Krivine's Theorem

A sequence $(x_n)_{n=1}^{\infty}$ in a Banach space X is said to be <u>spreading</u> if it is 1-equivalent to all of its subsequences, i.e., for all $1 \le p_1 < p_2 < \dots < p_n$ and scalars $a_1, \dots, a_n$ we have

$$\left\| \sum_{j=1}^{n} a_i x_i \right\| = \left\| \sum_{j=1}^{n} a_i x_{p_i} \right\| .$$

Let also $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be bounded sequences in Banach spaces X and Y resp. We say that $(y_n)_{n=1}^{\infty}$ is <u>block finitely representable</u> in $(x_n)_{n=1}^{\infty}$ if for all $\varepsilon > 0$ and N there is a sequence of blocks

$$u_j = \sum_{i=p_{j-1}+1}^{p_j} a_i x_i \qquad , \qquad 0 = p_0 < p_1 < \dots < p_N$$

and an operator $T : [y_j]_{j=1}^{N} \to [u_j]_{j=1}^{N}$ such that $T y_j = u_j$ and $\|T\| \|T^{-1}\| < 1 + \varepsilon$.

Theorem (Brunel - Sucheston)

Suppose $(x_n)_{n=1}^{\infty}$ is a normalised sequence in a Banach space X such that $\overline{\{x_n\}_{n=1}^{\infty}}$ is not compact. Then there is a normalised spreading sequence (e_j) in some space E which is block finitely representable in X.

More precisely, for some some subsequence (y_n) of (x_n) we have for all scalars $a_1, \dots, a_m$,

$$\lim_{\substack{p_1 < \dots < p_m \\ p_1 \to \infty}} \left\| \sum_{i=1}^{m} a_i y_i \right\| = \left\| \sum_{i=1}^{m} a_i e_i \right\|$$

[So asymptotically the sequence (y_i) behaves like the spreading sequence (e_i).]

Proof Before we begin let us recall Ramsey's original theorem in its infinite version:

If k, n are natural numbers and $C_1, \dots, C_k$ is a partition of the n-element subsets of $\mathbb{N}$, then there is an infinite set $M \subseteq \mathbb{N}$ and $i \le k$ such that all n-element subsets of M belong to C_i.

Since $\{x_n\}_{n\in\mathbb{N}}$ is not relatively compact, we can find an infinite M such that $\{x_n\}_{n\in M}$ has no convergent subsequences. 6.3

List now all finite sequences of rational numbers as $(a_1^{(i)}, \dots, a_{n_i}^{(i)})_{n\in\mathbb{N}}$ and define by induction on $i\in\mathbb{N}$ a sequence of infinite subsets of M, $M_1 \supseteq M_2 \supseteq M_3 \supseteq \dots$ such that for all $p_1, \dots p_{n_i}, q_1, \dots, q_{n_i}$ in M_i with $p_1 < \dots < p_{n_i}$, $q_1 < \dots < q_{n_i}$,

$$\left| \left\| \sum_{j=1}^{n_i} a_j^{(i)} x_{p_j} \right\| - \left\| \sum_{j=1}^{n_i} a_j^{(i)} x_{q_j} \right\| \right| < \frac{1}{i+1}.$$

This follows from the fact that for all $p_1, \dots, p_{n_i}$

$$0 \leq \left\| \sum_{j=1}^{n_i} a_j^{(i)} x_{p_j} \right\| \leq \sum_{j=1}^{n_i} |a_j^{(i)}|$$

and the interval $[0, \sum_{j=1}^{n_i} |a_j^{(i)}|]$ can be covered by finitely many intervals of length $< \frac{1}{i+1}$. So colour the n_i-element subsets $\{p_1 < p_2 < \dots < p_{n_i}\}$ of M_{i-1} according to which interval the norm falls into and apply Ramsey's theorem.

Now, find some infinite set M_∞ such that for all M_i, $M_\infty \setminus M_i$ is finite.

Then for all sequences $a_1, \dots, a_n$ of reals we have that

$$\lim_{\substack{p_1, \dots, p_n \in M_\infty \\ p_1 < \dots < p_n \\ p_1 \to \infty}} \left\| \sum_{i=1}^{n} a_i x_{p_i} \right\|$$

exists.

Now let V be the free vector space with basis $(e_i)_{i=1}^{\infty}$ and define a norm on V by setting

$$\left\| \sum_{i=1}^{n} a_i e_i \right\| = \lim_{\substack{p_1, \dots, p_n \in M_\infty \\ p_1 < \dots < p_n \\ p_1 \to \infty}} \left\| \sum a_i x_{p_i} \right\| .$$

It is not hard to see that this is a seminorm and moreover that

$$\left\| \sum_{i=1}^{n} a_i e_i \right\| = \left\| \sum_{i=1}^{n} a_i e_{q_i} \right\|$$

for all $q_1 < \dots < q_n$.

So to see that it is a norm, suppose that $\left\|\sum_{i=1}^{n} a_i e_i\right\| = 0$ with $a_n \neq 0$.
By spreading we also have $\left\|\sum_{i=1}^{n-1} a_i e_i + a_n e_{n+1}\right\| = 0$, whereby $\|e_1 - e_2\| = \|e_n - e_{n+1}\|$

$$= \frac{1}{|a_n|} \left\| \sum_{i=1}^{n} a_i e_i - \left(\sum_{i=1}^{n-1} a_i e_i + a_n e_{n+1}\right)\right\| = 0 .$$

By definition this implies that

$$\lim_{\substack{p_1, p_2 \in M_\infty \\ p_1 < p_2 \\ p_1 \to \infty}} \| x_{p_1} - x_{p_2} \| = 0$$

and so $(x_j)_{j \in M_\infty}$ is Cauchy and hence convergent contradicting our assumption. $\square$

Normalised spreading sequences are close to being basic :

<u>Proposition</u> Suppose $(e_n)_{n=1}^{\infty}$ is a non-constant spreading sequence in a Banach space E.

(i) If (e_n) is not weakly Cauchy, then (e_n) is a basic sequence equivalent to the unit vector basis of ℓ_1.

(ii) If (e_n) is weakly null, then (e_n) is a suppression unconditional basic sequence, i.e., for all scalars $(a_n)_{n=1}^{N}$ and $F \subseteq [1,N]$,

$$\Big\| \sum_{n\in F} a_n e_n \Big\| \leq \Big\| \sum_{n=1}^{N} a_n e_n \Big\| .$$

(iii) If is weakly Cauchy, then $(e_{2n+1} - e_{2n})_{n\in\mathbb{N}}$ is weakly null and spreading.

<u>Proof</u> Notice that since (e_n) is equivalent with all of its subsequences we can pass between it and its subsequences.

(i) So if (e_n) is not weakly Cauchy it has no weakly Cauchy subsequences and hence by Rosenthal's theorem it has a subsequence equivalent to ℓ_1, which again implies that itself must be equivalent to ℓ_1.

(iii) is obvious, since

(ii) By induction we can assume that

$F = [1, N] \setminus \{m\}$. So suppose $\varepsilon > 0$. Then by Mazur's theorem, $0 \in \overline{\{e_n\}}^{w}$

$\subseteq \overline{\text{conv}}^{w}\{e_n\} = \overline{\text{conv}}^{\|\cdot\|}\{e_n\}$, so for some

$c_1, \ldots, c_\ell \geq 0$, $\sum_{i=1}^{\ell} c_i = 1$, we have

$\left\|\sum_{i=1}^{\ell} c_i e_i\right\| < \varepsilon$. Now consider

$$x = \sum_{j=1}^{m-1} a_j e_j + a_m \sum_{j=m}^{m+\ell-1} c_{j-m+1} e_j + \sum_{j=m+\ell}^{N+\ell-1} a_{j-\ell+1} e_j$$

$$= \sum_{i=1}^{\ell} c_i \left(\sum_{j<m} a_j e_j + a_m e_{i+m} + \sum_{j=m+1}^{N} a_j e_{\ell+j-1} \right),$$

so $$\|x\| \leq \max_{1 \leq i \leq \ell} \left\| \sum_{j<m} a_j e_j + a_m e_{i+m} + \sum_{j=m+1}^{N} a_j e_{\ell+j-1} \right\|$$

$$= \left\| \sum_{j=1}^{N} a_j e_j \right\|.$$

But
$$\Big\|\sum_{j<m} a_j e_j + \sum_{m<j\le N} a_j e_j\Big\| \le \|x\| + |a_m|\,\Big\|\sum_{j=m}^{m+l-1} c_{j-m+1} e_j\Big\|$$
$$= \|x\| + |a_m|\,\Big\|\sum_{j=1}^{l} c_j e_j\Big\| < \|x\| + |a_m|\varepsilon .$$

And so

$$\Big\|\sum_{j<m} a_j e_j + \sum_{m<j\le m} a_j e_j\Big\| \le \Big\|\sum_{j=1}^{N} a_j e_j\Big\| + |a_m|\varepsilon ,$$

As $\varepsilon>0$ was arbitrary this proves (ii). $\square$

Krivine's theorem

Let (x_n) be a normalised sequence in a Banach space X such that $\overline{\{x_n\}}$ is not compact. Then for some $1\le p\le\infty$, l_p is block finitely representable in (x_n).

Notice first that by the theorem of Brunel-Sucheston it is enough to prove the theorem when (x_n) is replaced by a normalised spreading sequence that is non-constant, (e_n).

Secondly, if (e_n) is not weakly Cauchy, then it is equivalent to ℓ_1 and hence by James' theorem, ℓ_1 is block finitely representable in (e_n). If (e_n) is weakly Cauchy, but not weakly null, we replace it by $\left(\dfrac{e_{2n+1}-e_{2n}}{\|e_{2n+1}-e_{2n}\|}\right)$ and see that anything block finitely representable there is also block finitely representable in (e_n). So we can assume that (e_n) is a suppression unconditional spreading normalised basic sequence.

Denote by c_{00} the vector space with basis (e_n), i.e., the set of finite block vectors of (e_n). Two elements $\xi, \eta \in c_{00}$ are equivalent if their non-zero coordinates are identical and appear in the same order:

$$\sqrt{2}\, e_7 + 3e_8 + 10\, e_{11} \sim \sqrt{2}\, e_2 + 3e_{14} + 10\, e_{50}.$$

So equivalent elements have the same norm.

We also say that ξ and η are <u>ε-equivalent</u> if for some $u, v \in c_{00}$, $u+\xi$ and $v+\eta$ are equivalent and $\|u\| + \|v\| < \varepsilon$.

Let also $\xi \oplus \eta$ be any vector in c_{00} whose non-zero entries are those of ξ followed by those of η.

We say that ξ is <u>replaceable</u> by η if for all $u, v \in c_{00}$,

$$\|u \oplus \xi \oplus v\| = \|u \oplus \eta \oplus v\|$$

and ε-<u>replaceable</u> by η if for all $u, v \in c_{00}$

$$\big|\, \|u \oplus \xi \oplus v\| - \|u \oplus \xi \oplus v\| \,\big| < \varepsilon .$$

Clearly, (ε-)equivalent elements are (ε-)inter-replaceable.

<u>Lemma</u> Suppose (e_n) is a suppression unconditional normalised spreading basic sequence. Then there is a normalised spreading basic sequence (f_n) block finitely representable in (e_n) such that f_1 is replaceable by $-f_1$.

<u>Proof</u> Notice that if (e_n) is equivalent to c_0 then by James' stabilisation theorem for c_0 we can just let (f_n) be the standard basis for c_0. So assume (e_n) is not equivalent to c_0.

Notice also that by suppression unconditionality we have for all scalars (a_n) and signs $\varepsilon_n = \pm 1$,

$$\Big\| \sum_{n=1}^{\infty} \varepsilon_n a_n e_n \Big\| = \Big\| \sum_{\varepsilon_n = 1} a_n e_n - \sum_{\varepsilon_n = -1} a_n e_n \Big\|$$

$$\leq \Big\| \sum_{\varepsilon_n = 1} a_n e_n \Big\| + \Big\| \sum_{\varepsilon_n = -1} a_n e_n \Big\|$$

$$\leq \Big\| \sum_{n=1}^{\infty} a_n e_n \Big\| + \Big\| \sum_{n=1}^{\infty} a_n e_n \Big\| = 2 \Big\| \sum_{n=1}^{\infty} a_n e_n \Big\| .$$

It follows from this and that (e_n) is not equivalent to c_0 that for $y_n = \sum_{j=1}^{n} (-1)^j e_j$ we have $\|y_n\| \to \infty$ ((e_n) is not boundedly complete).

For each n, let $u_n = \frac{y_n}{\|y_n\|}$. Then as

$$\frac{e_1}{\|y_n\|} \oplus u_n \quad \text{and} \quad -u_n \oplus \frac{e_{n+1}}{\|y_n\|} \quad \text{are equivalent}$$

for n even and similarly $\frac{e_1}{\|y_n\|} \oplus u_n$ and $-u_n \oplus \frac{-e_{n+1}}{\|y_n\|}$ are equivalent for n odd,

we see that u_n and $-u_n$ are $\frac{2}{\|y_n\|}$-equivalent.

Now if (z_n) is a block sequence of (e_n) where each z_n is equivalent to u_k for some fixed k, then (z_n) is a normalised spreading basic sequence in which z_n is $\frac{2}{\|y_k\|}$-replaceable by $-z_n$.

We can thus define a sequence of norms on c_{00} by setting

$$\Big\| \sum_{i=1}^{n} a_i e_i \Big\|_k = \Big\| \sum_{i=1}^{n} a_i z_i \Big\|$$

where z_i is equivalent with u_k for all i.
By a simple diagonalisation over all finite sequences of rationals $a_1, \dots, a_n$, we can find $k_m \nearrow$ such that

$$\lim_{m \to \infty} \Big\| \sum_{i=1}^{n} a_i e_i \Big\|_{k_m} =: \Big\| \sum_{i=1}^{n} a_i e_i \Big\|_*$$

exists for all $a_1, \dots, a_n$. Clearly then $(f_i) = \big((e_i), \|\ \|_*\big)$ is a normalised suppression unconditional basic sequence block finitely representable in $\big((e_i), \|\cdot\|\big)$ and such that e_1 is replaceable by $-e_1$. $\boxed{\mathrm{V}}$

Notice that if f_1 is replaceable by $-f_1$, then for all $x = \sum_{i=1}^{\infty} a_i f_i$ and signs (ε_i)

$$\Big\| \sum_{i=1}^{\infty} a_i f_i \Big\| = \Big\| \sum_{i=1}^{\infty} \varepsilon_i a_i f_i \Big\| .$$

In this case we say that (f_i) is 1-unconditional.

Lemma Suppose (e_i) is a normalised spreading 1-unconditional basic sequence.

(i) If $2^{1/p}e_1$ is replaceable by e_1+e_2 for some $1 \le p < \infty$, then (e_i) is equivalent to ℓ_p.

(ii) If for some $1 \le p < \infty$ both $2^{1/p}e_1$ is replaceable by e_1+e_2 and also $3^{1/p}e_1$ is replaceable by $e_1+e_2+e_3$, then (e_i) is 1-equivalent with ℓ_p.

Proof (i) Suppose $(k_j)_{j=1}^{\infty}$ is an increasing sequence of non-negative integers. Then, if $N = \sum_{j=1}^{n} 2^{k_j}$, we have

$$\left\| \sum_{j=1}^{n} 2^{k_j/p} e_j \right\| = \left\| \sum_{j=1}^{n} 2^{k_j - 1/p} \left(2^{1/p} e_{2j} \right) \right\| = \left\| \sum_{j=1}^{n} 2^{k_j-1/p} \left(e_{2j} + e_{2j+1} \right) \right\|$$

$$\dots = \left\| \sum_{j=1}^{N} e_j \right\| .$$

Also, $\left\| \sum_{j=1}^{2^r} e_j \right\| = 2^{1/p} \left\| \sum_{j=1}^{2^{r-1}} e_j \right\| = \dots = 2^{r/p} \|e_1\| = 2^{r/p}$.

Now clearly $2^{k_n} \le N \le 2^{k_n+1}$, so

$$\left\| \sum_{j=1}^{N} e_j \right\| \le \left\| \sum_{j=1}^{2^{k_n+1}} e_j \right\| = 2^{k_n+1/p} = 2^{1/p} \left(2^{k_n} \right)^{1/p} \le 2^{1/p} N^{1/p},$$

and $$\left\| \sum_{j=1}^{N} e_j \right\| \ge \left\| \sum_{j=1}^{2^{k_n}} e_j \right\| = 2^{k_n/p} \ge 2^{-1/p} N^{1/p} .$$

Suppose now ~~that~~ (a_j) are scalars with $\sum_{j=1}^{n} |a_j|^p = 1$ and let α be the least non-zero value of $|a_j|$. For every j find a non-negative k_j such that $2^{k_j/p} \leq \frac{|a_j|}{\alpha} \leq 2^{k_j+1/p}$ and let $N = \sum_{j=1}^{n} 2^{k_j}$. Then by unconditionality

$$\left\| \sum_{j=1}^{N} e_j \right\| = \left\| \sum_{j=1}^{n} 2^{k_j/p} e_j \right\| \leq \left\| \sum_{j=1}^{n} \frac{|a_j|}{\alpha} e_j \right\|$$

$$= \frac{1}{\alpha} \left\| \sum_{j=1}^{n} a_j e_j \right\| \leq \left\| \sum_{j=1}^{n} 2^{k_j+1/p} e_j \right\| \leq \left\| \sum_{j=1}^{N} 2^{1/p} e_j \right\|$$

$$= \left\| \sum_{j=1}^{2N} e_j \right\|,$$ and so

$$2^{-1/p} N^{1/p} \alpha \leq \left\| \sum_{j=1}^{n} a_j e_j \right\| \leq 2^{1/p} (2N)^{1/p} \alpha = 2^{2/p} N^{1/p} \alpha.$$

Now,

$$N \alpha^p = \alpha^p \sum_{j=1}^{n} 2^{k_j} \leq \alpha^p \sum_{j=1}^{n} \frac{|a_j|^p}{\alpha^p} = 1$$

$$\leq \alpha^p \sum_{j=1}^{n} 2^{k_j+1} = 2 \alpha^p N,$$ so finally

$$2^{-2/p} \leq \left\| \sum_{j=1}^{n} a_j e_j \right\| \leq 2^{2/p}.$$

Thus, (e_j) is $2^{2/p} \cdot 2^{2/p} = 16^{1/p}$ equivalent with l_p.

(ii) The proof of (ii) is similar to (i) except that now we use that $\{2^m 3^l\}_{l,m\in\mathbb{Z}}$ is dense in $]0,+\infty[$. We skip it. $\square$

For the following we let $Q = \mathbb{Q} \cap [0,1[$ and $c_{00}(Q)$ be the linear vectorspace over basis $(f_q)_{q\in Q}$. Suppose also that $(e_n)_{n=1}^{\infty}$ is a 1-unconditional spreading sequence space. Then we define a norm on $c_{00}(Q)$ as follows. If $q_1 < \dots < q_n$ are in Q and $a_{q_i} \in \mathbb{R}$ set

$$\left\| \sum_{j=1}^{n} a_j f_{q_j} \right\| = \left\| \sum_{j=1}^{n} a_j e_j \right\| .$$

Let $X(Q)$ be the completion wrt this norm. Then $(f_q)_{q\in Q}$ can be considered as a Q-ordered basis.

Let
$$T_2 f_q = f_{q/2} + f_{q+1/2}$$
and
$$T_3 f_q = f_{q/3} + f_{q+1/3} + f_{q+2/3}$$
for $q \in Q$. As $(e_n)_{n=1}^{\infty}$ is 1-unconditional

and spreading, $1 \le \|T_2\| \le 2$, $1 \le \|T_3\| \le 3$.

Now set $2^{\vartheta} = \lim_{n\to\infty} \|T_2^n\|^{1/n}$ = spectral radius of T_2.

<u>Lemma</u> Suppose (e_n) is a 1-unconditional spreading basic sequence. Then

(i) There exists $(x_n)_{n=1}^{\infty}$ in $X(Q)$ normalised such that $\lim_{n\to\infty} \|T_2 x_n - 2^{\vartheta} x_n\| = 0$

(ii) If (e_n) is equivalent to some ℓ_p, $1 \le p < \infty$, then $\vartheta = 1/p$ and there is a normalised sequence (x_n) in $X(Q)$ such that $\lim_{n\to\infty} \|T_2 x_n - 2^{1/p} x_n\| = 0$ and $\lim_{n\to\infty} \|T_3 x_n - 3^{1/p} x_n\| = 0$.

<u>Proof</u> (i) First $\lim_{n\to\infty} \|T_2^n\|^{1/n} = \inf_n \|T_2^n\|^{1/n}$

and so $\|T_2^n\| \ge 2^{n\vartheta}$ for all $n \ge 1$.

Therefore, $\lim_{n\to\infty} \|(n+1) 2^{-n\vartheta} T_2^n\| = \infty$

and so by the uniform boundedness principle we can find some $x \in X(Q)$, $\|x\| = 1$, such that $\left((n+1) 2^{-n\vartheta} T_2^n x\right)_{n=1}^{\infty}$ is unbounded.

By unconditionality we can assume x only has non-negative entries. Now, if $r < 2^{-\vartheta}$, then $\frac{1}{r} \notin sp(T_2)$ and so $\frac{1}{r} - T_2$ and also $1 - rT_2$ are invertible. Also, $(1-rT_2)^{-2}$ can be expanded in its binomial series:

$$(1-rT_2)^{-2}(x) = \sum_{n=0}^{\infty} (n+1) r^n T_2^n (x) .$$

By the unboundedness of $\left((n+1)2^{-n\vartheta} T_2^n x\right)_{n=1}^{\infty}$ and since $x \geq 0$ we have that

$$\lim_{r \to 2^{-\vartheta}} \left\| (1-rT_2)^{-2}(x) \right\| = \infty .$$

So we can find some $r_n \to 2^{-\vartheta}$ so that either

$$\lim_{n\to\infty} \frac{\left\| (1-r_nT_2)^{-2}(x) \right\|}{\left\| (1-r_nT_2)^{-1}(x) \right\|} = \infty$$

or

$$\lim_{n\to\infty} \left\| (1-r_nT_2)^{-1}(x) \right\| = \infty .$$

In any case, we can find some $x_n \in X(Q)$ of norm 1 st. $\lim_{n\to\infty} \left\| (1-r_nT_2)(x_n) \right\| = 0$ implying (i).

(ii) Let $\| \|_p$ denote the equivalent l_p norm on $X(Q)$. Then $\|T_2^n\|_p = (2^n)^{1/p} = 2^{n/p}$ and $\|T_3^n\| = 3^{n/p}$. Let also

$$x_n = n^{-2/p} \sum_{j=1}^{n} \sum_{k=1}^{n} 2^{-j/p} 3^{-k/p} T_2^j T_3^k e_0 .$$

Now, T_2 and T_3 commute and

$$T_2^j T_3^k e_q = e_{q/2^j3^k} + e_{(q+1)/2^j3^k} + \dots + e_{(q+2^j3^k-1)/2^j3^k} .$$

Moreover, $\|x_n\|_q = 1$ and

$$\|2^{-1/p} T_2 x_n - x_n\|_p = 2^{1/p} n^{-1/p}$$

$$\|3^{-1/p} T_3 x_n - x_n\|_p = 3^{1/p} n^{-1/p} .$$

This proves (ii) $\square$

We can now prove the theorem:

We can suppose (e_n) is a spreading, 1-unconditional basis. By the preceding lemma we can find a sequence of normalised blocks (x_n) of $X(Q)$ such that $\|T_2 x_n - 2^\vartheta x_n\| = 0$. Thus, $2^\vartheta x_n + (T_2 x_n - 2^\vartheta x_n)$ and $T_2 x_n$ are equivalent and thus $2^{\vartheta}x_n$ and T_2x_n are ε_n-equivalent for $\varepsilon_n \to 0$. Notice also that $x_n \oplus x_n$ and T_2x_n are equivalent, so by taking equivalent coplus of x_n and $x_n \oplus x_n$ over (e_k) we can that for a normalised block-sequence (u_n) of (e_n) we have that u_n and $u_n \oplus u_n$ are ε_n-equivalent.

We now define a new sequence of norms $\|\cdot\|_k$ on (e_n):

$$\Big\|\sum_{j=1}^{N} a_j e_j\Big\|_k = \|a_1 u_k \oplus a_2 u_k \oplus \dots \oplus a_N u_k\|.$$

Thus, $((e_n), \|\cdot\|_k)$ remains 1-unconditional and so by passing to a subsequence of the k, we can suppose

$$\lim_k \Big\|\sum_{j=1}^{N} a_j e_j\Big\|_k = \Big\|\sum_{j=1}^{N} a_j e_j\Big\|_* \text{ exists for}$$

all scalars.

Clearly, $((e_n), \|\ \|_*)$ is block finitely representable in $((e_n), \|\ \|)$ and, moreover, $2^{\vartheta}e_n$ is replaceable by $e_n \oplus e_n$ in $\|\ \|_*$.

Now, if $\vartheta = 0$, then $\|e_1 + \dots + e_n\|_* = 1$ and so $((e_n), \|\ \|_*)$ is 1-isometric to c_0.

If $\vartheta > 0$, let $1/p = \vartheta$ and notice that by the lemma, $((e_n), \|\ \|_*)$ is equivalent to ℓ_p.

Again by a lemma, we can repeat the above construction to produce a new norm $\|\cdot\|_{**}$ on (e_n) such that $((e_n), \|\cdot\|_{**})$ is block finitely representable in $((e_n), \|\cdot\|_*)$ and thus also in $((e_n), \|\cdot\|)$ and such that $3^{1/p} e_1$ and $2^{1/p} e_1$ are replaceable in $\|\cdot\|_{**}$ by $e_1 + e_2 + e_3$ and $e_1 + e_2$ respectively. Again $((e_n), \|\cdot\|_{**})$ is then isometric to ℓ_p. □