Infinite-dimensional Ramsey theory

We shall follow standard notation in this area:

So if $A \subseteq \mathbb{N}$ we let $[A]^\omega$ be the set of all infinite subsets of A, while $[A]^{<\omega}$ denotes the set of all finite subsets of A.

Thus for $A \subseteq B \subseteq \mathbb{N}$, $[A]^\omega \subseteq [B]^\omega \subseteq [\mathbb{N}]^\omega$.

Now, subsets A of $\mathbb{N}$ can be identified with their characteristic function

$$\chi_A : \mathbb{N} \longrightarrow 2 \quad , \text{ where } \quad 2 = \{0,1\} ,$$

and also with their increasing enumeration

(a_n) , $\{a_n\} = A$, $a_1 < a_2 < \dots$.

Identifying $A \subseteq \mathbb{N}$ with χ_A, we can see $[\mathbb{N}]^\omega$ as a subset of the space $2^{\mathbb{N}}$ of all infinite binary sequences given the Tychonoff product topology where $\{0,1\} = 2$ is taken discrete.

So the basic open sets in $2^{\mathbb{N}}$ are of the form

$$N_s = \{ \alpha \in 2^{\mathbb{N}} \mid \alpha_1 = s_1, \dots, \alpha_n = s_n \}$$

for some finite binary sequence $s = (s_1, \dots, s_n)$.

So $$[\mathbb{N}]^\omega = \{ \alpha \in 2^{\mathbb{N}} \mid \exists^\infty n \quad \alpha_n = 1 \}$$
$$= \bigcap_n \bigcup_{m \geq n} \{ \alpha \in 2^{\mathbb{N}} \mid \alpha_m = 1 \}.$$

Now, if $a \in [\mathbb{N}]^{<\omega}$ and $A \in [\mathbb{N}]^\omega$ we set

$$[a, A] = \{ B \in [\mathbb{N}]^\omega \mid B \cap \{0, \dots, \max a\} = a \;\&\; B \subseteq a \cup A \}.$$

So if $\max a < \min A$,

$$[a, A] = \{ B \in [\mathbb{N}]^\omega \mid a \subseteq B \subseteq a \cup A \}.$$

We recall that a subset F of a topological space X is <u>nowhere dense</u> if for all non-empty open $U \subseteq X$ there is a non-empty open $V \subseteq U \subseteq X$ such that $V \cap F = \emptyset$.

A set $M \subseteq X$ is <u>meagre</u> if it is the countable union of nowhere dense sets.

A set B has the <u>Baire property</u> if there is an open set $U \subseteq X$ such that $B \Delta U$ is meagre.

The subsets of X having the Baire property form a σ-algebra $BP(X) \subseteq \mathcal{P}(X)$ in which the meagre sets form a σ-ideal $\mathcal{M}(X) \subseteq BP(X)$.

A subset $\mathcal{A} \subseteq [N]^\omega$ is said to be <u>completely Ramsey</u> if for all $a \in [N]^{<\omega}$ and $A \in [N]^\omega$ there is $B \in [A]^N$ such that

- either $[a, B]^\omega \subseteq \mathcal{A}$
- or $[a, B]^\omega \cap \mathcal{A} = \varnothing$.

The <u>Ellentuck topology</u> on $[N]^\omega$ is the topology with basis

$$\{\varnothing\} \cup \{ [a, A] \mid a \in [N]^{<\omega} \ \& \ A \in [N]^\omega \}.$$

<u>Theorem</u> A subset $\mathbb{A} \subseteq [\mathbb{N}]^\omega$ is completely Ramsey if and only if it has the Baire property in the Ellentuck topology.

<u>Proof</u>

($\Rightarrow$) Suppose $\mathbb{A} \subseteq [\mathbb{N}]^\omega$ is completely Ramsey. Then we claim that $\mathbb{A} \setminus \operatorname{int}(\mathbb{A})$ is nowhere dense. For if $V = [a, A]$ is an open set, then, as $\mathbb{A}$ is completely Ramsey, there is some $B \in [A]^\omega$ such that either $[a, B] \subseteq \mathbb{A}$ or $\mathbb{A} \cap [a, B] = \emptyset$. Since $U = [a, B] \subseteq V$ is open, this shows that either $U \subseteq \operatorname{int} \mathbb{A}$ or $U \cap \mathbb{A} = \emptyset$ and so $U \cap (\mathbb{A} \setminus \operatorname{int} \mathbb{A}) = \emptyset$. Thus, $\mathbb{A} \setminus \operatorname{int} \mathbb{A}$ cannot be open in V and, as V is arbitrary, must be nowhere dense.

Now for simplicity write $A/n = A \cap \{n+1, n+2, \dots\}$.

Lemma Any open set U is completely Ramsey.

Proof We say that $[a, A]$ is good if for some $B \in [A]^\omega$, $[a, B] \subseteq U$, and otherwise it is bad. $[a, A]$ is very bad if it is bad and for all $n \in A$, $[a \cup \{n\}, A]$ is bad.

Notice that if $[a, A]$ is (very) bad and $B \in [A]^\omega$, then $[a, B]$ is (very) bad.

Claim If $[a, A]$ is bad, then for some $B \in [A]^\omega$, $[a, B]$ is very bad.

For otherwise there is some $n_0 \in B_0 = A/\max a$ such that $[a \cup \{n_0\}, B_0]$ is good, and so for some $B_1 \subseteq B_0/n_0$, $[a \cup \{n_0\}, B_1]^\omega \subseteq U$.

Since $[a, B_1]$ is not very bad there is some $n_1 \in B_1$ (and so $n_0 < n_1$) such that $[a \cup \{n_1\}, B_1]$ is good and hence for some $B_2 \subseteq B_1/n_1$, $[a \cup \{n_1\}, B_2] \subseteq U$.

Choose $n_2 \in B_2$ (so $n_0 < n_1 < n_2$) such that $[a \cup \{n_2\}, B_2]$ is good etc.

Thus, in the end we have infinite sets $A = B_0 \supseteq B_1 \supseteq B_2 \supseteq \ldots$ and integers $n_0 < n_1 < n_2 < \ldots$, $n_i \in B_i$, such that

$$[a \cup \{n_i\}; B_{i+1}] \subseteq U.$$

Then set $A_\infty = \{n_i\}_{i=0}^{\infty} \subseteq A$ and notice that if $C \in [a, A_\infty]$, then there is some i such that $C \in [a \cup \{n_i\}, B_{i+1}] \subseteq U$. So $[a, A_\infty] \subseteq U$, contradicting that $[a, A]$ is bad. ▲

<u>Claim</u> If $[a, A]$ is bad, then there is some $B \in [A]^\omega$ with $[a, B] \cap U = \emptyset$.

For using the previous claim repeatedly, we can construct $A \supseteq B_0 \supseteq B_1 \supseteq \ldots$ with $\max a < n_0 = \min B_0 < n_1 = \min B_1 < \ldots$

such that for all $b \subseteq \{n_0, n_1, \dots, n_{i-1}\}$,
$[a \cup b, B_i]$ is very bad.

To see how this is done, suppose $n_0, \dots, n_{i-1}$ and B_i are chosen as above. Then let $n_i = \min B_i$ and notice that for all $b \subseteq \{n_0, n_1, \dots, n_{i-1}\}$, $[a \cup b, B_i]$ is very bad and thus $[a \cup b \cup \{n_i\}, B_i]$ is bad. List $\mathcal{P}(\{n_0, n_1, \dots, n_{i-1}\})$ as $b_1, \dots, b_k$ and choose inductively $B_i \supseteq B_i^{(1)} \supseteq \dots \supseteq B_i^{(k)} = B_{i+1}$ such that $[a \cup b_j \cup \{n_i\}, B_i^{(j)}]$ is very bad. Then for all $b \subseteq \{n_0, \dots, n_{i-1}, n_i\}$, $[a \cup b, B_{i+1}]$ is very bad.

Let now $B = \{n_0, n_1, \dots\}$. Then $[a, B] \cap U = \emptyset$, or otherwise, as U is open, one could find some $c < C$ such that $[c, C] \subseteq [a, B] \cap U$. But then there are i and $b \subseteq \{n_0, \dots, n_{i-1}\}$ such that $c = a \cup b$, $C \subseteq \{n_i, n_{i+1}, \dots\} \subseteq B_i$, contradicting that $[a \cup b, B_i]$ is (very) bad. ▲

Now to see that $\mathcal{U}$ is completely Ramsey, suppose $[a, A]$ is given. Either $[a, A]$ is good and hence for some $B \in [A]^\omega$, $[a, B] \subseteq \mathcal{U}$, or $[a, A]$ is bad, whereby there is some $B \in [A]^\omega$ such that $[a, B] \cap \mathcal{U} = \emptyset$.

So $\mathcal{U}$ is completely Ramsey. $\square$

<u>Lemma</u> If $\mathbb{A} \subseteq [\mathbb{N}]^\omega$ is nowhere dense, then for all $a < A$ there is $B \in [A]^\omega$ with $[a, B] \cap \mathbb{A} = \emptyset$.

<u>Proof</u> Notice that $\overline{\mathbb{A}}$ is completely Ramsey so if the lemma fails there must be some $B \subseteq A$ such that $[a, B] \subseteq \overline{\mathbb{A}}$. But then as $\mathbb{A}$ is nowhere dense there is $[c, C] \subseteq [a, B] \setminus \mathbb{A} \subseteq \overline{\mathbb{A}} \setminus \mathbb{A}$, contradicting $\operatorname{int}(\overline{\mathbb{A}} \setminus \mathbb{A}) = \emptyset$. $\square$

<u>Lemma</u> If $\mathbb{A}$ is meagre, then for all $a < A$ there is $B \in [A]^\omega$ with $[a,B] \cap \mathbb{A} = \emptyset$.

<u>Proof</u> Let $\mathbb{A} = \bigcup_n \mathbb{A}_n$, where $\mathbb{A}_n$ is nowhere dense.

Let $A_0 \subseteq A / \max a$ be such that $[a, A_0] \cap \mathbb{A}_0 = \emptyset$. Set $n_0 = \min A_0$ and let $A_1 \subseteq A_0 / n_0$ be such that for all $b \subseteq \{n_0\}$, $[a \cup b, A_1] \cap \mathbb{A}_1 = \emptyset$. Set $n_1 = \min A_1$ and let $A_2 \subseteq A_1 / n_1$ be such that for all $b \subseteq \{n_0, n_1\}$, $[a \cup b, A_2] \cap \mathbb{A}_2 = \emptyset$. Etc.

Now let $B = \{n_0, n_1, n_2, \ldots\}$. $\square$

We can now prove the other implication of the theorem: So suppose $\mathbb{A}$ has the Baire property and find $\mathbb{U}$ open such that $\mathbb{A} \triangle \mathbb{U}$ is meagre. Then for all $a < B$ there is $B \in [A]^\omega$ such that $[a,B] \cap (\mathbb{A} \triangle \mathbb{U}) = \emptyset$. Since $\mathbb{U}$ is completely Ramsey there is $C \in [B]^\omega$ such that either $[a,C] \subseteq \mathbb{U}$ or $[a,C] \cap \mathbb{U} = \emptyset$. But then $[a,C] \subseteq \mathbb{A}$ or $[a,C] \cap \mathbb{A} = \emptyset$.

$\square$

Since the Ellentuck topology is finer than the usual topology on $[\mathbb{N}]^\omega$ and the sets with the Baire property form a σ-algebra, we see that the Borel sets in the usual topology all are completely Ramsey.

Suppose S is a set. We say that a pair (A, B) of subsets of S is <u>disjoint</u> if $A \cap B = \emptyset$. A sequence $((A_n, B_n))_{n=1}^\infty$ of disjoint pairs is said to be <u>independent</u> if for all finite disjoint sets $F, G \subseteq \mathbb{N}$,

$$\bigcap_{n \in F} A_n \cap \bigcap_{n \in G} B_n \neq \emptyset .$$

Finally, $((A_n, B_n))_{n=1}^\infty$ is <u>convergent</u> if

$$\forall x \in S \ \left(\left(\forall^\infty_n \ x \notin A_n\right) \text{ or } \left(\forall^\infty_n \ x \notin B_n\right) \right).$$

Proposition Suppose $((A_n, B_n))_{n=1}^{\infty}$ is a sequence of disjoint pairs of subsets of a set S. Then $((A_n, B_n))_{n=1}^{\infty}$ has either an independent or a convergent subsequence.

Proof We define $\mathcal{A} \subseteq [\mathbb{N}]^{\omega}$ by

$$\{n_1, n_2, n_3, \ldots\}_< \in \mathcal{A} \iff$$

$$\forall k \left(\bigcap_{\substack{i \leq k \\ i \text{ even}}} A_{n_i} \cap \bigcap_{\substack{i \leq k \\ i \text{ odd}}} B_{n_i} \neq \emptyset \right).$$

Then $\mathcal{A}$ is closed in the usual topology and hence is completely Ramsey. Therefore, there is some $\{m_i\}_< \in [\mathbb{N}]^{\omega}$ such that either whenever $\{n_i\}_< \subseteq \{m_j\}_<$, we have $\{n_i\}_< \in \mathcal{A}$, or whenever $\{n_i\}_< \subseteq \{m_j\}_<$ we have $\{n_i\}_< \notin \mathcal{A}$.

In the first case we claim that $((A_{m_{2i}}, B_{m_{2i}}))_{i=1}^{\infty}$ is independent.

For if F and G are finite disjoint sets of $\{m_2, m_4, m_6, \dots\}$ we can find a finite subsequence $(n_1, n_2, \dots, n_k)$ of $\{m_i\}_{i=1}^{\infty}$ such that $F \subseteq \{n_i\}_{i \le k,\, i \text{ even}}$ and $G \subseteq \{n_i\}_{i \le k,\, i \text{ odd}}$, whereby

$$\emptyset \neq \bigcap_{\substack{i \le k \\ i \text{ even}}} A_{n_i} \cap \bigcap_{\substack{i \le k \\ i \text{ odd}}} B_{n_i} \subseteq \bigcap_{n \in F} A_n \cap \bigcap_{n \in G} B_n .$$

On the other hand, in the second case we claim that $((A_{m_i}, B_{m_i}))_{i=1}^{\infty}$ converges.

Otherwise there must be some $x \in S$ belonging to infinitely many A_{m_i} and to infinitely many B_{m_i}. This clearly means that we can pick a subsequence $\{n_i\}_{i=1}^{\infty}$ of $\{m_i\}_{i=1}^{\infty}$ such that $x \in A_{n_{2i}}$ and $x \in B_{n_{2i-1}}$ for $i \in \mathbb{N}$. But this contradicts that $\{n_i\}_{i=1}^{\infty} \notin \mathbb{A}$ $\square$