Schauder bases

A sequence $(e_n)_{n=1}^{\infty}$ in an infinite-dimensional Banach space X is said to be a basis for X if for all $x \in X$ there is a unique representation

$$x = \sum_{i=1}^{\infty} a_i e_i$$

as a norm-convergent series.

Theorem Let (e_i) be a sequence in a Banach space X. Then the following are equivalent:

(i) (e_i) is a basis for X

(ii) there is a sequence (e_i^*) in X^* s.t.

(a) $e_k^*(e_i) = \delta_{ik}$

(b) $$x = \sum_{i=1}^{\infty} e_i^*(x) e_i .$$

In this case (e_i^*) are called the biorthogonal functionals associated with (e_i).

<u>Proof</u> The implication (ii) $\Rightarrow$ (i) is easy.

So assume that (e_i) is a basis for X and let $e_n^* : X \to \mathbb{R}$ be the functional defined by $e_n^*\left(\sum_{i=1}^{\infty} a_i e_i\right) = a_n$. Let also $S_n : X \to X$ be the linear operator

$$S_n\left(\sum_{i=1}^{\infty} a_i e_i\right) = \sum_{i=1}^{n} a_i e_i .$$

Both e_n^* and S_n are potentially unbounded. But since for each $x \in X$, $S_n x \xrightarrow[\|\cdot\|]{} x$, the following defines a norm on X

$$|||x||| = \sup_{n \geq 1} \|S_n x\| \geq \|x\| .$$

We claim that $(X, |||\cdot|||)$ is complete. So suppose (x_n) is $|||\cdot|||$-Cauchy. Then $(x_n)_n$ and $(S_k x_n)_n$ are also $\|\cdot\|$-Cauchy for all k and hence $\|\cdot\|$-converge to some x and y_k in X respectively.

We wish to prove that also $x_n \xrightarrow[|||\cdot|||]{} x$, which would show completeness.

Now, for each k, the $S_k x_n$ belong to the finite dimensional space $E_k = [e_1, \dots, e_k]$ on which the functionals $e_1^*, \dots, e_k^*$ are continuous, so for $j \le k$,

$$\lim_n e_j^*(x_n) = \lim_n e_j^*(S_k x_n) = e_j^*\left(\lim_n S_k x_n\right)$$

$$= e_j^*(y_k) =: a_j$$

and thus a_j is independent of $k \geq j$.

So as $y_k \in E_k$ we have $y_k = \sum_{i=1}^{k} a_i e_i$, and therefore, if $y_k \xrightarrow[\|\cdot\|]{} x$, then $x = \sum_{i=1}^{\infty} a_i e_i$.

Now given $\varepsilon > 0$ pick some N such that for $n \geq N$, we have $|||x_n - x_N||| < \varepsilon$. Find then some K such that for $k \geq K$, we have $\|x_N - S_k x_N\| < \varepsilon$. Then for $k \geq K$ we have

$$\|y_k - x\| = \lim_n \|S_k x_n - x_n\|$$

$$\le \lim_n \left(\|S_k x_n - S_k x_N\| + \|S_k x_N - x_N\| + \|x_N - x\|\right)$$

$$< 3\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, $\|y_k - x\| \to 0$.

Now,

$$|||x_n - x||| = \sup_{k\geq 1} \|S_k x_n - S_k x\|$$

$$\leq \sup_{k\geq 1} \limsup_m \|S_k x_n - S_k x_m\|$$

$$\leq \limsup_m \sup_k \|S_k x_n - S_k x_m\|$$

$$= \limsup_m |||x_n - x_m||| \xrightarrow[n]{} 0.$$

So $(X, |||\cdot|||)$ is complete.

Since also $\|\cdot\| \leq |||\cdot|||$ we have that

$$id : (X, |||\cdot|||) \to (X ; \|\cdot\|)$$

is continuous and hence an isomorphism by the open mapping theorem.

So there is some C such that $|||\cdot||| \leq C\|\cdot\|$, which implies that $\|S_k\| \leq C$ for all k.

As $|e_n^*(x)| \leq \|S_n(x) - S_{n-1}(x)\| \, \|e_n\|^{-1}$

$$\leq 2C\|e_n\|^{-1} \cdot \|x\|$$

we see that each e_n^* is bounded and thus belongs to X^*. $\square$

Notice that if (e_n) is a basis for X then it follows from the proof of the theorem that the partial sum operators $S_n : X \to X$ are uniformly bounded in norm. We call $C = \sup_n \|S_n\|$ the <u>basis constant</u> of (e_n) and say that (e_n) is <u>monotone</u> in case $C = 1$.

It turns out that in order for a space X to have a basis it is enough to have a sequence of bounded projections as the S_n.

<u>Theorem</u> Suppose X is a Banach space and $S_n : X \to X$ is a sequence of bounded linear projections (i.e., idempotents $S_n \circ S_n = S_n$) such that

(a) $\dim S_n(X) = n$

(b) $S_n S_m = S_m S_n = S_{\min\{n,m\}}$

(c) $S_n(x) \to x$ for all $x \in X$.

Then if $e_k \in S_k(X) \cap S_{k-1}^{-1}(0)$ and $e_k \neq 0$, the sequence (e_k) is a basis for X with projections S_n.

Proof We construct biorthogonal functionals to (e_n) as follows :

Let e_1^* be defined by $e_1^*(x)e_1 = S_1(x)$.

Now, suppose e_n^* has been defined. Then we let $e_{n+1}^*(x)e_{n+1} = S_{n+1}(x) - S_n(x)$.

Clearly, $e_n^* \in X^*$ for all n, and also

$$|e_n^*(e_i)| \leq \| S_n(e_i) - S_{n-1}(e_i)\| \cdot \|e_i\|^{-1} = \delta_{ni}.$$

So to see that $x = \sum_{i=1}^{\infty} e_i^*(x)e_i$, notice that

$$S_n(x) = \sum_{i=1}^{n} S_i(x) - S_{i-1}(x) = \sum_{i=1}^{n} e_i^*(x)e_i$$

which by (iii) converges to x. $\square$

We say that a sequence (e_i) in a Banach space X is a __basic__ __sequence__ if it is a basis for its closed linear span $[e_i]$. We define the __basis__ __constant__ correspondingly.

Grunblum's criterion

A sequence (e_i) of non-zero elements of a Banach space X is basic if and only if there is a constant K such that for all sequences of scalars (a_i) and integers $m \leq n$

$$\left\| \sum_{i=1}^{m} a_i e_i \right\| \leq K \left\| \sum_{i=1}^{n} a_i e_i \right\| .$$

Proof If (e_i) is a basic sequence, then we can just let K be the basis constant of (e_i).

Conversely, if K exists we let E be the vector subspace of X spanned by (e_i) and define for $\sum_{i=1}^{n} a_i e_i \in E$ and m,

$$S_m \left(\sum_{i=1}^{n} a_i e_i \right) = \sum_{i=1}^{\min(m,n)} a_i e_i .$$

Then the S_m have norm bounded by K and hence extend uniquely to continuous operators $S_m : [e_k] \longrightarrow [e_1, \ldots, e_m]$ with $\|S_m\| \leq K$. Clearly (a), (b) of the preceding theorem are true and also $S_n(x) \to x$ for all $x \in E$.

But consider the operator $T : X \to \ell_\infty(X)$ defined by $T(x) = (S_n(x) - x)_n$, then $E \subseteq T^{-1}(c_0(X))$, which is closed, so $X \subseteq T^{-1}(c_0(X))$, whence $S_n(x) \to x$ for all $x \in X$. $\square$

Equivalence of basic sequences

Two basic sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ in Banach spaces X and Y respectively are equivalent, written $(x_n) \sim (y_n)$, if for all sequences of scalars (a_n), the series $\sum_{i=1}^\infty a_i x_i$ converges if and only if $\sum_{i=1}^\infty a_i y_i$ converges.

Theorem Two basic sequences (x_n) and (y_n) are equivalent if and only if there is a (necessarily unique) isomorphism $T : [x_n] \to [y_n]$ such that $Tx_n = y_n$ for all n.

<u>Proof</u> If such an isomorphism exists, then clearly (x_n) and (y_n) are equivalent. So suppose instead that (x_n) and (y_n) are equivalent and define a linear transformation

$T : [x_n] \to [y_n]$ by setting

$T\left(\sum_{i=1}^{\infty} a_i x_i\right) = \sum_{i=1}^{\infty} a_i y_i$. By assumption,

T is well-defined and bijective, so to prove that T is continuous we show that it has closed graph $\subseteq X \times Y$.

So suppose $u_j \to u$ in X and $Tu_j \to v$ in Y. Write $u_j = \sum_{i=1}^{\infty} x_i^*(u_j) x_i$ and

$u = \sum_{i=1}^{\infty} x_i^*(u) x_i$. Then it follows from the continuity of the biorthogonal functionals x_n^* and y_n^* that for all i,

$$x_i^*(u) = \lim_j x_i^*(u_j) = \lim_j y_i^*(Tu_j) = y_i^*(v)$$

whence $$v = \sum_{i=1}^{\infty} y_i^*(v) y_i = \sum_{i=1}^{\infty} x_i^*(u) y_i$$

and thus $v = Tu$. $\square$

<u>Corollary</u> Two basic sequences (x_n) and (y_n) are equivalent if and only if there is a constant $C > 0$ such that for all finite sequences of scalars $(a_i)_{i=1}^{k}$ we have

$$C^{-1} \left\| \sum_{i=1}^{k} a_i x_i \right\| \le \left\| \sum_{i=1}^{k} a_i y_i \right\| \le C \left\| \sum_{i=1}^{k} a_i x_i \right\| .$$

In this case the basic sequences are said to be <u>C-equivalent</u>.

<u>Block basic sequences</u>

Suppose (e_n) is a basic sequence. If (a_n) is a sequence of scalars with only finitely many non-zero terms, we say that $x = \sum_{n=1}^{\infty} a_n e_n$ is a <u>block</u> <u>vector</u> of (e_n) and let the <u>support</u> of x be the set

$$\operatorname{supp}(x) = \{ n \in \mathbb{N} \mid a_n \neq 0 \} .$$

Now, if A and B are non-empty subsets of $\mathbb{N}$ and $n, m \in \mathbb{N}$, we write

$n \le A < B \le m$ to denote that

$n \le \min A$, $\sup A < \min B$, and

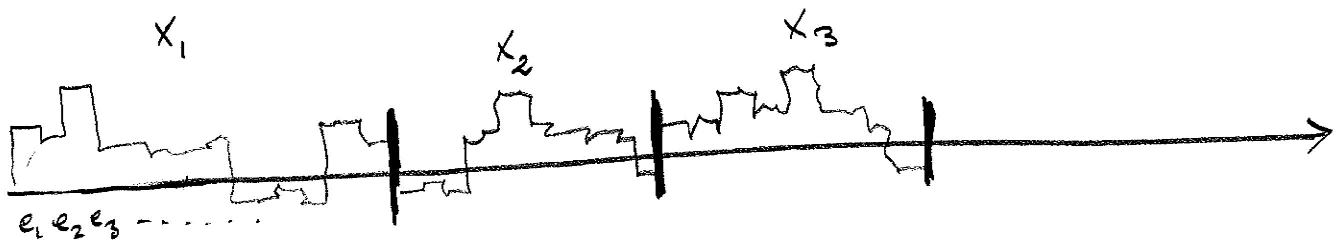
$\sup B \le m$.

Also, if x and y are non-zero block vectors of (e_n) we write

$n \le x < y \le m$ instead of

$n \le \operatorname{supp}(x) < \operatorname{supp}(y) \le m$.

A non-zero finite or infinite sequence of non-zero block vectors (x_n) of (e_n) is said to be a <u>block (basic) sequence</u> or a <u>blocking</u> of (e_n) if for all n, we have $x_n < x_{n+1}$.



<u>Proposition</u> Suppose (e_n) is a basic sequence with basis constant K and (x_n) is an infinite block basic sequence of (e_n). Then (x_n) is itself a basic sequence with basis constant $\leq K$.

Proof We can write $x_n = \sum_{i=p_{n-1}+1}^{p_n} a_i e_i$ for some sequence (a_i) of scalars and strictly increasing sequence (p_n) of integers with $p_0 = 0$. Thus for any finite sequence $(b_n)_{n=1}^{k}$ of scalars and $m \leq k$,

$$\left\| \sum_{n=1}^{m} b_n x_n \right\| = \left\| \sum_{n=1}^{m} b_n \left(\sum_{i=p_{n-1}+1}^{p_n} a_i e_i \right) \right\|$$

$$= \left\| \sum_{n=1}^{m} \sum_{i=p_{n-1}+1}^{p_n} b_n a_i e_i \right\| = \left\| \sum_{i=1}^{p_m} c_i e_i \right\|$$

$$\leq K \left\| \sum_{i=1}^{p_k} c_i e_i \right\| = K \left\| \sum_{n=1}^{k} b_n x_n \right\| ,$$

where $c_i = b_n a_i$ for all $i \in]p_{n-1}, p_n]$.

So by Grunblum's criterion, (x_n) is a basic sequence with constant $\leq K$. $\square$

Definition Two sequences (x_n) and (y_n) in Banach spaces X and Y respectively are said to be congruent with respect to X and Y if there is an isomorphism $T: X \to Y$ such that $Tx_n = y_n$ for all n.

In particular, if two basic sequences are congruent, then they are also equivalent. However, the converse may not hold. For if (e_n) denotes the standard unit vector basis of ℓ_1, then the sequences $(x_n)_{n=1}^{\infty} = (e_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty} = (e_{n+1})_{n=1}^{\infty}$ are clearly equivalent but they are not congruent with respect to $[e_n]_{n=1}^{\infty}$ and $[e_n]_{n=1}^{\infty}$.

Principle of small perturbations (Krein-Milman-Rutman)

Let (x_n) be a basic sequence in a Banach space X with basis constant K. If (y_n) is another sequence in X such that

$$2K \sum_{n=1}^{\infty} \frac{\|x_n - y_n\|}{\|x_n\|} =: \vartheta < 1,$$

then (x_n) and (y_n) are congruent with respect to X and X by an operator A s.t. $\|A\| < 1+\vartheta$, $\|A^{-1}\| < (1-\vartheta)^{-1}$.

It follows that

(i) (y_n) is a basic sequence with basis constant at most $K(1+\vartheta)(1-\vartheta)^{-1}$.

(ii) If (x_n) is a basis for X, then so is (y_n).

(iii) If $[x_n]$ is complemented in X, then so is $[y_n]$.

Proof If $x \in [x_i]$ and $m \geq 2$, then

$$x_m^*(x)\,x_m = \sum_{i=1}^{m} x_i^*(x)\,x_i - \sum_{i=1}^{m-1} x_i^*(x)\,x_i .$$

So $\|x_m^*(x)\,x_m\| \leq 2K\,\|x\|$ for all $x \in X$, which implies that $\|x_m^*\|\,\|x_m\| \leq 2K$. Also, for $m=1$, we have evidently $\|x_m^*\|\,\|x_m\| \leq K$. So by Hahn-Banach we can extend x_m^* to elements of X^* of norm $\leq \dfrac{2K}{\|x_m\|}$ that we denote by $\hat{x}_m^*$.

Define now for $x \in X$

$$A(x) = x + \sum_{m=1}^{\infty} \hat{x}_m^*(x)\,(y_m - x_m)$$

Then A is a perturbation of the identity from X to X.

Clearly, $A : X \to X$ satisfies

$$\|A\| \le 1 + \sum_{m=1}^{\infty} \|\hat{x}_m^*\| \, \|y_m - x_m\| \le 1 + 2K \sum_{m=1}^{\infty} \frac{\|y_m - x_m\|}{\|x_m\|}$$

$$= 1 + \vartheta$$

so is bounded. But also $\|A - I\| \le \vartheta < 1$, so A is invertible and $\|A^{-1}\| \le (1-\vartheta)^{-1}$.

Recall : $|r| < 1 \Rightarrow \dfrac{1}{r+1} = \dfrac{\sum_{n=0}^{k} (-1)^n r^n}{(-1)^n r^{n+1} + 1} \longrightarrow \sum_{n=0}^{\infty} (-1)^n r^n$

So as $\|A - I\| \le \vartheta < 1$, we have

$$A^{-1} = \frac{1}{(A-I)+I} = \sum_{n=0}^{\infty} (-1)^n (A-I)^n .$$

Also, clearly $A(x_n) = y_n$, so (x_n) and (y_n) are congruent with respect to X and X.

Thus (i) and (ii) follow. Now suppose $[x_n]$ is complemented in X and let $P : X \to [x_n]$ be a projection. Extend x_m^* to all of X by setting $u_m^*(x) = x_m^* \circ P(x)$. Then the u_m^* satisfy for $x \in X$

$$(*) \qquad P(x) = \sum_{m=1}^{\infty} u_m^*(x) \, x_m ,$$

in other words, (*) defines a projection of X onto $[x_n]$. Since A is an isomorphism of X such that $Ax_n = y_n$, also

$$Qx = \sum_{m=1}^{\infty} A^* x_m^* (x)\, y_m$$

defines a projection onto $[y_m]$. $\square$

<u>Bessaga - Pełczyński Selection Principle</u>

Let (e_n) be a basis for a Banach space X with basis constant K and biorthogonal functionals (e_n^*). Suppose (x_n) is a sequence in X bounded away from 0 but such that for all k, $\lim_n e_k^*(x_n) = 0$.

Then (x_n) has a subsequence (z_n) congruent in X to a block basic sequence (y_n) of (e_n). Moreover, (z_n) and (y_n) can be chosen such that the congruence is arbitrarily close to I.

Proof

Let $\alpha = \inf_n \|x_n\|$ and choose $0 < \nu < 1/4$.

Choose $z_1 = x_1$ and find r_1 large enough that $\|z_1 - S_{r_1} z_1\| < \frac{\nu\alpha}{2K}$. By assumption, $\|S_{r_1} x_n\| \xrightarrow[n]{} 0$, so we can find some $z_2 = x_n$ such that $\|S_{r_1} z_2\| < \frac{\nu^2\alpha}{2K}$.

So pick $r_2 > r_1$ such that $\|z_2 - S_{r_2} z_2\| < \frac{\nu^2\alpha}{2K}$.

Again, since $\|S_{r_2} x_n\| \xrightarrow[n]{} 0$, we can find $z_3 = x_n$ such that $\|S_{r_2} z_3\| < \frac{\nu^3\alpha}{2K}$ and $r_3 > r_2$ with $\|z_3 - S_{r_3} z_3\| < \frac{\nu^3\alpha}{2K}$ etc.

Thus, in this way we have a subsequence (z_n) of (x_n) and $r_1 < r_2 < \dots$ such that

$$\|z_n - S_{r_n} z_n\| < \frac{\nu^n\alpha}{2K}, \quad \|S_{r_{n-1}} z_n\| < \frac{\nu^n\alpha}{2K}.$$

Now, set $y_n = S_{r_n} z_n - S_{r_{n-1}} z_n$, whereby $\|y_n - z_n\| < \frac{\nu^n\alpha}{K}$ and (y_n) is a block sequence of (e_n) with

$$\|y_n\| \geq \alpha - 2\frac{\nu^n\alpha}{2K} \geq \alpha(1-\nu).$$

Also,
$$2K \sum_{n=1}^{\infty} \frac{\|z_n - y_n\|}{\|y_n\|} \le 2K \sum_{n=1}^{\infty} \frac{\nu^n \alpha}{K} \Big/ \alpha(1-\nu)$$

$$= 2\nu(1-\nu)^{-1} \sum_{n=1}^{\infty} \nu^n = 2\nu(1-\nu)^{-2} =: \vartheta < 1.$$

So by the principle of small perturbations (z_n) and (y_n) are equivalent. Moreover, by choosing ν sufficiently small, we can make ϑ arbitrarily small and hence make the congruence arbitrarily close to I. $\square$

Existence of basic sequences (Mazur)

Lemma Suppose that $S \subseteq X^*$ satisfies $0 \in \overline{S}^{w^*}$ and $0 \notin \overline{S}^{\|\|}$. Let $E \subseteq X^*$ be finite-dimensional. Then for all $\varepsilon > 0$ there is $x^* \in S$ such that

$$\|e^* + \lambda x^*\| \geq (1-\varepsilon)\|e^*\|$$

for all $\lambda \in \mathbb{R}$ and $e^* \in E$.

[So $\mathbb{R}x^*$ can be seen as being almost orthogonal to E^*.]

Proof Pick $\alpha > 0$ such that $\alpha < \|x^*\|$ for all $x^* \in S$. Set now $\bar{\varepsilon} = \dfrac{\alpha\varepsilon}{2(1+\alpha)}$ and notice that since E is finite-dimensional its unit sphere $S_E = \{e^* \in E \mid \|e^*\| = 1\}$ is compact.

So fix an $\bar{\varepsilon}$-dense finite set $D \subseteq S_E$.

For each $y^* \in D$ pick $x \in S_X$ such that $y^*(x) > 1 - \bar{\varepsilon}$ and let $F \subseteq S_X$ be the set of these.

Since $0 \in \overline{S}^{w^*}$, there is $x^* \in S$ such that

$|x^*(x)| < \bar{\varepsilon}$ for all $x \in F$.

Now if $e^* \in S_E$ we can find some $y^* \in D$ such that $\|e^* - y^*\| < \bar{\varepsilon}$ and find some $x \in F$ such that $y^*(x) > 1 - \bar{\varepsilon}$.

Then if $|\lambda| > \frac{2}{\alpha}$, we have

$$\|e^* + \lambda x^*\| \geq \frac{2}{\alpha} \cdot \alpha - 1 = 1 > (1-\varepsilon)\|e^*\|$$

and if $|\lambda| \leq \frac{2}{\alpha}$, we have

$$\|e^* + \lambda x^*\| \geq \|y^* + \lambda x^*\| - \bar{\varepsilon} \geq (y^* + \lambda x^*)(x) - \bar{\varepsilon}$$

$$\geq 1 - \bar{\varepsilon} - \lambda \bar{\varepsilon} - \bar{\varepsilon} \geq 1 - 2\bar{\varepsilon} - \frac{\varepsilon}{1+\alpha} = 1 - \frac{\alpha\varepsilon + \varepsilon}{1+\alpha}$$

$$= 1 - \varepsilon = (1-\varepsilon)\|e^*\|.$$

To see that $\|e^* + \lambda x^*\| \geq (1-\varepsilon)\|e^*\|$ for all $e^* \in E$ and $\lambda \in \mathbb{R}$, we can just work with $\frac{e^*}{\|e^*\|}$ and $\frac{\lambda}{\|e^*\|} \in \mathbb{R}$. $\square$

<u>Theorem</u> (Mazur) Suppose $S \subseteq X^*$ is bounded away from 0 but $0 \in \overline{S}^{w*}$. Then for any $\varepsilon > 0$, S contains a basic sequence with basis constant $\leq 1+\varepsilon$.

<u>Proof</u> Fix a decreasing sequence of strictly positive real numbers (ε_n) such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ and $\prod_{n=1}^{\infty} (1-\varepsilon_n) > (1+\varepsilon)^{-1}$.

Choose $x_1^* \in S$ arbitrarily and let $E_1 = [x_1^*]$. By the preceding lemma there is $x_2^* \in S$ such that $\|e^* + \lambda x_2^*\| \geq (1-\varepsilon_1)\|e^*\|$ for all $e^* \in E_1$ and $\lambda \in \mathbb{R}$.

Again let $E_2 = [x_1^*, x_2^*]$ and choose $x_3^* \in S$ such that for $e^* \in E_2$ and $\lambda \in \mathbb{R}$, we have

$$\|e^* + \lambda x_2^*\| \geq (1-\varepsilon_2)\|e^*\|,$$

etc. We thus construct (x_n^*) in S such that for all scalars (a_n) we have

$$\left\| \sum_{n=1}^{k+1} a_n x_n^* \right\| \geq (1-\varepsilon_k) \left\| \sum_{n=1}^{k} a_n x_n^* \right\|$$

So for all $m \leq k$, $\left\| \sum_{n=1}^{m} a_n x_n^* \right\| \leq \prod_{n=1}^{\infty} (1-\varepsilon_n)^{-1} \left\| \sum_{n=1}^{k} a_n x_n^* \right\|$

$$\le (1+\varepsilon)\left\| \sum_{n=1}^{k} a_n x_n^* \right\| .$$

So by Grunblum's condition, (x_n^*) is a basic sequence with basis constant $\le 1+\varepsilon$. $\square$

<u>Theorem</u> Every infinite dimensional Banach space X contains a basic sequence with basis constant arbitrarily close to 1.

<u>Proof</u> Let $S = S_X \subseteq X^{**}$. $\square$