<u>Theorem</u> Let X be an infinite-dimensional Banach space. Then for every $\varepsilon > 0$, X either contains an unconditional basic sequence of constant $\leq 4/\varepsilon$ or an $HI(\varepsilon)$ subspace.

<u>Proof</u> We can suppose that X is separable and $\varepsilon > 0$ is fixed.

For all finite dimensional $E, F \subseteq X$ and infinite-dimensional $Z \subseteq X$ we let

$$A(E, F; Z) := \sup_{\substack{V, U \subseteq Z \\ \text{inf. dim.}}} \alpha(E+U, F+V)$$

$$\leq \alpha(E, F).$$

We say that (E, F) <u>accepts</u> Z if $A(E, F; Z) < \varepsilon$.

Thus for any choice of U, V there are $z_1 \in S_{E+U}$ $z_2 \in S_{F+V}$ such that $\|z_1 - z_2\| < \varepsilon$, meaning that $E+U$ and $F+V$ are close to being <u>parallel</u> in Z.

Notice that $(\{0\}, \{0\})$ accepts Z if and only if Z is $HI(\varepsilon')$ for some $\varepsilon' < \varepsilon$.

We remark that (E,F) accepts Z if and only if (F,E) accepts Z and that if $Z' \subseteq Z$ then $A(E,F;Z') \leq A(E,F;Z)$ and so if (E,F) accepts Z it also accepts Z'.

Moreover, if (E,F) accepts Z, then it also accepts $Z+G$ for all finite-dimensional $G \subseteq X$. For if $U,V \subseteq Z+G$ are infinite-dimensional, then so are $U' = U \cap Z$ and $V' = V \cap Z$ and clearly

$$\alpha(E+U, F+V) \leq \alpha(E+U', F+V') \leq A(E,F;Z) < \varepsilon$$

so $A(E,F;Z+G) < \varepsilon$ too.

In fact, this argument shows that

$$A(E,F;Z) = A(E,F;Z+G)$$

for all finite-dimensional $G \subseteq X$.

Notice that if (E,F) does not accept Z, then in particular $\alpha(E,F) \geq \varepsilon$.

We say that (E,F) <u>rejects</u> Z if no infinite-dimensional subspace $Z' \subseteq Z$ is accepted by (E,F).

<u>Lemma</u> There is an infinite-dimensional subspace $Z_0 \subseteq X$ such that for all finite-dimensional $E, F \subseteq X$, either (E, F) accepts or rejects Z_0

<u>Proof</u> Let $\mathcal{F}$ denote the collection of all finite-dimensional subspace of X and let for $E, E' \in \mathcal{F}$;

$$\delta(E, E') = \sup\Big\{\sup_{e \in S_E}(e, S_{E'}),\ \sup_{e' \in E'}(e', S_E)\Big\},$$

i.e., $\delta(E, E')$ is the Hausdorff distance between S_E and $S_{E'}$.

Since X is separable so is $(\mathcal{F}, \delta)$ and so we can pick a countable dense subset $\mathcal{E} \subseteq \mathcal{F}$ in the δ-metric. We first prove the lemma for pairs (E, F) in $\mathcal{E}$:

<u>Sublemma</u> There is a subspace Z_0 of X of infinite dimension such that for all $(E, F) \in \mathcal{E}^2$ and rational $a \in]0, \varepsilon[$ we have

- either $A(E, F; Z_0) < a$
- or $A(E, F; Y) \geq a$ for all infinite-dimensional $Y \subseteq Z_0$.

<u>Proof</u> Enumerate all triples $(E, F, a) \in \mathcal{E}^2 \times (\mathbb{Q} \cap]0, \varepsilon[)$ as (E_n, F_n, a_n). We construct $X = X_0 \supseteq X_1 \supseteq \ldots$ as follows:

- if $A(E_{n+1}, F_{n+1}; Y) \geq a_{n+1}$ for every infinite-dimensional $Y \subseteq X_n$, let $X_{n+1} = X_n$.
- otherwise there is $X_{n+1} \subseteq X_n$ with
$$A(E_{n+1}, F_{n+1}; X_{n+1}) < a_{n+1} .$$

We diagonalise $X_0 \supseteq X_1 \supseteq \ldots$ by choosing by induction on n, $x_{n+1} \in X_{n+1} \setminus \operatorname{span}\{x_0, \ldots, x_n\}$. Let then $Z_0 = \overline{\operatorname{span}}\{x_n\}_{n=0}^{\infty}$. We have $Z_0 \subseteq X_n + \operatorname{span}\{x_0, \ldots, x_{n-1}\}$ for all n and so
$$A(E_n, F_n; Z_0) \leq A(E_n, F_n; X_n + \operatorname{span}\{x_0, \ldots, x_{n-1}\})$$
$$= A(E_n, F_n; X_n) .$$
Thus for all n, either $A(E_n, F_n; Z_0) < a_n$ or $A(E_n, F_n; Y) \geq a_n$ for all $Y \subseteq Z_0$. $\square$

<u>Sublemma</u> $A(E, F; Z) \leq A(E', F'; Z) + 2\delta(E', E) + 2\delta(F', F)$.

Proof It is enough to show that if $E, E' \subseteq X$ are finite-dimensional and $M, Z \subseteq X$ infinite-dimensional then

$$\sup_{\substack{U \subseteq Z \\ \text{inf. dim.}}} \alpha(E'+U, M) \leq \sup_{\substack{U \subseteq Z \\ \text{inf. dim.}}} \alpha(E+U, M) + 2\delta(E', E).$$

So suppose $s > \sup_{U \subseteq Z} \alpha(E+U, M)$ and let $U \subseteq Z$ be an arbitrary infinite-dimensional subspace.

Then for all $t > 1$ we can find $\varphi_1, \dots, \varphi_k \in S_{X^*}$ such that

$$t \cdot \max\{|\varphi_1(e)|, \dots, |\varphi_k(e)|\} \geq \|e\|$$

for all $e \in E$ (since $\dim E < \infty$).

Let $U' = U \cap \ker \varphi_1 \cap \dots \cap \ker \varphi_k$. Then

$$\begin{aligned} t\|u'+e\| &\geq t \cdot \max\{|\varphi_1(u'+e)|, \dots, |\varphi_k(u'+e)|\} \\ &\geq \|e\| \end{aligned}$$

for all $e \in E$ and $u' \in U'$. Also, as

$s > \alpha(E+U', M)$, we can find $e+u'$, m of norm 1 with $\|e+u'-m\| < s$. But $\|e+u'\| = 1$ implies ~~that~~ $\|e\| \leq t$ and so for some $e' \in E'$, $\|e-e'\| \leq t\delta$, whereby for some $x \in S_{E'+U'}$ $\|x-(e'+u')\| \leq t\delta$.

Therefore,

$$\alpha(E'+U, M) \leq \alpha(E'+U', M) \leq \|x - m\| < \delta + 2t\delta. \quad \square$$

Proof of lemma: Let Z_0 be the subspace of the sublemma. We claim that Z_0 works. For if (E, F) does not reject Z_0 then it accepts some subspace $Y \subseteq Z_0$ and hence for some rational $a < \varepsilon$ we have $A(E, F; Y) < a$. Let $b \in \mathbb{Q} \cap]0, \frac{\varepsilon - a}{8}[$ and let $E', F' \in \mathcal{E}$ be such that $\delta(E, E'), \delta(F, F') < b$. Then $A(E', F'; Y) < a + 4b < \varepsilon - 4b$ and hence also $A(E', F'; Z_0) < a + 4b$ by choice of Z_0. Therefore, $A(E, F; Z_0) < a + 8b < \varepsilon$ and (E, F) accepts Z_0. $\square$

Lemma If (E,F) rejects Z_0, then for all infinite-dimensional $Y \subseteq Z_0$ there is an infinite-dimensional $W \subseteq Y$ such that for all finite-dimensional $G \subseteq W$, $(E+G,F)$ rejects Z_0.

Proof Assume not. Then there is $Y \subseteq Z_0$ such that for all $W \subseteq Y$ there is $G \subseteq W$ such that $(E+G,F)$ does not reject Z_0, whence, by assumption on Z_0, $(E+G,F)$ accepts Z_0.

So for all $V \subseteq Z_0$ infinite-dimensional, since $E+G+W = E+W$,

$$\alpha(E+W,F+V) = \alpha(E+G+W,F+V)$$
$$\leq A(E+G,F;Z_0) < \varepsilon,$$

and hence $\sup\limits_{\substack{W,V \subseteq Y \\ \text{infinite dim.}}} \alpha(E+W,F+V) < \varepsilon$.

So (E,F) accepts Y contradicting the hypothesis. $\square$

Iterating this lemma we get:

<u>Lemma</u> Suppose $(E_l, F_l)_{l=1}^N$ reject Z_0.

Then for all infinite-dimensional $Y \subseteq Z_0$ there is an infinite-dimensional $W \subseteq Y$ such that for all finite-dimensional $G \subseteq W$ and all $l = 1, \dots, N$, $(E_l + G, F_l)$ rejects Z_0.

In particular:

<u>Lemma</u> If $(E_l, F_l)_{l=1}^N$ reject Z_0, then for all infinite-dimensional $Y \subseteq Z_0$ there is a normalised vector $y \in Y$ such that for all $l = 1, \dots, N$

$$(E_l + [y], F_l) \text{ rejects } Z_0 .$$

Construction of the unconditional basic sequence:

We can suppose that $(\{0\}, \{0\})$ rejects Z_0.

From this, we wish to construct by induction on n a sequence $(e_n)_{n=1}^\infty$ of normalised vectors such that if $E_I = \operatorname{span}\{e_i ; i \in I\}$, then for all partitions $I \cup J = \{1, \dots, n\}$, (E_I, E_J) rejects Z_0. Moreover, $\dim \operatorname{span}\{e_1, \dots, e_n\} = n$.

The choice of e_1 is given directly by the preceding lemma.

Now, suppose $e_1, \dots, e_n$ have been chosen. List all partitions $I \cup J = \{1, \dots, n\}$ as $(I_\ell, J_\ell)_{\ell=1}^N$ and let $(E_\ell, F_\ell) = (E_{I_\ell}, E_{J_\ell})$. Then every pair (E_ℓ, F_ℓ) rejects Z_0. Fix a subspace $Y \subseteq Z_0$ of infinite dimension such that

$Y \cap \operatorname{span}\{e_1, \dots, e_n\} = \{0\}$ and find by the lemma some normalised $y \in Y$ such that for all ℓ,

$(E_\ell + [y], F_\ell)$ rejects Z_0. Now, notice that also $(F_\ell + [y], E_\ell)$ can be written as $(E_{\ell'} + [y], F_{\ell'})$ for some ℓ', so also $(E_\ell, F_\ell + [y])$ rejects Z_0. So we can take $e_{n+1} = y$.

<u>Lemma</u> If $\alpha(E, F) \geq \varepsilon$, then for all $x \in E$, $y \in F$,

$$\|x + y\| \leq \frac{4}{\varepsilon} \|x - y\|.$$

<u>Proof</u> By and homogeneity we can suppose $\|y\| \leq \|x\| = 1$. Thus $\|y + x\| \leq 2$ and we need only show that $\|x - y\| \geq \frac{\varepsilon}{2}$.

- Either $\|y\| \leq 1 - \frac{\varepsilon}{2}$, whereby $\|x-y\| \geq \|x\| - \|y\|$
$\geq 1 - \left(1 - \frac{\varepsilon}{2}\right) \geq \frac{\varepsilon}{2}$,

- or $1 - \frac{\varepsilon}{2} < \|y\| \leq 1$ and hence

$$\varepsilon \leq \left\|x - \frac{y}{\|y\|}\right\| \leq \|x-y\| + \left\|y - \frac{y}{\|y\|}\right\|$$

$$= \|x-y\| + \left(1 - \|y\|\right) \leq \|x-y\| + \frac{\varepsilon}{2}$$ and

thus $\|x-y\| \geq \frac{\varepsilon}{2}$. $\square$

Suppose now that $a_1, \dots, a_n$ are scalars and $\varepsilon_i \in \{-1, 1\}$. Let $I = \{i \leq n \mid \varepsilon_i = 1\}$ and $J = \{i \leq n \mid \varepsilon_i = -1\}$. Then $x := \sum_{i \in I} a_i e_i \in E_I$ and $y := \sum_{i \in J} a_i e_i \in E_J$ and so

$$\left\| \sum_{i=1}^{n} \varepsilon_i a_i e_i \right\| = \|x + y\| \leq \frac{4}{\varepsilon} \|x - y\| = \frac{4}{\varepsilon} \left\| \sum_{i=1}^{n} a_i e_i \right\|.$$

This shows that (e_n) is unconditional with constant of unconditionality $\leq \frac{4}{\varepsilon}$. $\square$