Embeddability of ℓ_1

Theorem (Rosenthal's ℓ_1-Theorem)

Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence in a Banach space X. Then either

(a) (x_n) has a subsequence which is weakly Cauchy, or

(b) (x_n) has a subsequence which is basic and equivalent to the standard unit vector basis of ℓ_1.

We shall deduce this result from a more general theorem concerning bounded sequences in $\ell_\infty(S)$ = bounded real valued functions on S with supremum norm $\|\cdot\|_\infty$.

Theorem If (f_n) is a bounded sequence in $\ell_\infty(S)$, then there is a subsequence (g_n) that is either pointwise convergent or is basic and equivalent to the standard unit vector basis of ℓ_1.

Let us see how this implies the l_1-Theorem:

So suppose (x_n) in X is given. Then X embeds isometrically into $l_\infty(B_{X^*})$ by identifying $x \in X$ with the bounded real valued function $x^* \in B_{X^*} \longmapsto x^*(x) \in \mathbb{R}$.

So now if (y_n) is a subsequence of (x_n) that is pointwise convergent we see that for all $x^* \in B_{X^*}$, $(y_n(x^*))$ is convergent whereby (y_n) is weakly Cauchy.

So let us prove the second theorem:

<u>Proof</u>

For rational numbers $r < s$, let

$$A_n = A_n(r,s) = \{x \in S \mid f_n(x) < r\},$$

$$B_n = B_n(r,s) = \{x \in S \mid f_n(x) > s\},$$

which are clearly disjoint.

Now if $((A_n, B_n))_{n=1}^{\infty}$ is independent, then (f_n) is basic and equivalent with l_1.

For if $\|f_n\| \leq K$, then for all scalars $c_1, \dots, c_n$, $\quad \| \sum_{i=1}^{n} c_i f_i \|_\infty \leq K \sum_{i=1}^{n} |c_i|$; so only the lower bound is non-trivial.

Let now $F = \{i \leq n \mid c_i \geq 0\}$, $G = \{i \leq n \mid c_i < 0\}$.

Then by independence there are some

$$x \in \bigcap_{n \in F} A_n \cap \bigcap_{n \in G} B_n, \quad y \in \bigcap_{n \in G} A_n \cap \bigcap_{n \in F} B_n,$$

whereby

$$c := \sum_{i=1}^{n} c_i f_i(x) = \sum_{i \in F} |c_i| f_i(x) - \sum_{i \in G} |c_i| f_i(x)$$

$$\leq r \sum_{i \in F} |c_i| - s \sum_{i \in G} |c_i|$$

and $$d := \sum_{i=1}^{n} c_i f_i(y) = \sum_{i \in F} |c_i| f_i(y) - \sum_{i \in G} |c_i| f_i(y)$$

$$\geq s \sum_{i \in F} |c_i| - r \sum_{i \in G} |c_i| .$$

So $d - c \geq (s-r) \sum_{i=1}^{n} |c_i|$ and hence

$$2 \left\| \sum_{i=1}^{n} c_i f_i \right\|_\infty \geq (s-r) \sum_{i=1}^{n} |c_i|$$

proving the lower bound. $\square$

Consider now the case when S is a Polish space, i.e., a separable topological space whose topology can be given by a complete metric. For example, $S = (B_{X^*}, w^*)$, where X is separable.

We then consider the space $\mathbb{R}^S$ of all functions from S to $\mathbb{R}$ with the pointwise convergence topology.

We denote by $\mathcal{B}_1(S)$ the set of functions $f \in \mathbb{R}^S$ that are the pointwise limits of a <u>sequence</u> (f_n) of continuous maps $f_n : S \to \mathbb{R}$. These are called <u>Baire class</u> <u>1</u> functions.

<u>Theorem</u> (Baire) Let S be a Polish space and $f : S \to \mathbb{R}$ a function. TFAE:

(i) f is a Baire class 1 function,

(ii) for all closed non-empty $F \subseteq S$, $f|_F$ has a point of continuity.

<u>Theorem</u> (Bourgain - Fremlin - Talagrand)

Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous functions from S to $\mathbb{R}$ and let $\mathcal{K} = \overline{\{f_n\}_{n=1}^{\infty}}$ be the closure of $\{f_n\}_{n=1}^{\infty}$ in $\mathbb{R}^S$.

Then the following are equivalent:

(i) Every $f \in \mathcal{K} \setminus \{f_n\}_{n=1}^{\infty}$ is the limit of a subsequence of (f_n).

(ii) Every $f \in \mathcal{K}$ is a Borel function.

(iii) $\mathcal{K} \subseteq \mathcal{B}_1(S)$.

Notice that obviously (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii).

<u>Proof</u> (ii) $\Rightarrow$ (i): Suppose $f \in \mathcal{K} \setminus \{f_n\}_{n=1}^{\infty}$ and let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ such that for all $x \in S$,

$$\lim_{n \to \mathcal{U}} f_n(x) = f(x).$$

We define for all rationals $p < q$ a derivation

$$D_{p,q} : \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$$

as follows:

For $A \subseteq S$ closed let

$$x \in D_{p,q}(A) \iff \text{for all neighbourhoods } V \ni x$$

$$\{n \geq 1 \mid \inf_{V \cap A} f_n < p < q < \sup_{V \cap A} f_n\} \in \mathcal{U}.$$

Then $D_{p,q}(A)$ is closed, $D_{p,q}(A) \subseteq A$, and if $A \subseteq B$, ~~then~~ $D_{p,q}(A) \subseteq D_{p,q}(B)$.

Now, suppose for a contradiction that $A \neq \emptyset$ is closed and $D_{p,q}(A) = A$. Then we define a scheme of non-empty of subsets of A, $(V_s)_{s \in 2^{<\mathbb{N}}}$, such that for some integers $(k_n)_{n \geq -1}$

(i) $\operatorname{diam}(V_s) < 2^{-|s|}$ for $s \neq \emptyset$.

(ii) $\overline{V}_s \subseteq V_r$ if s extends r

(iii) if $|s| = n$, then $V_{s0} \subseteq \{x \in S \mid f_{k_n}(x) < p\}$

and $V_{s1} \subseteq \{x \in S \mid f_{k_n}(x) > q\}$.

To begin, let $V_\emptyset = A$ and $k_{-1} = 1$.

Now, assume V_s and k_{n-1} have been defined for $s \in 2^n$. Then for all $s \in 2^n$

$$B_s = \{n \geq 1 \mid \inf_{V_s \cap A} f_n < p < q < \sup_{V_s \cap A} f_n\} \in \mathcal{U}$$

so we can find some $k_n > k_{n-1}$ such that

$k_n \in \bigcap_{s \in 2^n} B_s$ and then define V_{s0}, V_{s1} correspondingly.

Now define a continuous injection $h : 2^{\mathbb{N}} \to A$

by $\{h(\alpha)\} = \bigcap_{n \in \mathbb{N}} \overline{V_{\alpha|n}}$, so

$$h(\alpha) \in \bigcap_{\alpha_n = 0} \{x \in A \mid f_{k_n}(x) < p\} \cap \bigcap_{\alpha_n = 1} \{x \in A \mid f_{k_n}(x) > q\}$$

and hence $f_{k_n} \circ h(\alpha) > q \iff \alpha_n = 1$

and $f_{k_n} \circ h(\alpha) < p \iff \alpha_n = 0$.

Now, define a Borel function $g : 2^{\mathbb{N}} \to \mathbb{R}$

$$g = \lim_{n \to \mathcal{U}} (f_{k_n} \circ h) = \left(\lim_{n \to \mathcal{U}} f_{k_n}\right) \circ h \in \mathcal{K} \circ h.$$

Then for $D \subseteq \mathbb{N}$,

$$D \in \mathcal{U} \iff g(\chi_D) > \frac{q+p}{2},$$

which shows that $\mathcal{U}$ is a Borel ultrafilter on $\mathbb{N}$, which is impossible.

This shows that for all closed $A \neq \emptyset$,

$D_{p,q}(A) \subsetneq A$.

Define now by induction on ordinals α, $F^{p,q}_\alpha \subseteq S$ by $F_0^{p,q} = S$, $F^{p,q}_{\alpha+1} = D_{p,q}(F^{p,q}_\alpha)$,

$$F_\lambda^{p,q} = \bigcap_{\xi<\lambda} F_\xi^{p,q}$$ for λ a limit. Then $(F^{p,q}_\alpha)_\alpha$ is a decreasing well-ordered sequence of closed sets in a second countable space and so for some minimal $\xi(p,q) < \omega_1$ we have

$$F^{p,q}_{\xi(p,q)} = \emptyset .$$

To see this suppose ~~that~~ for all $\alpha < \omega_1$, $F^{p,q}_\alpha \setminus F^{p,q}_{\alpha+1} \neq \emptyset$. Then if $\{U_n\}_{n\in N}$ is a basis for S we can find for each $\alpha < \omega_1$ some $n_\alpha \in N$ such that

$$U_{n_\alpha} \cap F^{p,q}_\alpha \neq \emptyset , \quad U_{n_\alpha} \cap F^{p,q}_{\alpha+1} = \emptyset .$$

But then $\alpha \in \omega_1 \mapsto n_\alpha \in N$ is an injection, which is impossible.

Thus, for all $p<q$ rational,

$$S = \bigcup_{\alpha<\xi(p,q)} F^{p,q}_\alpha \setminus F^{p,q}_{\alpha+1} = \bigcup_{\alpha<\xi(p,q)} F^{p,q}_\alpha \setminus D_{p,q}(F^{p,q}_\alpha) .$$

Now, let $\{U_n\}_{n=1}^{\infty}$ be a basis for the topology on S and define for all $p < q$, $\alpha < \xi(p,q)$, and $l \in N$, the sets

$$A(p,q,\alpha,l) = \{n \geq 1 \mid \inf_{U_l \cap F_\alpha^{p,q}} f_n \geq p\}$$

and

$$B(p,q,\alpha,l) = \{n \geq 1 \mid \sup_{U_l \cap F_\alpha^{p,q}} f_n \leq q\}.$$

Let $\mathcal{F}$ be the filter on N with the countable basis

$$\mathcal{U} \cap \{A(p,q,\alpha,l), B(p,q,\alpha,l) \mid p<q, \alpha < \xi(p,q), l \in N\}.$$

We claim that $\lim_{n \to \mathcal{F}} f_n = f$ pointwise.

For if $x \in S$ and $f(x) < p < q$, then there is some $\alpha < \xi(p,q)$ such that $x \in F_\alpha^{p,q} \setminus D_{p,q}(F_\alpha^{p,q})$. Thus, for some neighbourhood U_l of x, either $A(p,q,\alpha,l) \in \mathcal{U}$ or $B(p,q,\alpha,l) \in \mathcal{U}$.

Since $\lim_{n \to \mathcal{U}} f_n(x) = f(x)$, we have

$\{n \geq 1 \mid f_n(x) < p\} \in \mathcal{U}$ and hence

as $x \in U_\ell \cap F_\alpha^{p,q}$, $A(p,q,\alpha,\ell) \notin \mathcal{U}$.

So $B(p,q,\alpha,\ell) \in \mathcal{U}$, whereby

$\lim_{n \to \mathcal{F}} f_n(x) \leq q$.

A similar argument shows $\lim_{n \to \mathcal{F}} f_n(x) \geq s$ for all $s < r < f(x)$ rationals, so $\lim_{n \to \mathcal{F}} f_n(x) = x$ for all $x \in \mathcal{S}$.

Now, let $\mathbb{N} = D_0 \supseteq D_1 \supseteq \dots$ be a countable basis for $\mathcal{F}$, i.e., $E \in \mathcal{F} \Leftrightarrow \exists n\ D_n \subseteq E$. Let also $(k_n)_{n=1}^{\infty}$ be increasing such that $k_n \in D_n$ for all $n \in \mathbb{N}$. Then $f_{k_n} \to f$ pointwise. $\square$

Notice that if $\{f_n\}_{n=1}^{\infty}$ is a <u>pointwise bounded</u> sequence of continuous functions such that for all rational $p < q$,

$$S = \bigcup_{\alpha < \xi(p,q)} F_\alpha^{p,q} \setminus D_{p,q}\left(F_\alpha^{p,q}\right), \quad \text{then } \{f_n\}_{n=1}^{\infty}$$

has a pointwise convergent subsequence.

For define $\mathcal{F}$ as in the proof and notice that if $x \in S$ and $p < q$ are given then either

$$\limsup_{n \to \mathcal{F}} f_n(x) \leq q$$

or

$$\liminf_{n \to \mathcal{F}} f_n(x) \geq p.$$

As $(f_n(x))_{n=1}^{\infty}$ is a bounded sequence this shows ~~that~~ $\lim_{n \to \mathcal{F}} f_n(x)$ exists. So we can extract a convergent subsequence.

So if $\{f_n\}_{n=1}^{\infty}$ has no convergent subsequence then there must be $\emptyset \neq A \subseteq S$ closed and $p < q$ such that $D_{p,q}(A) = A$.

Define $h : 2^{\mathbb{N}} \to A$ and $\{k_n\}_{n=1}^{\infty}$ as in the proof. Then

$$f_{k_n} \circ h(\alpha) > q \iff \alpha_n = 1 \quad \& \quad f_{k_n} \circ h(\alpha) < p \iff \alpha_n = 0 .$$

Now, if $\{f_n\}_{n=1}^{\infty} \subseteq B_{\ell_\infty}(S)$, then $(f_{k_n})_{n=1}^{\infty}$ will be equivalent to the standard basis for ℓ_1. This gives a different proof of the ℓ_1-theorem.

Consider the special case where $\{f_n\}_{n=1}^{\infty}$ is a dense sequence in B_X considered as a subset of the set of continuous functions on $S = (B_{X^*}, w^*)$. As B_X is dense in $B_{X^{**}}$, we have that

$$\mathcal{K} = \overline{\{f_n\}_{n=1}^{\infty}} = B_{X^{**}} .$$

So either $\{f_n\}_{n=1}^{\infty}$ has a subsequence equivalent to the unit vector basis in ℓ_1 or, by (i) of the theorem of Bourgain - Fremlin - Talagrand, every $x^{**} \in B_{X^{**}}$ is the w^*-limit of a sequence in X.

This proves

<u>Theorem</u> (Odell - Rosenthal)

Let X be a separable Banach space not containing a copy of ℓ_1. Then every element of X^{**} is the w^*-limit of a sequence in X.