Finite representability

Let X and Y be infinite-dimensional Banach spaces. We say that Y is finitely representable in X if for all finite-dimensional subspaces $E \subseteq Y$ and $\varepsilon > 0$ there is a finite-dimensional subspace $F \subseteq X$ and an isomorphism $T : E \to F$ such that $\|T\| \cdot \|T^{-1}\| < 1 + \varepsilon$.

Remark Finite representability induces a quasiorder on the class of infinite-dimensional Banach spaces. I.e., if Y is finitely representable in X and Z finitely representable in Y, then Z is finitely representable in X.

Lemma Suppose X and Y are Banach spaces and $T : X \to Y$ a bounded linear operator TFAE:

(i) $T(X)$ is closed in Y.

(ii) The induced map $\tilde{T} : X/\ker(T) \to Y$ is an isomorphic embedding.

(iii) For some C and all $x \in X$, $d(x, \ker(T)) \leq C\|Tx\|$.

<u>Proof</u> (i) $\Rightarrow$ (ii) : If $T(X)$ is closed in Y, then $\tilde{T} : X/_{\ker(T)} \longrightarrow Y$ is a bounded injective operator with closed range. So by the open mapping theorem, $\tilde{T}$ must be an isomorphism with its image.

(ii) $\Rightarrow$ (iii) : If $\tilde{T} : X/_{\ker(T)} \longrightarrow Y$ is an isomorphic embedding, there is a constant C such that for all $x + \ker(T) \in X/_{\ker(T)}$,

$$d(x, \ker(T)) = \|x + \ker(T)\| \leq C \|\tilde{T}(x + \ker(T))\|$$

$$= C\|Tx\| .$$

(iii) $\Rightarrow$ (i) : If $d(x, \ker(T)) \leq C\|Tx\|$ and $(Tx_n)_{n=1}^{\infty}$ is a Cauchy sequence in Y, then $(x_n + \ker(T))_{n=1}^{\infty}$ is a Cauchy sequence in $X/\ker(T)$ and hence converges to some $x + \ker(T)$. Then (Tx_n) also converges to Tx and thus $T(X)$ is closed. $\square$

<u>Proposition</u> Let $T : X \to Y$ be a bounded operator with closed range, and suppose that $x^{**} \in X^{**}$, $\|x^{**}\| < 1$, satisfies $y = T^{**}x^{**} \in Y$. Then there is some $x \in X$, $\|x\| < 1$, with $Tx = y$.

<u>Proof</u> Suppose first towards a contradiction that $y \notin T(X)$. Then by Hahn-Banach there is some $y^* \in Y^*$ such that $y^*(y) = 1$ but $y^*|_{T(X)} \equiv 0$, i.e., $T^*y^* = 0$. But this is impossible since $x^{**}(T^*y^*) = T^{**}x^{**}(y^*) = y(y^*)$ $= y^*(y) = 1$.

Next, suppose $y \in T(X) \setminus T(\operatorname{int} B_X)$. By the open mapping theorem, $T(\operatorname{int} B_X)$ is open in Y, and hence by Hahn-Banach there is some $y^* \in Y^*$ such that $y^*(y) \geq 1$, while $y^*(T(x)) < 1$ for all $x \in X$, $\|x\| < 1$. In particular, $\|T^*y^*\| \leq 1$, and so $|x^{**}(T^*y^*)| < 1$, which contradicts

$$x^{**}(T^* y^*) = T^{**}x^{**}(y^*) = y(y^*) = y^*(y) = 1 . \quad \square$$

Proposition Suppose $T : X \to Y$ is a bounded operator with closed range and $K : X \to Y$ has finite rank. Then $T + K$ has closed range.

Proof If $T+K$ does not have closed range, then by the lemma we can find a bounded sequence (x_n) in X such that $T+K(x_n) \to 0$, but $d(x_n, \ker(T+K)) \geq 1$. By passing to a subsequence we can also suppose that $Kx_n \to y \in Y$ and thus that $Tx_n \to -y$. Since T has closed range, we must have some $x \in X$ such that $-y = Tx$ and hence $T(x_n - x) \to 0$. By the lemma this again implies that $d(x_n - x, \ker(T)) \to 0$. So, as $K(x_n - x) \to y - Kx$, we have $y - Kx \in \overline{K(\ker T)} = K(\ker T)$ and so for some $u \in \ker T$, $y - Kx = Ku$.

Now

$$d(x_n - x - u, \ker(T)) \to 0$$

and

$$\| Kx_n - Kx - u \| \to 0$$

and since $K|_{\ker(T)}$ has closed range this implies that

$$d(x_n - x - u, \ker(T) \cap \ker(K)) \to 0.$$

But $T(x+u) = Tx = -y = -K(u+x)$ and thus $x+u \in \ker(T+K)$, whence

$$d(x_n, \ker(T+K)) = d(x_n - x - u, \ker(T+K))$$

$$\le d(x_n - x - u, \ker(T) \cap \ker(K)) \to 0,$$

which is a contradiction. $\square$

<u>Theorem</u> Let X be a Banach space, $A = (a_{jk})$ and $B = (b_{jk})$ $m \times n$ and $p \times n$ real matrices. Suppose $y_1, \dots, y_m \in X$, $y_1^*, \dots, y_p^* \in X^*$, and $\xi_1, \dots, \xi_p \in \mathbb{R}$. Assume that there exists vectors $x_1^{**}, \dots, x_n^{**} \in X^{**}$ with $\max_{1 \le k \le n} \|x_k^{**}\| < 1$ such that

$$A \begin{pmatrix} x_1^{**} \\ \vdots \\ x_n^{**} \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

and

$$y_j^* \left(\sum_{k=1}^{n} b_{jk} x_k^{**} \right) = \xi_j \quad , \quad 1 \le j \le p .$$

Then there are $x_1, \dots, x_n \in X$ with $\max_{1 \le k \le n} \|x_k\| < 1$ satisfying the same equations as the $x_1^{**}, \dots, x_n^{**}$.

<u>Proof</u> Consider the operator $T_0 : \ell_\infty^n(X) \to \ell_\infty^m(X)$ defined

$$T_0(x_1, \dots, x_n) = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} .$$

We claim that T_0 has closed range. For $A = PDQ$, where P and Q are invertible $m \times m$ and $n \times n$ matrices and D is of the form

$$D = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

where $r = \operatorname{rank} A$. This is done by first putting A in reduced row echelon form B, which corresponds to producing an invertible P such that $A = PB$. Now by doing elementary column operations on B we finally write $B = DQ$ as above.

Now, define $T : \ell_\infty^n(X) \to \ell_\infty^m(X) \oplus_\infty \ell_\infty^p$

by $T(x_1, \dots, x_n) = \left(T_0(x_1, \dots, x_n), \left(y_j^*\left(\sum_{k=1}^n b_{jk} x_k \right)\right)_{j=1}^p \right)$.

Then T is a finite rank perturbation of T_0 and hence also has closed range.

By assumption, there is $\vec{x}^{**} = (x_1^{**}, \dots, x_n^{**}) \in \ell_\infty^n(X)^{**} = \ell_\infty^n(X^{**})$ such that $\|\vec{x}^{**}\|_\infty < 1$ and

$$T^{**}\vec{x}^{**} = (y_1, \dots, y_m, \xi_1, \dots, \xi_p).$$

By the proposition, this now also has a solution in $\ell_\infty^n(X)$. $\square$

The principle of local reflexivity (Lindenstrauss - Rosenthal)

Let X be a Banach space, $F \subseteq X^{**}$ and $G \subseteq X^*$ finite-dimensional subspaces. Then for all $\varepsilon > 0$ there is a subspace $E \subseteq X$ with $X \cap F \subseteq E$, $\dim E = \dim F$, and a linear isomorphism $T : F \to E$ with $\|T\| \cdot \|T^{-1}\| < 1+\varepsilon$ and such that

$$Tx = x \quad \text{for } x \in X \cap F,$$

$$x^*(Tx^{**}) = x^{**}(x^*) \quad \text{for } x^* \in G, \ x^{**} \in F$$

<u>Proof</u> Given $\varepsilon > 0$ find $\nu > 0$ so that $\frac{(1+\nu)}{(1-3\nu)} < 1+\varepsilon$ and find a ν-net $(x_j^{**})_{j=1}^N$ in S_F.

Let $S : \mathbb{R}^N \twoheadrightarrow F$ be defined by $S(\xi_1, \dots, \xi_N) = \sum_{j=1}^N \xi_j x_j^{**}$.

Set $H = S^{-1}(X \cap F)$ and suppose $(\vec{a}(j))_{j=1}^N$ is a basis for H. Let $y_j = S(\vec{a}(j)) \in X \cap F$.

Let A be the $m \times N$ matrix whose rows are the $\vec{a}(j)$ and pick $x_1^*, \dots, x_N^* \in X^*$, $\|x_j^*\| = 1$, $x_j^{**}(x_j^*) > 1-\nu$. Finally choose a basis $\{g_1^*, \dots, g_\ell^*\}$ for G.

Consider now the system of equations in $(x_1, \ldots, x_N)$:

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \quad , \quad \text{i.e.}, \quad \vec{a}^{(j)} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = y_j .$$

$$x_j^*(x_j) = x_j^{**}(x_j^*) = \alpha_j \quad , \quad j = 1, \ldots, N$$

$$g_j^*(x_j) = x_j^{**}(g_j^*) = \beta_j \quad , \quad j = 1, \ldots, N .$$

This system has a solution $(x_1^{**}, \ldots, x_N^{**})$ in X^{**} with $\max_j \|x_j^{**}\| = 1$. By the preceding theorem we can then find some solution $(x_1, \ldots, x_N)$ in X with $\max_j \|x_j\| < 1 + \vartheta$.

Now define $S_1 : \mathbb{R}^N \to X$ by $S_1(\xi_1, \ldots, \xi_N) = \sum_{j=1}^N \xi_j x_j$.

Then if $S(\xi_1, \ldots, \xi_N) = 0$ also $S_1(\xi_1, \ldots, \xi_N) = 0$.

For if $S(\xi_1, \ldots, \xi_N) = 0$, then $(\xi_1, \ldots, \xi_N) \in H$ and hence $(\xi_1, \ldots, \xi_N)$ can be written as a linear combination of the $\vec{a}^{(j)}$:

$$(\xi_1, \ldots, \xi_N) = \sum_{j=1}^m \lambda_j \vec{a}^{(j)} \quad , \quad \text{whereby}$$

$$S_1(\xi_1, \ldots, \xi_N) = \sum_{i=1}^N \xi_i x_i = \sum_{j=1}^m \lambda_j \left(\vec{a}^{(j)} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \right)$$

$$= \sum_{j=1}^m \lambda_j y_j = S(\xi_1, \ldots, \xi_N) = 0 .$$

Thus, S_1 factors through S, i.e., for some operator $T : F \to X$ we have $S_1 = TS$.

Now, let $E = T(F)$ and note that for $1 \le j \le N$ we have $1 - \nu \le \|x_j\| < 1 + \nu$.

So by the next lemma we have $\|T\| \|T^{-1}\| < 1 + \varepsilon$. The other properties are clear. $\square$

<u>Lemma</u> Let E be a finite-dimensional normed space and $(x_j)_{j=1}^N$ an ε-net in S_E for some $0 < \varepsilon < 1$. Suppose $T : E \to X$ is a linear operator such that for all j

$$1 - \varepsilon \leq \|Tx_j\| \leq 1 + \varepsilon .$$

Then for all $x \in E$,

$$\left(\frac{1-3\varepsilon}{1-\varepsilon}\right)\|x\| \leq \|Tx\| \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right)\|x\| .$$

<u>Proof</u> We can assume $\|x\| = 1$. Pick j such that $\|x - x_j\| < \varepsilon$. Then

$$\|Tx\| \leq \|Tx_j\| + \|Tx - Tx_j\| \leq (1+\varepsilon) + \|Tx - Tx_j\| .$$

So $\|T\| \leq (1+\varepsilon) + \|T\|\varepsilon$ and hence

$$(1-\varepsilon)\|T\| \leq (1+\varepsilon) \quad \text{or} \quad \|T\| \leq \frac{1+\varepsilon}{1-\varepsilon} .$$

Conversely, $\|T\| \geq (1-\varepsilon) - \varepsilon\|T\| \geq \dfrac{1-3\varepsilon}{1-\varepsilon}$. $\square$