taking an affine translation of h we can suppose that $h : S_Y \to [0,1[$ and is 1-Lipschitz.

Set $A_i = \{ x \in S_Y \mid \frac{i}{N} \leq h(x) < \frac{i+1}{N} \}$

for $i = 0, \dots, N-1$ and N such that $\frac{\varepsilon}{4} > \frac{1}{2N}$.

Let $Z \subseteq Y$ be such that S_Z intersects a minimal number of A_i. Then there must be i and $j \geq i+2$ such that

$S_V \cap A_i \neq \emptyset$ and $S_V \cap A_j \neq \emptyset$ for all $V \subseteq Z$.

Notice that A_i and A_j are symmetric and $\mathrm{dist}(A_i, A_j) \geq \frac{1}{N}$.

We are ready to define the norm.

Let $\mathcal{B} = \operatorname{conv}\left(A_i \cup \frac{1}{4} B_X\right)$ and let $|||\cdot|||$ be the $\mathcal{B}$-gauge defined on X:

$$|||x||| = \inf\{ r \geq 0 \mid x \in r\mathcal{B} \}.$$

Since A_i is symmetric this defines a norm on X and since $\frac{1}{4} B_X \subseteq \mathcal{B} \subseteq B_X$, we have

$$\|x\| \leq |||x||| \leq 4\|x\|.$$

We claim that $|||\cdot|||$ is a disturbed norm on Z. For if $V \subseteq Z$ we can find $x_1 \in S_V \cap A_i \subseteq \mathcal{B}$ and $x_2 \in S_V \cap A_j$.

Thus $|||x_1||| \leq 1$, while

$$|||x_2||| = \inf\{ r \geq 0 \mid x_2 \in r \cdot \operatorname{conv}(A_i \cup 1/4 B_X)\}$$

$$\geq \inf\{ r \geq 0 \mid x_2 \in r \cdot \operatorname{conv}(B_X \setminus B(x_2, 1/N))\},$$

since $x_2 \in S_V \cap A_j$ and $\operatorname{dist}(S_V \cap A_j, A_i \cup 1/4 B_X) \geq 1/N$.

Thus $|||x_2||| \geq 1 + \frac{\eta(1/N)}{2}$ and $|||\cdot|||$ is $1 + \frac{\eta(1/N)}{2}$ disturbed on Z. $\square$

Theorem (James) ℓ_1 and c_0 are not distortable.

Proof We only do the proof for ℓ_1. Let (e_i) be the standard unit vector basis for ℓ_1 and let $\|\cdot\|$ be its norm.

Suppose $|||\cdot|||$ is an equivalent norm on ℓ_1 and put

$$r = \lim_n \left(\inf \{ |||x||| \mid x \in [e_i]_{i=n}^{\infty} , \|x\| = 1 \} \right) > 0 .$$

Assume $0 < \varepsilon < \frac{r}{10}$ and pick N so large that $\forall n \geq N$

$$r - \varepsilon < \inf \{ |||x||| \mid x \in [e_i]_{i=n}^{\infty} , \|x\| = 1 \} \leq r .$$

Choose now a block basic sequence (x_n) in $[e_i]_{i=N}^{\infty}$ such that $\|x_n\| = 1$ and $r - \varepsilon < |||x_n||| < r + \varepsilon$.

Then for all finite sequences of scalars $(a_i)_{i=1}^{n}$ with $\sum_{i=1}^{n} |a_i| = 1$, we have by the triangle inequality for $|||\cdot|||$

$$||| \sum_{i=1}^{n} a_i x_i ||| \leq \sum_{i=1}^{n} |a_i| \, |||x_i||| \leq r + \varepsilon .$$

Also, since $\sum_{i=1}^{n} a_i x_i \in [e_i]_{i=N}^{\infty} \cap S_{\ell_1}$, we have $|||\sum_{i=1}^{n} a_i x_i||| \geq r - \varepsilon$ and so

$$\sup \left(\frac{|||y_1|||}{|||y_2|||} \;\middle|\; y_i \in [x_n],\ \|y_i\| = 1 \right) \leq \frac{r+\varepsilon}{r-\varepsilon}.$$

As ε is arbitrary, this shows that ℓ_1 is not distortable. $\square$